# CLASE 7

Martes 29/09/2020

En esta clase:

- Representación de **Heisenberg**

- **Función de Green** de una partícula

Many-Particle Theory

E K U Gross
Universität Würzburg

E Runge
Harvard University

O Heinonen
University of Central Florida

**Chapter 14**

**Pictures**

**Chapter 15**

**The single-particle Green's function**

Sea un observable O. En mecánica cuántica todo depende de los elementos de matriz:

$$\langle \psi \mid O \mid \phi \rangle$$

Sea $A(t)$ un operador unitario que puede depender del tiempo:

$$\longrightarrow \quad A^\dagger(t)A(t) = 1 = A(t)A^\dagger(t)$$

Podemos hacer:

$$\begin{aligned} \langle \psi \mid O \mid \phi \rangle &= \langle \psi \mid \underbrace{A^\dagger(t)A(t)}_{1} O \underbrace{A^\dagger(t)A(t)}_{1} \mid \phi \rangle \\ &= \langle A(t)\psi \mid A(t)OA^\dagger(t) \mid A(t)\phi \rangle. \end{aligned}$$

Entonces, si transformamos todos los estados y los operadores según:

$$\begin{aligned} \mid \psi(t)\rangle_A &\equiv A(t)|\psi\rangle \\ O(t)_A &\equiv A(t)OA^\dagger(t) \end{aligned}$$

Todos los elementos de matriz de operadores se conservan:

$$\langle \psi \mid O \mid \phi \rangle = \langle \psi_A \mid O_A \mid \phi_A \rangle$$

# Pictures

## Schrödinger Picture: $A(t) \equiv 1$

Los estados y los operadores tienen la dependencia temporal “normal” que surge de aplicar las reglas de cuantización para obtener operadores a partir de las variables clásicas.

Evolución temporal del estado – ecuación de Schrödinger dependiente del tiempo:

$$i\hbar \frac{\partial}{\partial t} \mid \Psi(t)\rangle_S = H_S \mid \Psi(t)\rangle_S$$

El operador de evolución nos da el ket evolucionado:

$$\mid \Psi(t)\rangle_S \equiv U(t,t_0)_S \mid \Psi(t_0)\rangle$$

Reemplazando en la ec. de Schrödinger:

$$i\hbar \frac{\partial}{\partial t} U(t,t_0)_S \mid \Psi(t_0)\rangle_S = H_S U(t,t_0)_S \mid \Psi(t_0)\rangle_S$$

Pictures

# Schrödinger Picture: $A(t) \equiv 1$

$$i\hbar \frac{\partial}{\partial t} U(t,t_0)_S \mid \Psi(t_0)\rangle_S = H_S U(t,t_0)_S \mid \Psi(t_0)\rangle_S$$

Como es válido para todo estado inicial:

$$\longrightarrow \quad i\hbar \frac{\partial}{\partial t} U(t,t_0)_S = H_S U(t,t_0)_S \qquad \text{con} \quad U(t_0,t_0) = 1$$

$$\mid \Psi(t)\rangle_S = U(t,t')_S \mid \Psi(t')\rangle_S = U(t,t')_S U(t',t_0)_S \mid \Psi(t_0)\rangle_S$$

$$\longrightarrow \quad U(t,t_0)_S = U(t,t')_S U(t',t_0)_S \qquad \text{intuitivo}$$

Si ponemos: $t = t_0 \quad \longrightarrow \quad 1 = U(t_0,t')_S U(t',t_0)_S$

$$\longrightarrow \quad U(t,t_0)_S^{-1} = U(t_0,t)_S$$

**Pictures**

# Schrödinger Picture: $A(t) \equiv 1$

**Unitariedad** del operador de evolución

$$\begin{aligned}\frac{\partial U_S}{\partial t} &= -\frac{i}{\hbar} H_S U_S \\ \frac{\partial U_S^\dagger}{\partial t} &= \frac{i}{\hbar} U_S^\dagger H_S\end{aligned} \longrightarrow \frac{\partial}{\partial t}\left[U_S^\dagger U_S\right] = \frac{\partial U_S^\dagger}{\partial t} U_S + U_S^\dagger \frac{\partial U_S}{\partial t} = 0$$

$$\longrightarrow \left.\begin{aligned} U^\dagger(t,t_0)_S U(t,t_0)_S &= \text{constant.} \\ U(t_0,t_0)_S &= 1 \\ U^\dagger(t_0,t_0)_S &= 1 \end{aligned}\right\} \quad U^\dagger(t,t_0)_S U(t,t_0)_S = 1$$

unitario

Si $H_S$ no depende del tiempo: $\boxed{U(t,t_0)_S = \exp\left[-\frac{i}{\hbar} H_S (t-t_0)\right]}$

Pictures

# Heisenberg picture

Está definida por: $A(t) \equiv U^{\dagger}(t,t_0)_S = U(t_0,t)_S$

Vemos que el estado está "frozen":

$$|\Psi(t)\rangle_H = A(t)\,|\Psi(t)\rangle_S = U(t_0,t)_S\,|\Psi(t)\rangle_S = |\Psi(t_0)\rangle_S = \text{constant.}$$

Los operadores están dados por:

$$O(t)_H = U^{\dagger}(t,t_0)_S O(t)_S U(t,t_0)_S$$

Y como dijimos, si $H_S$ es independiente del tiempo:

$$O(t)_H = e^{+\frac{i}{\hbar}H_S t} O(t)_S e^{-\frac{i}{\hbar}H_S t} \qquad t_0 = 0$$

**Pictures**

# Heisenberg picture

Los estados están fijos y toda la evolución temporal está en los operadores.

Ecuación de movimiento de los operadores:

$$
\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}O(t)_H &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(U_S^\dagger O_S U_S\right) \\
&= i\hbar\left(\frac{\partial U_S^\dagger}{\partial t}\right)O_S U_S + i\hbar U_S^\dagger\left(\frac{\partial O_S}{\partial t}\right)U_S + i\hbar U_S^\dagger O_S\left(\frac{\partial U_S}{\partial t}\right)
\end{aligned}
$$

Teníamos: $i\hbar\frac{\partial U_S}{\partial t} = H_S U_S \longrightarrow -i\hbar\frac{\partial U_S^\dagger}{\partial t} = U_S^\dagger H_S$

$$
\begin{aligned}
& i\hbar\frac{\mathrm{d}}{\mathrm{d}t}O(t)_H \\
&= -U_S^\dagger H_S U_S U_S^\dagger O_S U_S + i\hbar U_S^\dagger\left(\frac{\partial O_S}{\partial t}\right)U_S + U_S^\dagger O_S U_S U_S^\dagger H_S U_S \\
&= -H_H O_H + O_H H_H + i\hbar U_S^\dagger\left(\frac{\partial O_S}{\partial t}\right)U_S.
\end{aligned}
$$

# Heisenberg picture

$$
\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}O(t)_H & \\
&= -U_S^\dagger H_S U_S U_S^\dagger O_S U_S + i\hbar U_S^\dagger \left(\frac{\partial O_S}{\partial t}\right) U_S + U_S^\dagger O_S U_S U_S^\dagger H_S U_S \\
&= -H_H O_H + O_H H_H + i\hbar U_S^\dagger \left(\frac{\partial O_S}{\partial t}\right) U_S .
\end{aligned}
$$

Definimos: $\left[\frac{\partial O}{\partial t}\right]_H \equiv U_S^\dagger \left(\frac{\partial O_S}{\partial t}\right) U_S$

$$
\boxed{i\hbar\frac{\mathrm{d}}{\mathrm{d}t}O(t)_H = [O_H, H_H] + i\hbar\left[\frac{\partial O}{\partial t}\right]_H}
$$

Ecuación de movimiento de Heisenberg

# Interaction Picture

Supongamos: $H_S = H_0 + V_S$ $\qquad H_0 \neq H_0(t)$

$$A(t) \equiv e^{\frac{i}{\hbar}H_0 t} \longrightarrow \begin{cases} |\Psi(t)\rangle_I = e^{\frac{i}{\hbar}H_0 t}\,|\Psi(t)\rangle_S \\ O(t)_I = e^{\frac{i}{\hbar}H_0 t} O(t)_S e^{-\frac{i}{\hbar}H_0 t} \end{cases}$$

Notar que, en particular, el $H_0$ no cambia:

$$[H_0]_I = e^{\frac{i}{\hbar}H_0 t}\,[H_0]_S\,e^{-\frac{i}{\hbar}H_0 t} = [H_0]_S \equiv H_0$$

# Interaction Picture

En el picture de interacción **evolucionan los estados y los operadores**

Operadores

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} O(t)_I = [O(t)_I, H_0] + i\hbar \left[\frac{\partial O}{\partial t}\right]_I$$

$$\left[\frac{\partial O}{\partial t}\right]_I \equiv \mathrm{e}^{\frac{i}{\hbar} H_0 t} \left[\frac{\partial O_S}{\partial t}\right] \mathrm{e}^{-\frac{i}{\hbar} H_0 t}.$$

Se demuestra igual que en el picture de Heisenberg

# Funciones de Green

Hamiltoniano
con interacción

Estado fundamental

Representación
de Heisenberg

$$H|\Psi_0\rangle = E_0|\Psi_0\rangle$$

Operadores de campo en
representación de Heisenberg:

$$\boxed{\begin{aligned} \psi(x,t)_H &= \psi_s(\mathbf{r},t)_H \\ \psi^\dagger(x,t)_H &= \psi_s^\dagger(\mathbf{r},t)_H \end{aligned}}$$

# Funciones de Green

$$iG(xt,x't') \equiv \frac{1}{\langle\Psi_0|\Psi_0\rangle}\langle\Psi_0\,|\,T[\psi(x,t)_H\psi^\dagger(x',t')_H]|\,\Psi_0\rangle$$

**Función de Green de una partícula (ordenada temporalemente)**

**Producto ordenado temporalemente de operadores de creación y destrucción:**

$$T[A(t)B(t')] = \begin{cases} A(t)B(t') & \text{for } t > t' \\ \pm B(t')A(t) & \text{for } t' > t \end{cases}$$

**O sea:**

$$iG(xt,x't') = \frac{1}{\langle\Psi_0|\Psi_0\rangle}\begin{cases} \langle\Psi_0\,|\,\psi(x,t)_H\psi^\dagger(x't')_H|\,\Psi_0\rangle & t > t' \\ -\langle\Psi_0\,|\,\psi^\dagger(x',t')_H\psi(x,t)_H|\,\Psi_0\rangle & t' > t \end{cases}$$

FERMIONES

# Funciones de Green

FERMIONES

$$iG(xt, x't') = \frac{1}{\langle\Psi_0|\Psi_0\rangle}\begin{cases}\langle\Psi_0 \mid \psi(x,t)_H\psi^\dagger(x't')_H \mid \Psi_0\rangle & t > t' \\ -\langle\Psi_0 \mid \psi^\dagger(x',t')_H\psi(x,t)_H \mid \Psi_0\rangle & t' > t\end{cases}$$

$$= \frac{1}{\langle\Psi_0|\Psi_0\rangle}\Big[\theta(t-t')\langle\Psi_0 \mid \psi(x,t)_H\psi^\dagger(x',t')_H \mid \Psi_0\rangle$$

$$-\theta(t'-t)\langle\Psi_0 \mid \psi^\dagger(x',t')_H\psi(x,t)_H \mid \Psi_0\rangle\Big]$$

**donde:** $$\theta(y) = \begin{cases}1 & y > 0 \\ 0 & y < 0\end{cases}$$

# Funciones de Green

**Dijimos:**

$$iG(xt, x't') \equiv \frac{1}{\langle \Psi_0 | \Psi_0 \rangle} \langle \Psi_0 \, | \, T[\psi(x,t)_H \psi^\dagger(x',t')_H] | \, \Psi_0 \rangle$$

**Notación con el spin separado:**

$$iG_{ss'}(\mathbf{r}t, \mathbf{r}'t') \equiv \frac{1}{\langle \Psi_0 | \Psi_0 \rangle} \langle \Psi_0 \, | \, T[\psi_s(\mathbf{r}t)_H \psi^\dagger_{s'}(\mathbf{r}'t')_H] | \, \Psi_0 \rangle$$

# Funciones de Green

Cambio de base,
a la base de momento:

$$
\left\{
\begin{aligned}
\psi_\alpha(\mathbf{r},t)_H &= \sum_{\mathbf{k}} \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k}\cdot\mathbf{r}} c_{\mathbf{k}\alpha}(t)_H \\
\psi^\dagger_\beta(\mathbf{r}',t')_H &= \sum_{\mathbf{k}} \frac{1}{\sqrt{\Omega}} e^{-i\mathbf{k}'\cdot\mathbf{r}'} c^\dagger_{\mathbf{k}'\beta}(t')_H
\end{aligned}
\right.
$$

$$
iG_{ss'}(\mathbf{r}t,\mathbf{r}'t') \equiv \frac{1}{\langle\Psi_0|\,\Psi_0\rangle}\langle\Psi_0\mid T[\psi_s(\mathbf{r}t)_H\psi^\dagger_{s'}(\mathbf{r}'t')_H]|\,\Psi_0\rangle
$$

Entonces:

$$
\begin{aligned}
iG_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = &\sum_{\mathbf{k},\mathbf{k}'} \frac{1}{\Omega} e^{i(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}')} \\
&\times \underbrace{\frac{1}{\langle\Psi_0|\,\Psi_0\rangle}\langle\Psi_0\mid T[c_{\mathbf{k}\alpha}(t)_H c^\dagger_{\mathbf{k}'\beta}(t')_H]|\,\Psi_0\rangle}_{iG_{\alpha\beta}(\mathbf{k}t,\mathbf{k}'t')}
\end{aligned}
$$

# Funciones de Green

**Entonces:**

$$iG_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \sum_{\mathbf{k},\mathbf{k}'} \frac{1}{\Omega} e^{i(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}')}\, iG_{\alpha\beta}(\mathbf{k}t,\mathbf{k}'t')$$

**Podemos generalizar a cualquier base de partícula única:**

$$iG(\lambda t,\lambda' t') \equiv \frac{1}{\langle\Psi_0|\Psi_0\rangle}\langle\Psi_0 \,|\, T[c_\lambda(t)_H c^\dagger_{\lambda'}(t')_H]|\,\Psi_0\rangle$$