Clase 8

Many-Particle Theory

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La clase pasada vimos:

Representaciones o “pictures” en mecánica cuántica, que se obtienen a través de transformaciones unitarias de estados y operadores.

Schrödinger: evolucionan los **estados**

Heisenberg: evolucionan los **operadores**

Interaction: evolucionan **ambos**

Definición de las **funciones de Green** de una partícula

Funciones de Green (de una partícula)

$$iG(xt, x't') \equiv \frac{1}{\langle\Psi_0|\Psi_0\rangle}\langle\Psi_0 \mid T[\psi(x,t)_H \psi^\dagger(x',t')_H]|\Psi_0\rangle$$

REPASO

$$T[A(t)B(t')] = \begin{cases} A(t)B(t') & \text{for } t > t' \\ \pm B(t')A(t) & \text{for } t' > t \end{cases}$$

$$iG_{ss'}(\mathbf{r}t, \mathbf{r}'t') \equiv \frac{1}{\langle\Psi_0|\Psi_0\rangle}\langle\Psi_0 \mid T[\psi_s(\mathbf{r}t)_H \psi^\dagger_{s'}(\mathbf{r}'t')_H]|\Psi_0\rangle$$

$$iG_{\alpha\beta}(\mathbf{k}t, \mathbf{k}'t') \equiv \frac{1}{\langle\Psi_0|\Psi_0\rangle}\langle\Psi_0 \mid T[\underbrace{c_{\mathbf{k}\alpha}(t)_H c^\dagger_{\mathbf{k}'\beta}(t)_H}]|\Psi_0\rangle$$

$$iG(\lambda t, \lambda't') \equiv \frac{1}{\langle\Psi_0|\Psi_0\rangle}\langle\Psi_0 \mid T[\underbrace{c_\lambda(t)_H c^\dagger_{\lambda'}(t')_H}]|\Psi_0\rangle$$

Hamiltoniano independiente del tiempo (invariancia temporal):

Teorema: $H \neq H(t) \quad \Rightarrow \quad G(\lambda t, \lambda' t') = G(\lambda(t - t'), \lambda' 0)$

Demostración:

$$\psi(xt)_H = e^{iHt}\psi(x)_S e^{-iHt} \qquad \hbar = 1$$
$$\psi^\dagger(xt)_H = e^{iHt}\psi^\dagger(x)_S e^{-iHt}$$

Para $t > t'$ la función de Green tiene:

$$\underbrace{\langle \Psi_0 | e^{iHt}}_{\langle \Psi_0 | e^{iE_0 t}} \overbrace{\psi(x)_S e^{-iHt}}^{\psi(x,t)_H} \overbrace{e^{iHt'} \psi^\dagger(x')_S e^{-iHt'}}^{\psi^\dagger(x't')_H} \underbrace{| \Psi_0 \rangle}_{e^{-iE_0 t'} | \Psi_0 \rangle}$$

Funciones de Green

Para $t > t'$ la función de Green tiene:

$$\overbrace{\underbrace{\langle\Psi_0 | e^{iHt}}_{\langle\Psi_0|e^{iE_0 t}} \psi(x)_S e^{-iHt}}^{\psi(x,t)_H} \overbrace{e^{iHt'} \psi^\dagger(x')_S \underbrace{e^{-iHt'} | \Psi_0\rangle}_{e^{-iE_0 t'}|\Psi_0\rangle}}^{\psi^\dagger(x't')_H}$$

Combinando con la expresión para $t' > t$ obtenemos:

$$\begin{aligned} iG(xt, x't') &= \frac{1}{\langle\Psi_0 | \Psi_0\rangle} \Big[\theta(t-t') e^{iE_0(t-t')} \langle\Psi_0 | \psi(x)_S e^{-iH(t-t')} \psi^\dagger(x')_S | \Psi_0\rangle \\ &\quad - \theta(t'-t) e^{iE_0(t'-t)} \langle\Psi_0 | \psi^\dagger(x')_S e^{-iH(t'-t)} \psi(x)_S | \Psi_0\rangle\Big]. \end{aligned}$$

III

Listo

Funciones de Green

$$iG(xt, x't') = \frac{1}{\langle \Psi_0 | \Psi_0 \rangle} \Big[\theta(t-t') e^{iE_0(t-t')} \langle \Psi_0 | \psi(x)_S e^{-iH(t-t')} \psi^\dagger(x')_S | \Psi_0 \rangle - \theta(t'-t) e^{iE_0(t'-t)} \langle \Psi_0 | \psi^\dagger(x')_S e^{-iH(t'-t)} \psi(x)_S | \Psi_0 \rangle \Big] .$$

Esta expresión facilita la interpretación física de la función de Green.

Pasemos un momento a la representación de Schrödinger:

$$| \Psi_0 \rangle_S = e^{-iE_0 t} | \Psi_0 \rangle \longrightarrow$$

$$iG(xt, x't') = \frac{1}{\langle \Psi_0 | \Psi_0 \rangle} \Big[\theta(t-t') \langle \Psi_0(t) {}_S | \psi(x)_S e^{-iH(t-t')} \psi^\dagger(x')_S | \Psi_0(t') \rangle_S - \theta(t'-t) \langle \Psi_0(t') {}_S | \psi^\dagger(x')_S e^{-iH(t'-t)} \psi(x)_S | \Psi_0(t) \rangle_S \Big] .$$

Funciones de Green

$H \neq H(t)$ Invariancia temporal

$$iG(xt, x't') = \frac{1}{\langle\Psi_0|\Psi_0\rangle}\Big[\theta(t-t')\langle\Psi_0(t)_S \mid \psi(x)_S e^{-iH(t-t')}\psi^\dagger(x')_S \mid \Psi_0(t')\rangle_S - \theta(t'-t)\langle\Psi_0(t')_S \mid \psi^\dagger(x')_S e^{-iH(t'-t)}\psi(x)_S \mid \Psi_0(t)\rangle_S\Big] .$$

Interpretación:

t > t' : Se **crea** una particula en x', las N+1 partículas evolucionan de t' a t,

se **destruye** una partícula en x y se calcula el overlap con el ground state.

t' > t : Se **destruye** una particula en x, las N-1 partículas evolucionan de t a t',

se **crea** una partícula en x y se calcula el overlap con el ground state.

Tenemos así propagación de una partícula o de un hueco.

También tenemos una función de correlación entre x y x'

Hamiltoniano con invariancia traslacional:

Teorema: $\left[\hat{H},\hat{\mathbf{P}}\right]=0 \quad \Rightarrow \quad G_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t')=G\left((\mathbf{r}-\mathbf{r}')t,0\,t\right)$

Demostración:

La dejamos para después, pero notemos la expresión del momento

en términos de los operadores de campo:

$$\hat{\mathbf{P}}=\sum_{\alpha}\int \mathrm{d}^3r\,\hat{\psi}^{\dagger}_{\alpha}(\mathbf{r})\,(-i\nabla)\,\hat{\psi}_{\alpha}(\mathbf{r})$$

Funciones de Green

Consecuencia de la invariancia traslacional:

$$G_{\alpha\beta}(\mathbf{k},t,t') = \int \mathrm{d}^3(\mathbf{r}-\mathbf{r}')\mathrm{e}^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}G_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t')$$

$$G_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}G_{\alpha\beta}(\mathbf{k},t,t')$$

En otras palabras, la función de Green del momento:

$$iG_{\alpha\beta}(\mathbf{k}t,\mathbf{k}'t') \equiv \frac{1}{\langle\Psi_0|\Psi_0\rangle}\langle\Psi_0\,|\,T[c_{\mathbf{k}\alpha}(t)_H c^\dagger_{\mathbf{k}'\beta}(t)_H]|\,\Psi_0\rangle$$

Es diagonal en los índices **k** y **k'**, o sea:

$$\left[\hat{H},\hat{\mathbf{P}}\right]=0 \quad\Rightarrow\quad G_{\alpha\beta}(\mathbf{k}t,\mathbf{k}'t') = \delta_{\mathbf{k},\mathbf{k}'}G_{\alpha\beta}(\mathbf{k},t,t')$$

Funciones de Green

Invariancia temporal y traslacional:

$$G_{\alpha\beta}(\mathbf{r}t, \mathbf{r}'t') = G_{\alpha\beta}\left((\mathbf{r}-\mathbf{r}')(t-t'), 0\,0\right) \longrightarrow$$

$$G_{\alpha\beta}(\mathbf{k}\omega) = \int \mathrm{d}^3(\mathbf{r}-\mathbf{r}') \int \mathrm{d}(t-t') \mathrm{e}^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \mathrm{e}^{i\omega(t-t')} G_{\alpha\beta}(\mathbf{r}t, \mathbf{r}'t')$$

Con la inversa:

$$G_{\alpha\beta}(\mathbf{r}t, \mathbf{r}'t') = \frac{1}{(2\pi)^4} \int \mathrm{d}^3k \int \mathrm{d}\omega\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \mathrm{e}^{-i\omega(t-t')} G_{\alpha\beta}(\mathbf{k}\omega)$$

A formalism which is manifestly covariant can be constructed with this Green's function. This formalism has obvious special importance for relativistic many-particle systems (see Chapter 24).

Ejemplo: fermiones libres y no interactuantes

$$H_0 = T \qquad V \equiv 0$$

En este caso las representaciones de Heisenberg e Interacción son iguales:

$$iG^{(0)}_{\alpha\beta}(\mathbf{r}t, \mathbf{r}'t') = \langle \Phi_0 \mid T[\psi_\alpha(\mathbf{r}t)_I \psi^\dagger_\beta(\mathbf{r}'t')_I] \mid \Phi_0 \rangle$$

Estado fundamental:
Determinante de Slater

Pasamos a la base de momento:

$$\psi_\alpha(\mathbf{r}t)_I = \sum_{\mathbf{k}} \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k}\cdot\mathbf{r}} c_{\mathbf{k}\alpha}(t)_I = \sum_{\mathbf{k}} \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\epsilon_k t} c_{\mathbf{k}\alpha}$$

$$\psi^\dagger_\beta(\mathbf{r}'t')_I = \sum_{\mathbf{k}'} \frac{1}{\sqrt{\Omega}} e^{-i\mathbf{k}'\cdot\mathbf{r}'} e^{i\epsilon_{k'} t'} c^\dagger_{\mathbf{k}'\beta}$$

$$\epsilon_k = \frac{k^2}{2m}$$

$$H_0 = T$$

Reemplazando se obtiene:

$$iG^{(0)}_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \frac{1}{\Omega}\sum_{\mathbf{k}\mathbf{k}'} e^{i(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}')} e^{-i(\epsilon_k t-\epsilon_{k'} t')}$$

$$\times\left[\theta(t-t')\langle\Phi_0 \mid c_{\mathbf{k}\alpha}c^\dagger_{\mathbf{k}'\beta} \mid \Phi_0\rangle - \theta(t'-t)\langle\Phi_0 \mid c^\dagger_{\mathbf{k}'\beta}c_{\mathbf{k}\alpha} \mid \Phi_0\rangle\right]$$

$$iG^{(0)}_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \langle\Phi_0 \mid T[\psi_\alpha(\mathbf{r}t)_I\psi^\dagger_\beta(\mathbf{r}'t')_I] \mid \Phi_0\rangle$$

$$\psi_\alpha(\mathbf{r}t)_I = \sum_{\mathbf{k}} \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\epsilon_k t} c_{\mathbf{k}\alpha}$$

$$\psi^\dagger_\beta(\mathbf{r}'t')_I = \sum_{\mathbf{k}'} \frac{1}{\sqrt{\Omega}} e^{-i\mathbf{k}'\cdot\mathbf{r}'} e^{i\epsilon_{k'} t'} c^\dagger_{\mathbf{k}'\beta}$$

Aparecen deltas en el espín y en el momento porque el GS $|\Phi_0\rangle$ es un Det. de Slater:

$$iG^{(0)}_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \frac{\delta_{\alpha\beta}}{\Omega}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} e^{-i\epsilon_k(t-t')}$$

$$\times\left[\theta(t-t')\langle\Phi_0 \mid c_{\mathbf{k}\alpha}c^\dagger_{\mathbf{k}\alpha} \mid \Phi_0\rangle - \theta(t'-t)\langle\Phi_0 \mid c^\dagger_{\mathbf{k}\alpha}c_{\mathbf{k}\alpha} \mid \Phi_0\rangle\right]$$

$H_0 = T$

Hasta ahora tenemos:

$$iG^{(0)}_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \frac{\delta_{\alpha\beta}}{\Omega}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}e^{-i\epsilon_k(t-t')}$$
$$\times\left[\theta(t-t')\langle\Phi_0\,|\,c_{\mathbf{k}\alpha}c^\dagger_{\mathbf{k}\alpha}\,|\,\Phi_0\rangle - \theta(t'-t)\langle\Phi_0\,|\,c^\dagger_{\mathbf{k}\alpha}c_{\mathbf{k}\alpha}\,|\,\Phi_0\rangle\right]$$

Y como el estado fundamental $|\,\Phi_0\rangle$ es una esfera de Fermi, tenemos:

$$\langle\Phi_0\,|\,c_{\mathbf{k}\alpha}c^\dagger_{\mathbf{k}\alpha}\,|\,\Phi_0\rangle = \langle c^\dagger_{\mathbf{k}\alpha}\Phi_0\,|\,c^\dagger_{\mathbf{k}\alpha}\Phi_0\rangle \left\{\begin{array}{ll} 1: & k > k_F \\ 0: & k \le k_F \end{array}\right\} = \theta(k-k_F)$$

$$\langle\Phi_0\,|\,c^\dagger_{\mathbf{k}\alpha}c_{\mathbf{k}\alpha}\,|\,\Phi_0\rangle = \langle c_{\mathbf{k}\alpha}\Phi_0\,|\,c_{\mathbf{k}\alpha}\Phi_0\rangle = \left\{\begin{array}{ll} 1: & k \le k_F \\ 0: & k > k_F \end{array}\right\} = \theta(k_F-k)$$

$\longrightarrow$

$$iG^{(0)}_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \delta_{\alpha\beta}\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}e^{-i\epsilon_k(t-t')}$$
$$\times\left[\theta(t-t')\theta(k-k_F) - \theta(t'-t)\theta(k_F-k)\right]$$

Funciones de Green

$H_0 = T$

$$iG^{(0)}_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \delta_{\alpha\beta}\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} e^{-i\epsilon_k(t-t')} \times \left[\theta(t-t')\theta(k-k_F) - \theta(t'-t)\theta(k_F-k)\right]$$

$$\longrightarrow \boxed{iG^{(0)}_{\alpha\beta}(\mathbf{k},t-t') = \delta_{\alpha\beta} e^{-i\epsilon_k(t-t')}\left[\theta(t-t')\theta(k-k_F) - \theta(t'-t)\theta(k_F-k)\right]}$$

Finalmente querríamos pasar del dominio tiempo al dominio frecuencia:

$$G^{(0)}_{\alpha\beta}(\mathbf{k},\omega) = \int d(t-t') e^{i\omega(t-t')} G^{(0)}_{\alpha\beta}(\mathbf{k},t-t')$$

$$iG^{(0)}_{\alpha\beta}(\mathbf{k},t-t') = \delta_{\alpha\beta} e^{-i\epsilon_k(t-t')}\left[\theta(t-t')\theta(k-k_F) - \theta(t'-t)\theta(k_F-k)\right]$$

Funciones de Green

$H_0 = T$

$$\begin{cases} \theta(\tau) = \lim_{\eta\to 0^+} -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega + i\eta} \\ \theta(-\tau) = \lim_{\eta\to 0^+} +\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega - i\eta} \end{cases}$$

$$iG^{(0)}_{\alpha\beta}(\mathbf{k}, t - t') = \delta_{\alpha\beta} e^{-i\epsilon_k(t-t')}\left[\theta(t-t')\theta(k-k_F) - \theta(t'-t)\theta(k_F-k)\right] \longrightarrow$$

$$iG^{(0)}_{\alpha\beta}(\mathbf{r}t, \mathbf{r}'t') = i\lim_{\eta\to 0^+}\delta_{\alpha\beta}\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \times \int_{-\infty}^{\infty}\frac{dw}{2\pi}e^{-i(\epsilon_k+w)(t-t')}\left[\frac{\theta(k-k_F)}{w+i\eta} + \frac{\theta(k_F-k)}{w-i\eta}\right]$$

Cambio de variable: $\omega \equiv \epsilon_k + w$

Funciones de Green

$H_0 = T$

$$\omega \equiv \epsilon_k + w$$

$$G^{(0)}_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \lim_{\eta\to 0^+} \delta_{\alpha\beta} \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \times e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} e^{-i\omega(t-t')} \left[\frac{\theta(k-k_F)}{\omega-\epsilon_k+i\eta} + \frac{\theta(k_F-k)}{\omega-\epsilon_k-i\eta}\right]$$

$$\longrightarrow \quad \boxed{G^{(0)}_{\alpha\beta}(\mathbf{k}\omega) = \lim_{\eta\to 0^+} \delta_{\alpha\beta} \left[\frac{\theta(k-k_F)}{\omega-\epsilon_k+i\eta} + \frac{\theta(k_F-k)}{\omega-\epsilon_k-i\eta}\right]}$$

Si $k > k_F$ $\longrightarrow$ polo simple en $\omega = \epsilon_k - i\eta$

Si $k < k_F$ $\longrightarrow$ polo simple en $\omega = \epsilon_k + i\eta$

Funciones de Green

Introduction to

Quantum field theory in condensed matter physics

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Función de Green retardada :

$$G^R(\mathbf{r}\sigma t, \mathbf{r}'\sigma' t') = -i\theta\left(t-t'\right)\left\langle[\Psi_\sigma(\mathbf{r}t), \Psi^\dagger_{\sigma'}(\mathbf{r}'t')]_{B,F}\right\rangle, \qquad \left\{ \begin{array}{ll} B: & \text{bosons} \\ F: & \text{fermions} \end{array} \right\}$$

Funciones de Green mayor (greater) y menor (lesser) :

$$G^>(\mathbf{r}\sigma t, \sigma'\mathbf{r}'t') = -i\langle\Psi_\sigma(\mathbf{r}t)\Psi^\dagger_{\sigma'}(\mathbf{r}'t')\rangle,$$

$$G^<(\mathbf{r}\sigma t, \sigma'\mathbf{r}'t') = -i\left(\pm 1\right)\langle\Psi^\dagger_{\sigma'}(\mathbf{r}'t')\Psi_\sigma(\mathbf{r}t)\rangle$$

Se ve que :

$$G^R(\mathbf{r}\sigma t, \mathbf{r}'\sigma' t') = \theta\left(t-t'\right)\left[G^>(\mathbf{r}\sigma t, \mathbf{r}'\sigma' t') - G^<(\mathbf{r}\sigma t, \mathbf{r}'\sigma' t')\right]$$

Ejercicio

(1) Demostrar las expresiones:

$$\theta(\tau) = \lim_{\eta \to 0^+} -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\mathrm{e}^{-i\omega\tau}}{\omega + i\eta}$$

$$\theta(-\tau) = \lim_{\eta \to 0^+} +\frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\mathrm{e}^{-i\omega\tau}}{\omega - i\eta}$$

(2) Hallar la función de Green G^R para el gas de electrones libres y no interactuantes.