Física de muchos cuerpos
Año 2020

Guía 2 - Ejercicio 2

Demostrar que para operadores fermiónicos se satisface:

$$\left[a_i^\dagger a_j, a_k^\dagger a_l\right] = \delta_{jk} a_i^\dagger a_l - \delta_{il} a_k^\dagger a_j.$$

Calcular el mismo conmutador para bosones.

Bosones (resolución Emiliano):

$$\begin{aligned}
\left[a_i^\dagger a_j, a_k^\dagger a_l\right] &= \left[a_i^\dagger a_j, a_k^\dagger\right] a_l + a_k^\dagger \left[a_i^\dagger a_j, a_l\right] \\
&= -\left(\left[a_k^\dagger, a_i^\dagger a_j\right]\right) a_l - a_k^\dagger \left(\left[a_l, a_i^\dagger a_j\right]\right) \\
&= -\left(\overbrace{\left[a_k^\dagger, a_i^\dagger\right]}^{=0} a_j + a_i^\dagger \overbrace{\left[a_k^\dagger, a_j\right]}^{=-\delta_{kj}}\right) a_l - a_k^\dagger \left(\overbrace{\left[a_l, a_i^\dagger\right]}^{\delta_{il}} a_j + a_i^\dagger \overbrace{\left[a_l, a_j\right]}^{=0}\right) \\
&= \delta_{jk} a_i^\dagger a_l - \delta_{il} a_k^\dagger a_j
\end{aligned}$$

Fermiones (resolución Eric):

$$\left[a_i^\dagger a_j, a_k^\dagger a_l\right] = a_i^\dagger \left[a_j, a_k^\dagger a_l\right] + \left[a_i^\dagger, a_k^\dagger a_l\right] a_j = a_i^\dagger \left(a_k^\dagger \left[a_j, a_l\right] + \left[a_j, a_k^\dagger\right] a_l\right) + \left(a_k^\dagger \left[a_i^\dagger, a_l\right] + \left[a_i^\dagger, a_k^\dagger\right] a_l\right) a_j.$$

Si $AB = \frac{1}{2}\left([A,B] + \{A,B\}\right) \implies [A,B] = 2AB - \{A,B\}$.

Además, recordar que:

- $\{A,B\} = \{B,A\}$.
- $\{a_m, a_n\} = 0 \ \wedge \ \{a_m^\dagger, a_n^\dagger\} = 0 \ \wedge \ \{a_m, a_n^\dagger\} = \delta_{mn}$.

$$\implies \left[a_i^\dagger a_j, a_k^\dagger a_l\right] = a_i^\dagger \left(2a_k^\dagger a_j a_l - 0 + 2a_j a_k^\dagger a_l - \delta_{jk} a_l\right) + \left(2a_k^\dagger a_i^\dagger a_l - a_k^\dagger \delta_{il} + 2a_i^\dagger a_k^\dagger a_l - 0\right) a_j =$$

$$2a_i^\dagger a_k^\dagger a_j a_l + 2a_i^\dagger a_j a_k^\dagger a_l - a_i^\dagger \delta_{jk} a_l + 2a_k^\dagger a_i^\dagger a_l a_j - a_k^\dagger \delta_{il} a_j + 2a_i^\dagger a_k^\dagger a_l a_j =$$

$$-\delta_{jk} a_i^\dagger a_l - \delta_{il} a_k^\dagger a_j + \underbrace{2a_i^\dagger a_k^\dagger a_j a_l + \overbrace{2a_i^\dagger a_j a_k^\dagger a_l}^{=2a_i^\dagger(\delta_{jk} - a_k^\dagger a_j) a_l}}_{=2\delta_{jk} a_i^\dagger a_l} + \overbrace{2a_k^\dagger a_i^\dagger a_l a_j + 2a_i^\dagger a_k^\dagger a_l a_j}^{=2\left\{a_k^\dagger, a_i^\dagger\right\} a_l a_j = 0} =$$

$$\delta_{jk} a_i^\dagger a_l - \delta_{il} a_k^\dagger a_j.$$