Física de muchos cuerpos
Año 2020

Guía 3 - Ejercicio 2

(a) Escribir en segunda cuantización el Hamiltoniano de un gas de electrones quasi-bidimensional, cuyo Hamiltoniano en primera cuantización es:

$$H^{(1)} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i\neq j} \frac{e^2}{\|\mathbf{r}_i - \mathbf{r}_j\|} + \sum_{i=1}^{N} u(z_i).$$

Sugerencia: utilizar como base de partícula única los estados:

$$\Psi_{n\mathbf{k}\sigma}(\mathbf{r}; s) = \frac{1}{L}\, e^{i\mathbf{k}.\boldsymbol{\rho}}\ \varphi(z)\ \chi_\sigma(s),$$

donde $\mathbf{k} = (k_x; k_y)$, $\boldsymbol{\rho} = (x; y)$ y $\varphi_n(z)$ satisface:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + u(z)\right]\varphi_n(z) = \epsilon_n \varphi_n(z).$$

Ayuda:

$$\int d^2\rho\ \frac{e^{i\mathbf{q}.\boldsymbol{\rho}}}{\sqrt{\rho^2 + (z-z')^2}} = \int_0^\infty d\rho \int_0^{2\pi} d\phi\ \frac{\rho\ e^{i\ q\ p\ cos(\phi)}}{\sqrt{\rho^2 + (z-z')^2}} =$$

$$2\pi \int_0^\infty d\rho\ \frac{\rho\ J_0(q\rho)}{\sqrt{\rho^2 + (z-z')^2}} = \frac{2\pi}{q}\ e^{-q|z-z'|}.$$

(b) Escribir en segunda cuantización el Hamiltoniano para el *jellium model* en dos dimensiones (obtener una expresión equivalente a la Ex. (3,19) de Fetter-Walecka).

Aclaraciones preliminares

Operador en 2da cuantización expresado en términos de operadores de campo

$$F = \sum_{q=1}^{N} f(q) \xrightarrow{2da\ cuantiz.} \sum_{k,l} \langle u_k| f |u_l\rangle a_k^\dagger a_l.$$

$$F = \sum_{k,l} \langle u_k| f |u_l\rangle a_k^\dagger a_l = \int d^3x\ d^3x' \sum_{k,l} \overbrace{\langle u_k|\mathbf{x}\rangle}^{=:\varphi_k^*(\mathbf{x})} \langle \mathbf{x}| f |\mathbf{x}'\rangle \overbrace{\langle \mathbf{x}'|u_l\rangle}^{=:\varphi_l(\mathbf{x}')} a_k^\dagger a_l =$$

$$\int d^3x\ d^3x' \left(\sum_k \varphi_k^*(\mathbf{x}) a_k^\dagger\right) \langle \mathbf{x}| f |\mathbf{x}'\rangle \left(\sum_l \varphi_l(\mathbf{x}') a_l\right),$$

quedando:

$$F = \int d^3x\ d^3x'\ \Psi^\dagger(\mathbf{x})\ \langle \mathbf{x}| f |\mathbf{x}'\rangle\ \Psi(\mathbf{x}'). \tag{1}$$

Si F admite representación en el espacio de posición, o bien es una función de la posición:

$$\langle \mathbf{x}| f |\mathbf{x}'\rangle = f(\mathbf{x}')\ \delta(\mathbf{x} - \mathbf{x}'),$$

así que la Ec. (1) queda:

$$F = \int d^3x\ \Psi^\dagger(\mathbf{x})\ f(\mathbf{x})\ \Psi(\mathbf{x}). \tag{2}$$

Operador sin representación en el espacio de spin extendido por la id en el mismo

El estado de una partícula con momento $\mathbf{p} = \hbar\mathbf{k}$ y spin σ puede escribirse como:

$$|\mathbf{p}, \sigma\rangle = |\mathbf{p}\rangle \otimes |\sigma\rangle\,. \tag{3}$$

Si aplicamos un operador que no tiene representación en el espacio de spin, por ejemplo $T = \frac{p^2}{2m}$, sucede:

$$T|\mathbf{p},\sigma\rangle = \frac{p^2}{2m}|\mathbf{p},\sigma\rangle = \left(\frac{p^2}{2m} \otimes id_{H_s}\right)|\mathbf{p},\sigma\rangle = \left(\frac{p^2}{2m}|\mathbf{p}\rangle\right) \otimes (id_{H_s}|\sigma\rangle) = \frac{p^2}{2m}|\mathbf{p}\rangle \otimes |\sigma\rangle = \frac{p^2}{2m}|\mathbf{p},\sigma\rangle,$$

luego:

$$\langle \mathbf{p}_i, \sigma_i| T |\mathbf{p}_j, \sigma_j\rangle = \langle \mathbf{p}_i, \sigma_i| T \otimes id_{H_s} |\mathbf{p}_j, \sigma_j\rangle = \langle \mathbf{p}_i| T |\mathbf{p}_j\rangle \otimes \langle \sigma_i| id_{H_s} |\sigma_j\rangle,$$

con lo cual

$$\langle \mathbf{p}_i, \sigma_i| T |\mathbf{p}_j, \sigma_j\rangle = \delta_{\sigma_i\sigma_j} \langle \mathbf{p}_i| T |\mathbf{p}_j\rangle\,. \tag{4}$$

Por otro lado, podemos desarrollar un estado cualquiera de la siguiente forma:

$$|\mathbf{p}, \lambda\rangle = \sum_{\lambda=-s}^{s} \int d^3x\, |\mathbf{x}, \lambda\rangle \langle \mathbf{x}, \lambda|\mathbf{p}, \sigma\rangle\,, \tag{5}$$

de modo que desarrolando la Ec. (4) en bases de spin y de coordenadas de posición como en la Ec. (5), queda:

$$\langle \mathbf{p}_i, \sigma_i| T |\mathbf{p}_j, \sigma_j\rangle = \left(-\frac{\hbar^2}{2m}\right) \delta_{\sigma_i\sigma_j} \int d^3x\ \langle \mathbf{p}_i|\mathbf{x}\rangle \nabla^2 \langle \mathbf{x}|\mathbf{p}_j\rangle\,. \tag{6}$$

Resolución (a)

$$H^{(1)} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} \frac{e^2}{\|\mathbf{r}_i - \mathbf{r}_j\|} + \sum_{i=1}^{N} u(z_i).$$

$$\Psi_{n\mathbf{k}\sigma}(\mathbf{r}; s) = \overbrace{\frac{1}{L}}^{=(1/\sqrt{L})^2} \quad e^{i\mathbf{k}.\boldsymbol{\rho}} \ \varphi_n(z) \ \chi_\lambda(s) = (\langle \mathbf{r}| \otimes \langle s|) \,. \ \overbrace{(|n, \mathbf{k}, \lambda\rangle)}^{=|n\rangle \otimes |\mathbf{k}\rangle \otimes |\lambda\rangle} \ = \langle \mathbf{r}|n, \mathbf{k}\rangle \otimes \langle s|\lambda\rangle,$$

Pasamos cada término de $H^{(1)}$ a segunda cuantización:

- $T = \sum_{i=1}^{N} \frac{p_i^2}{2m}$, $p_i \equiv p(i)$.

$$\langle n_1, \mathbf{k}_1, \lambda_1| \ \frac{p^2}{2m} \ |n_2, \mathbf{k}_2, \lambda_2\rangle = \left(-\frac{\hbar^2}{2m}\right) \delta_{\lambda_1\lambda_2} \int d^3r \ \ \langle n_1, \mathbf{k}_1|\mathbf{r}\rangle \, \nabla^2 \, \langle \mathbf{r}|n_2, \mathbf{k}_2\rangle =$$

$$\left(-\frac{\hbar^2}{2m}\right) \delta_{\lambda_1\lambda_2} \int d^2\rho \int dz \ \left(\frac{1}{L} \ e^{-i\mathbf{k}_1 \cdot \boldsymbol{\rho}} \ \varphi_{n_1}^*(z)\right) \ \nabla^2 \ \left(\frac{1}{L} \ e^{i\mathbf{k}_2 \cdot \boldsymbol{\rho}} \ \varphi_{n_2}(z)\right) =$$

$$\left(-\frac{\hbar^2}{2m}\right) \delta_{\lambda_1\lambda_2} \int d^2\rho \int dz \ \left(\frac{1}{L} \ e^{-i\mathbf{k}_1 \cdot \boldsymbol{\rho}} \ \varphi_{n_1}^*(z)\right) \left[\overbrace{(-k_{2x}^2 - k_{2y}^2)}^{=-k_2^2} \ \left(\frac{1}{L} \ e^{i\mathbf{k}_2 \cdot \boldsymbol{\rho}} \ \varphi_{n_2}(z)\right) + \ \left(\frac{1}{L} \ e^{i\mathbf{k}_2 \cdot \boldsymbol{\rho}} \ \frac{d^2}{dz^2} \varphi_{n_2}(z)\right)\right] =$$

$$\left(-\frac{\hbar^2}{2m}\right) \delta_{\lambda_1\lambda_2} (-k_2^2) \overbrace{\int d^2\rho \ \frac{1}{L^2} \ e^{-i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \boldsymbol{\rho}}}^{=\delta_{\mathbf{k_1 k_2}}} \ \overbrace{\int dz \ \varphi_{n_1}^*(z) \ \varphi_{n_2}(z)}^{=\langle n_1|n_2\rangle = \delta_{n_1 n_2}} \ + ...$$

$$... \left(-\frac{\hbar^2}{2m}\right) \delta_{\lambda_1\lambda_2} \overbrace{\int d^2\rho \ \frac{1}{L^2} \ e^{-i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \boldsymbol{\rho}}}^{=\delta_{\mathbf{k_1 k_2}}} \ \overbrace{\int dz \ \varphi_{n_1}^*(z) \ \frac{d^2}{dz^2} \varphi_{n_2}(z)}^{=(-1/\hbar^2) \ \langle n_1|p_z^2|n_2\rangle} =$$

$$\frac{\hbar^2 k_2^2}{2m} \ \delta_{\lambda_1\lambda_2} \ \delta_{\mathbf{k_1 k_2}} \ \delta_{n_1 n_2} + \delta_{\lambda_1\lambda_2} \ \delta_{\mathbf{k_1 k_2}} \ \langle n_1| \ \frac{p_z^2}{2m} \ |n_2\rangle.$$

$$\Longrightarrow T = \sum_{n_1, \mathbf{k}_1, \lambda_1} \ \sum_{n_2, \mathbf{k}_2, \lambda_2} \delta_{\mathbf{k_1 k_2}} \left[\frac{\hbar^2 k_2^2}{2m} \ \delta_{n_1 n_2} + \langle n_1| \ \frac{p_z^2}{2m} \ |n_2\rangle\right] \ \delta_{\lambda_1\lambda_2} a_{n_1\mathbf{k}_1\lambda_1}^\dagger a_{n_2\mathbf{k}_2\lambda_2},$$

quedando:

$$T = \sum_{n, \mathbf{k}, \lambda} \frac{\hbar^2 k^2}{2m} a_{n\mathbf{k}\lambda}^\dagger a_{n\mathbf{k}\lambda} + \sum_{n_1, n_2, \mathbf{k}, \lambda} \langle n_1| \ \frac{p_z^2}{2m} \ |n_2\rangle \ \ a_{n_1\mathbf{k}\lambda}^\dagger a_{n_2\mathbf{k}\lambda}. \tag{7}$$

- $V_1 = \dfrac{1}{2} \sum_{i \neq j} \dfrac{e^2}{\|\mathbf{r}_i - \mathbf{r}_j\|}$.

$$\langle n_1, \mathbf{k}_1, \lambda_1; n_2, \mathbf{k}_2, \lambda_2 | \frac{e^2}{2} \; \frac{1}{\|\mathbf{r}_1 - \mathbf{r}_2\|} \, | n_3, \mathbf{k}_3, \lambda_3; n_4, \mathbf{k}_4, \lambda_4 \rangle =$$

$$\frac{e^2}{2} \; \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \; \int d\rho_1^2 \int dz_1 \int d\rho_2^2 \int dz_2 \; \frac{1}{L^2} \; e^{-i(\mathbf{k}_1-\mathbf{k}_3).\boldsymbol{\rho}_1} \; \frac{1}{\|(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) + (\mathbf{z}_1 - \mathbf{z}_2)\|} \; \times ...$$

$$... \; \frac{1}{L^2} \; e^{-i(\mathbf{k}_2-\mathbf{k}_4).\boldsymbol{\rho}_2} \; \varphi^*_{n_1}(z_1) \; \varphi_{n_3}(z_1) \; \varphi^*_{n_2}(z_2) \; \varphi_{n_4}(z_2).$$

$\boldsymbol{\rho} := \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2 \; \wedge \; \boldsymbol{\rho}' := \boldsymbol{\rho}_2 \; \wedge \; z := z_1 \; \wedge \; z' := z_2.$

$$\Longrightarrow \langle n_1, \mathbf{k}_1, \lambda_1; n_2, \mathbf{k}_2, \lambda_2 | \frac{e^2}{2} \; \frac{1}{\|\mathbf{r}_1 - \mathbf{r}_2\|} \, | n_3, \mathbf{k}_3, \lambda_3; n_4, \mathbf{k}_4, \lambda_4 \rangle =$$

$$\frac{e^2}{2} \; \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \; \int d\rho^2 \int dz \int d\rho'^2 \int dz' \; \frac{1}{L^2} \; e^{-i(\mathbf{k}_1-\mathbf{k}_3).(\boldsymbol{\rho}+\boldsymbol{\rho}')} \; \frac{1}{\|\boldsymbol{\rho} + (\mathbf{z} - \mathbf{z}')\|} \; \times ...$$

$$... \; \frac{1}{L^2} \; e^{-i(\mathbf{k}_2-\mathbf{k}_4).\boldsymbol{\rho}'} \; \varphi^*_{n_1}(z) \; \varphi_{n_3}(z) \; \varphi^*_{n_2}(z') \; \varphi_{n_4}(z') =$$

$$\frac{e^2}{2} \; \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \; \frac{1}{L^4} \; \overbrace{\int d\rho'^2 e^{-i[(\mathbf{k}_1+\mathbf{k}_2)-(\mathbf{k}_3+\mathbf{k}_4)].\boldsymbol{\rho}'}}^{=L^2 \; \delta_{(\mathbf{k}_1+\mathbf{k}_2)(\mathbf{k}_3+\mathbf{k}_4)}} \; \int dz \int dz' \int d\rho^2 \; \frac{e^{-i(\mathbf{k}_1-\mathbf{k}_3).\boldsymbol{\rho}}}{\|\boldsymbol{\rho} + (\mathbf{z} - \mathbf{z}')\|} \times \; ...$$

$$... \; \varphi^*_{n_1}(z) \; \varphi_{n_3}(z) \; \varphi^*_{n_2}(z') \; \varphi_{n_4}(z') =$$

$$\frac{e^2}{2} \; \frac{1}{L^2} \; \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \; \delta_{(\mathbf{k}_1+\mathbf{k}_2)(\mathbf{k}_3+\mathbf{k}_4)} \int dz \int dz' \varphi^*_{n_1}(z) \; \varphi_{n_3}(z) \; \varphi^*_{n_2}(z') \; \varphi_{n_4}(z') \; \times \; ...$$

$$... \; \int_0^\infty d\rho \int_0^{2\pi} d\phi \; \rho \; \frac{e^{i||\mathbf{k}_3-\mathbf{k}_1|| \;\; \rho \;\; cos(\phi)}}{\sqrt{\rho^2 + (z-z')^2}}.$$

Auxiliar:

$$\int_0^\infty d\rho \int_0^{2\pi} d\phi \; \rho \; \frac{e^{i||\mathbf{k}_3-\mathbf{k}_1|| \;\; \rho \;\; cos(\phi)}}{\sqrt{\rho^2 + (z-z')^2}} = 2\pi \int_0^\infty d\rho \; \rho \; \frac{J_0(||\mathbf{k}_3 - \mathbf{k}_1|| \, .\rho)}{\sqrt{\rho^2 + (z-z')^2}} = \frac{2\pi}{||\mathbf{k}_3 - \mathbf{k}_1||} \; e^{-||\mathbf{k}_3-\mathbf{k}_1||.|z-z'|}.$$

$$\Longrightarrow \langle n_1, \mathbf{k}_1, \lambda_1; n_2, \mathbf{k}_2, \lambda_2 | \frac{e^2}{2} \; \frac{1}{\|\mathbf{r}_1 - \mathbf{r}_2\|} \, | n_3, \mathbf{k}_3, \lambda_3; n_4, \mathbf{k}_4, \lambda_4 \rangle =$$

$$\frac{e^2}{2} \; \frac{1}{L^2} \; \frac{2\pi}{||\mathbf{k}_3 - \mathbf{k}_1||} \; \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \; \delta_{(\mathbf{k}_1+\mathbf{k}_2)(\mathbf{k}_3+\mathbf{k}_4)} \; \overbrace{\int dz \int dz' \varphi^*_{n_1}(z) \; \varphi_{n_3}(z) \; \varphi^*_{n_2}(z') \; \varphi_{n_4}(z') \; e^{-||\mathbf{k}_3-\mathbf{k}_1||.|z-z'|}}^{=:I_{n_1 n_2 n_3 n_4 \mathbf{k}_1 \mathbf{k}_3}}.$$

$$\Longrightarrow V_1 = \frac{e^2}{2} \; \frac{1}{L^2} \; \sum_{n_1,\mathbf{k}_1,\lambda_1} \sum_{n_2,\mathbf{k}_2,\lambda_2} \sum_{n_3,\mathbf{k}_3,\lambda_3} \sum_{n_4,\mathbf{k}_4,\lambda_4} \frac{2\pi}{||\mathbf{k}_3 - \mathbf{k}_1||} \; \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \; \delta_{(\mathbf{k}_1+\mathbf{k}_2)(\mathbf{k}_3+\mathbf{k}_4)} \; I_{n_1 n_2 n_3 n_4 \mathbf{k}_1 \mathbf{k}_3} \;\; \times ...$$

$$... \; a^\dagger_{n_1\mathbf{k}_1\lambda_1} \; a^\dagger_{n_2\mathbf{k}_2\lambda_2} \; a_{n_4\mathbf{k}_4\lambda_4} \; a_{n_3\mathbf{k}_3\lambda_3}.$$

$\mathbf{k}_1 =: \mathbf{k} + \mathbf{p} \; \wedge \; \mathbf{k}_2 =: \mathbf{q} - \mathbf{p} \; \wedge \; \mathbf{k}_3 =: \mathbf{k} \; \wedge \; \mathbf{k}_4 =: \mathbf{q}$, con $\mathbf{p}$ proporcional al momento intercambiado.

$\Longrightarrow \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 = \mathbf{k} + \mathbf{q}$ (conservación del momento lineal).

Con este cambio de variables, queda:

$$V_1 = \frac{e^2}{2}\ \frac{1}{L^2} \sum_{n_1,n_2,n_3,n_4} \sum_{\mathbf{k},\mathbf{p},\mathbf{q}} \sum_{\lambda_1,\lambda_2} \frac{2\pi}{||\mathbf{p}||}\ I_{n_1n_2n_3n_4\mathbf{p}}\ a^\dagger_{n_1(\mathbf{k}+\mathbf{p})\lambda_1}\ a^\dagger_{n_2(\mathbf{q}-\mathbf{p})\lambda_2}\ a_{n_4\mathbf{q}\lambda_2}\ a_{n_3\mathbf{k}\lambda_1} \tag{8}$$

- $V_2 = \sum_{i=1}^{N} u(z_i)$.

$$\langle n_1, \mathbf{k}_1, \lambda_1|\, u(z)\, |n_2, \mathbf{k}_2, \lambda_2\rangle = \delta_{\lambda_1\lambda_2} \int d^2\rho \int dz\ \left(\frac{1}{L}\ e^{-i\mathbf{k}_1\cdot\boldsymbol{\rho}}\ \varphi^*_{n_1}(z)\right) u(z) \left(\frac{1}{L}\ e^{i\mathbf{k}_2\cdot\boldsymbol{\rho}}\ \varphi_{n_2}(z)\right) =$$

$$\delta_{\lambda_1\lambda_2} \overbrace{\int d^2\rho\ \frac{1}{L^2}\ e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\boldsymbol{\rho}}}^{=\delta_{\mathbf{k}_1\mathbf{k}_2}} \overbrace{\int dz\ \varphi^*_{n_1}(z)\ u(z)\ \varphi_{n_2}(z)}^{=\langle n_1|u|n_2\rangle},$$

de modo que V_2 queda expresado en 2da cuantización como:

$$V_2 = \sum_{n_1,n_2,\mathbf{k},\lambda} \langle n_1|\, u(z)\, |n_2\rangle\, a^\dagger_{n_1\mathbf{k}\lambda} a_{n_2\mathbf{k}\lambda} \tag{9}$$

Vemos que si sumamos el segundo término de la Ec. (7) con la Ec. (9) se satisface la Ec. de Schrödinger en z con autofunciones φ_n y autovalores ϵ_n, quedándonos el hamiltoniano en z expresado en 2da cuantización:

$$H^{(2)}_z := \sum_{n_1,n_2,\mathbf{k},\lambda} \left(\langle n_1|\, \frac{p_z^2}{2m}\, |n_2\rangle\ a^\dagger_{n_1\mathbf{k}\lambda} a_{n_2\mathbf{k}\lambda} + \langle n_1|\, u(z)\, |n_2\rangle\, a^\dagger_{n_1\mathbf{k}\lambda} a_{n_2\mathbf{k}\lambda}\right) =$$

$$\sum_{n_1,n_2,\mathbf{k},\lambda} \langle n_1| \left(\frac{p_z^2}{2m} + u(z)\right) |n_2\rangle\, a^\dagger_{n_1\mathbf{k}\lambda} a_{n_2\mathbf{k}\lambda} = \sum_{n_1,n_2,\mathbf{k},\lambda} \epsilon_2\ \delta_{n_1n_2}\ a^\dagger_{n_1\mathbf{k}\lambda} a_{n_2\mathbf{k}\lambda},$$

con lo cual

$$H^{(2)}_z = \sum_{n,\mathbf{k},\lambda} \epsilon_n\ a^\dagger_{n\mathbf{k}\lambda} a_{n\mathbf{k}\lambda} \tag{10}$$

Unimos todas las expresiones que fuimos obteniendo, y queda H en 2da cuantización como:

$$\begin{aligned} H^{(2)} = &\sum_{n,\mathbf{k},\lambda} \left(\frac{\hbar^2k^2}{2m} + \epsilon_n\right)\ a^\dagger_{n\mathbf{k}\lambda} a_{n\mathbf{k}\lambda}\ + \\ &\frac{e^2}{2}\ \frac{1}{L^2} \sum_{n_1,n_2,n_3,n_4} \sum_{\mathbf{k},\mathbf{p},\mathbf{q}} \sum_{\lambda_1,\lambda_2} \frac{2\pi}{||\mathbf{p}||}\ I_{n_1n_2n_3n_4\mathbf{p}}\ a^\dagger_{n_1(\mathbf{k}+\mathbf{p})\lambda_1}\ a^\dagger_{n_2(\mathbf{q}-\mathbf{p})\lambda_2}\ a_{n_4\mathbf{q}\lambda_2}\ a_{n_3\mathbf{k}\lambda_1}. \end{aligned} \tag{11}$$

Si $e^{-||\mathbf{k}_3-\mathbf{k}_1||.|z-z'|} = e^{-||\mathbf{p}||.|z-z'|} = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{||\mathbf{p}||^m}{m!}\ |z-z'|^m\ \wedge\ I_{n_1n_2n_3n_4\mathbf{p}} < \infty$, luego:

$$I_{n_1n_2n_3n_4\mathbf{p}} = I^{(0)}_{n_1n_2n_3n_4\mathbf{p}} + \sum_{m=1}^{\infty} (-1)^m \frac{||\mathbf{p}||^m}{m!} \overbrace{\int dz \int dz'\ |z-z'|^m\ \varphi^*_{n_1}(z)\ \varphi_{n_3}(z)\ \varphi^*_{n_2}(z')\ \varphi_{n_4}(z')}^{=:I^{(m)}_{n_1n_2n_3n_4\mathbf{p}}}.$$

$$I^{(0)}_{n_1n_2n_3n_4\mathbf{p}} = \int dz \int dz'\ \varphi^*_{n_1}(z)\ \varphi_{n_3}(z)\ \varphi^*_{n_2}(z')\ \varphi_{n_4}(z') = \delta_{n_1n_3}\delta_{n_2n_4}.$$

Con esto, finalmente:

$$\boxed{\begin{aligned} H^{(2)} = &\sum_{n,\mathbf{k},\lambda} \left(\frac{\hbar^2 k^2}{2m} + \epsilon_n \right) \; a^{\dagger}_{n\mathbf{k}\lambda} a_{n\mathbf{k}\lambda} \; + \\ &\frac{e^2}{2} \; \frac{1}{L^2} \sum_{n_1,n_2} \sum_{\mathbf{k},\mathbf{p},\mathbf{q}} \sum_{\lambda_1,\lambda_2} \; \frac{2\pi}{||\mathbf{p}||} \; a^{\dagger}_{n_1(\mathbf{k}+\mathbf{p})\lambda_1} \; a^{\dagger}_{n_2(\mathbf{q}-\mathbf{p})\lambda_2} \; a_{n_2\mathbf{q}\lambda_2} \; a_{n_1\mathbf{k}\lambda_1} \; + \\ &\frac{e^2}{2} \; \frac{1}{L^2} \sum_{n_1,n_2,n_3,n_4} \sum_{\mathbf{k},\mathbf{p},\mathbf{q}} \sum_{\lambda_1,\lambda_2} \; \frac{2\pi}{||\mathbf{p}||} \; \left(\sum_{m=1}^{\infty} (-1)^m \frac{||\mathbf{p}||^m}{m!} I^{(m)}_{n_1 n_2 n_3 n_4 \mathbf{p}} \right) \; \times \\ &a^{\dagger}_{n_1(\mathbf{k}+\mathbf{p})\lambda_1} \; a^{\dagger}_{n_2(\mathbf{q}-\mathbf{p})\lambda_2} \; a_{n_4\mathbf{q}\lambda_2} \; a_{n_3\mathbf{k}\lambda_1} \end{aligned}} \tag{12}$$