Física de muchos cuerpos
Año 2020

Guía 6 - Ejercicio 5

Demostrar que si $c_l(t) = exp\left(-i\frac{\epsilon_l t}{\hbar}\right)\ c_l$, los operadores fermiónicos de creación y destrucción satisfacen:

(a) $\overbrace{c_k^\dagger(t')c_j(t)} = -\overbrace{c_j(t)c_k^\dagger(t')}$.

(b) $\overbrace{c_k(t')c_j(t)} = 0 = \overbrace{c_k^\dagger(t')c_j^\dagger(t)}$.

(c) $\overbrace{\psi(x;t)\psi^\dagger(y;t')} = -\overbrace{\psi^\dagger(y;t')\psi(x;t)}$.

(d) $\overbrace{\psi(x;t)\psi(y;t')} = 0 = \overbrace{\psi^\dagger(x;t)\psi^\dagger(y;t')}$.

Preliminares

- Orden temporal:

  Si $t < t' \Longrightarrow T\left[A(t)B(t')\right] = s\ B(t')A(t)\ \ /\ \ s = -1$ para fermiones.

- Orden normal:

  $N\left[A_l(t')B_k^\dagger(t)\right] = s\ B_k^\dagger(t)A_l(t')\ \ /\ \ s = -1$ para fermiones.

  $N\left[A_l(t')B_k(t)\right] = A_l(t')B_k(t)$.

  $N\left[A_l^\dagger(t')B_k^\dagger(t)\right] = A_l^\dagger(t')B_k^\dagger(t)$.

- Contracción:

  $\overbrace{A(t)B(t')} = T\left[A(t)B(t')\right] - N\left[A(t)B(t')\right]$.

  $\overbrace{\sum_i \alpha_i A_i(t) \sum_j \beta_j B_j^\dagger(t')} = \sum_{i,j} \alpha_i\beta_j\ \overbrace{A_i(t)B_j^\dagger(t')}$ (linealidad que proviene de la linealidad de T, N).

Resolución (a)

- Caso $\epsilon_k, \epsilon_j \leq \epsilon_F : c_k^\dagger = b_k \ \wedge \ c_j = b_j^\dagger$.

$$\overline{c_k^\dagger(t')c_j(t)} = \overline{b_k(t')b_j^\dagger(t)} = T\left[b_k(t')b_j^\dagger(t)\right] - N\left[b_k(t')b_j^\dagger(t)\right] =$$

$$\left\{ \begin{array}{l} t < t' : b_k(t')b_j^\dagger(t) + b_j^\dagger(t)b_k(t') = -T\left[b_j^\dagger(t)b_k(t')\right] + N\left[b_j^\dagger(t)b_k(t')\right] \\ \\ t > t' : -b_j^\dagger(t)b_k(t') + b_j^\dagger(t)b_k(t') = -T\left[b_j^\dagger(t)b_k(t')\right] + N\left[b_j^\dagger(t)b_k(t')\right] \end{array} \right\} = -\overline{b_j^\dagger(t)b_k(t')} = -\overline{c_j(t)c_k^\dagger(t')}.$$

- Caso $\epsilon_k \leq \epsilon_F < \epsilon_j : c_k^\dagger = b_k \ \wedge \ c_j = a_j$.

$$\overline{c_k^\dagger(t')c_j(t)} = \overline{b_k(t')a_j(t)} = T\left[b_k(t')a_j(t)\right] - N\left[b_k(t')a_j(t)\right] =$$

$$\left\{ \begin{array}{l} t < t' : b_k(t')a_j(t) - b_k(t')a_j(t) = -T\left[a_j(t)b_k(t')\right] - b_k(t')a_j(t) \\ \\ t > t' : -a_j(t)b_k(t') - b_k(t')a_j(t) = -T\left[a_j(t)b_k(t')\right] - b_k(t')a_j(t) \end{array} \right\} = -T\left[a_j(t)b_k(t')\right] - b_k(t')a_j(t).$$

Auxiliar:

$$b_k(t')a_j(t) = c_k^\dagger(t')c_j(t) = exp\left(i\frac{\epsilon_k t'}{\hbar}\right) exp\left(-i\frac{\epsilon_j t}{\hbar}\right) c_k^\dagger c_j = exp\left(i\frac{\epsilon_k t'}{\hbar}\right) exp\left(-i\frac{\epsilon_j t}{\hbar}\right) \left(\delta_{jk} - c_j c_k^\dagger\right) =$$

$-c_j(t)c_k^\dagger(t')$, pues $j \neq k \Longrightarrow b_k(t')a_j(t) = -a_j(t)b_k(t')$.

$$-T\left[a_j(t)b_k(t')\right] - b_k(t')a_j(t) = -T\left[a_j(t)b_k(t')\right] + a_j(t)b_k(t') = -T\left[a_j(t)b_k(t')\right] + N\left[a_j(t)b_k(t')\right] =$$

$$-\overline{a_j(t)b_k(t')} = -\overline{c_j(t)c_k^\dagger(t')}.$$

- Caso $\epsilon_j \leq \epsilon_F < \epsilon_k : c_k^\dagger = a_k^\dagger \ \wedge \ c_j = b_j^\dagger$.

$$\overline{c_k^\dagger(t')c_j(t)} = \overline{a_k^\dagger(t')b_j^\dagger(t)} = T\left[a_k^\dagger(t')b_j^\dagger(t)\right] - N\left[a_k^\dagger(t')b_j^\dagger(t)\right] =$$

$$\left\{ \begin{array}{l} t < t' : a_k^\dagger(t')b_j^\dagger(t) - a_k^\dagger(t')b_j^\dagger(t) = -T\left[b_j^\dagger(t)a_k^\dagger(t')\right] + N\left[b_j^\dagger(t)a_k^\dagger(t')\right] \\ \\ t > t' : -b_j^\dagger(t)a_k^\dagger(t') - a_k^\dagger(t')b_j^\dagger(t) = -T\left[b_j^\dagger(t)a_k^\dagger(t')\right] + N\left[b_j^\dagger(t)a_k^\dagger(t')\right] \end{array} \right\} = -\overline{b_j^\dagger(t)a_k^\dagger(t')} = -\overline{c_j(t)c_k^\dagger(t')}.$$

- Caso $\epsilon_F < \epsilon_k, \epsilon_j : c_k^\dagger = a_k^\dagger \ \wedge \ c_j = a_j$.

$$\overline{c_k^\dagger(t')c_j(t)} = \overline{a_k^\dagger(t')a_j(t)} = T\left[a_k^\dagger(t')a_j(t)\right] - N\left[a_k^\dagger(t')a_j(t)\right] =$$

$$\left\{ \begin{array}{l} t < t' : a_k^\dagger(t')a_j(t) - a_k^\dagger(t')a_j(t) = -T\left[a_j(t)a_k^\dagger(t')\right] + N\left[a_j(t)a_k^\dagger(t')\right] \\ \\ t > t' : -a_j(t)a_k^\dagger(t') - a_k^\dagger(t')a_j(t) = -T\left[a_j(t)a_k^\dagger(t')\right] + N\left[a_j(t)a_k^\dagger(t')\right] \end{array} \right\} = -\overline{a_j(t)a_k^\dagger(t')} = -\overline{c_j(t)c_k^\dagger(t')}.$$

Resolución (b)

q.v.q $\overbrace{c_k(t')c_j(t)} = 0.$

- Caso $\epsilon_k, \epsilon_j \leq \epsilon_F : c_k = b_k^\dagger \;\wedge\; c_j = b_j^\dagger$.

$$\overbrace{c_k(t')c_j(t)} = \overbrace{b_k^\dagger(t')b_j^\dagger(t)} = T\left[b_k^\dagger(t')b_j^\dagger(t)\right] - N\left[b_k^\dagger(t')b_j^\dagger(t)\right] =$$

$$\left\{\begin{array}{l} t < t' : b_k^\dagger(t')b_j^\dagger(t) - b_k^\dagger(t')b_j^\dagger(t) \\ \\ t > t' : -b_j^\dagger(t)b_k^\dagger(t') - b_k^\dagger(t')b_j^\dagger(t) = -\left\{b_j^\dagger(t), b_k^\dagger(t')\right\} \end{array}\right\} = 0.$$

- Caso $\epsilon_k \leq \epsilon_F < \epsilon_j : c_k = b_k^\dagger \;\wedge\; c_j = a_j$.

$$\overbrace{c_k(t')c_j(t)} = \overbrace{b_k^\dagger(t')a_j(t)} = T\left[b_k^\dagger(t')a_j(t)\right] - N\left[b_k^\dagger(t')a_j(t)\right] =$$

$$\left\{\begin{array}{l} t < t' : b_k^\dagger(t')a_j(t) - b_k^\dagger(t')a_j(t) \\ \\ t > t' : -a_j(t)b_k^\dagger(t') - b_k^\dagger(t')a_j(t) = -\left\{a_j(t), b_k^\dagger(t')\right\} \end{array}\right\} = 0.$$

- Caso $\epsilon_j \leq \epsilon_F < \epsilon_k : c_k = a_k \;\wedge\; c_j = b_j^\dagger$.

$$\overbrace{c_k(t')c_j(t)} = \overbrace{a_k(t')b_j^\dagger(t)} = T\left[a_k(t')b_j^\dagger(t)\right] - N\left[a_k(t')b_j^\dagger(t)\right] =$$

$$\left\{\begin{array}{l} t < t' : a_k(t')b_j^\dagger(t) + b_j^\dagger(t)a_k(t') = \left\{a_k(t'), b_j^\dagger(t)\right\} \\ \\ t > t' : -b_j^\dagger(t)a_k(t') + b_j^\dagger(t)a_k(t') \end{array}\right\} = 0.$$

- Caso $\epsilon_F < \epsilon_k, \epsilon_j : c_k = a_k \;\wedge\; c_j = a_j$.

$$\overbrace{c_k(t')c_j(t)} = \overbrace{a_k(t')a_j(t)} = T\left[a_k(t')a_j(t)\right] - N\left[a_k(t')a_j(t)\right] =$$

$$\left\{\begin{array}{l} t < t' : a_k(t')a_j(t) - a_k(t')a_j(t) \\ \\ t > t' : -a_j(t)a_k(t') - a_k(t')a_j(t) = -\left\{a_j(t), a_k(t')\right\} \end{array}\right\} = 0.$$

Luego, por idéntico razonamiento, podemos ver que: $\overbrace{c_k^\dagger(t')c_j^\dagger(t)} = 0.$

Resolución (c)

$$\overbracket{\psi(x;t)\psi^\dagger(y;t')} = \overbracket{\sum_i \alpha_i c_i(t) \sum_j \beta_j c_j^\dagger(t')} = \sum_{i,j} \alpha_i \beta_j \, \overbracket{c_i(t) c_j^\dagger(t')} = - \sum_{i,j} \alpha_i \beta_j \, \overbracket{c_j^\dagger(t') c_i(t)} = - \overbracket{\psi(y;t')^\dagger \psi(x;t)}.$$

Observación:

Como $\langle \phi_0 | \, N[...] \, | \phi_0 \rangle = 0$, luego:

$$\overbracket{\psi(x;t)_H \psi^\dagger(y;t')_H} = \frac{1}{\langle \phi_0 | \phi_0 \rangle} \langle \phi_0 | \, T \left[\psi(x;t)_H \psi^\dagger(y;t')_H \right] | \phi_0 \rangle = i G^{(0)}(x,t;y,t').$$

Resolución (d)

$$\overbracket{\psi(x;t)\psi(y;t')} = \sum_{i,j} \alpha_i \beta_j \, \overbracket{c_i(t) c_j(t')} = 0.$$

$$\overbracket{\psi^\dagger(x;t)\psi^\dagger(y;t')} = \sum_{i,j} \alpha_i \beta_j \, \overbracket{c_i^\dagger(t) c_j^\dagger(t')} = 0.$$