Transformación de Jordan - Wigner

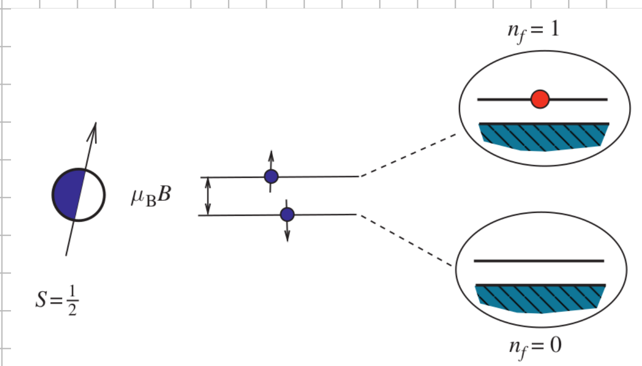
Intro: La representación clásica del espín dada por un vector apuntando en una dirección está bien para $S \gg 1$ pero para S chico el espín se comporta como un objeto muy distinto.

En general este objeto no se comporta como bosones ni como fermiones. Sin embargo en 1D resulta que espines con $S = 1/2$ se comportan como fermiones

1 espín

$$|\uparrow\rangle \equiv c^{\dagger}|0\rangle$$

$$|\downarrow\rangle \equiv |0\rangle$$



$$S^{+} = S_x + iS_y \equiv c^{\dagger} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

con $S_k = \frac{\sigma_k}{2}$

$$S^{-} = S_x - iS_y \equiv c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow S_z = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$S^{+} \cdot S^{-} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$S^+S^- - \frac{1}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$S_z = S^+S^- - \frac{1}{2} \equiv c^\dagger c - \frac{1}{2}$$

despejando S_x y S_y de $\begin{cases} S^+ = S_x + iS_y \equiv c^\dagger \\ S^- = S_x - iS_y \equiv c \end{cases}$

$$S_x = \frac{1}{2}\left(S^+ + S^-\right) \equiv \frac{1}{2}\left(c^\dagger + c\right)$$

$$S_y = \frac{1}{2i}\left(S^+ - S^-\right) \equiv \frac{1}{2i}\left(c^\dagger - c\right)$$

podemos ver que $\{S_a, S_b\} = \frac{1}{4}\underbrace{\{\sigma_a, \sigma_b\}}_{=2\delta_{ab}} = \frac{1}{2}\delta_{ab}$

¿Qué pasa con $\{c, c^\dagger\}$?

$$\{c, c^\dagger\} = \{S^-, S^+\} = \{S_x - iS_y, S_x + iS_y\}$$

$$= \underbrace{\{S_x, S_x\}}_{=\frac{1}{2}} + i\underbrace{\{S_x, S_y\}}_{=0} - i\underbrace{\{S_y, S_x\}}_{=0} + \underbrace{\{S_y, S_y\}}_{=\frac{1}{2}}$$

$$= 1$$

!! (relación de conmutación de fermiones)

Transformación de Jordan - Wigner

$$\begin{cases} S_j^z = c_j^\dagger c_j - \frac{1}{2} = n_j - \frac{1}{2} \\ S_j^+ = c_j^\dagger \, e^{i\pi \sum_{l<j} n_l} \\ S_j^- = c_j \, e^{i\pi \sum_{l<j} n_l} \end{cases}$$

Sigue abajo

muchos espines

Tenemos que considerar la fase correspondiente al sitio (recordar operador $\Theta_k = (-1)^{\sum_{i<k} n_i}$

Por ej: 3 sitios

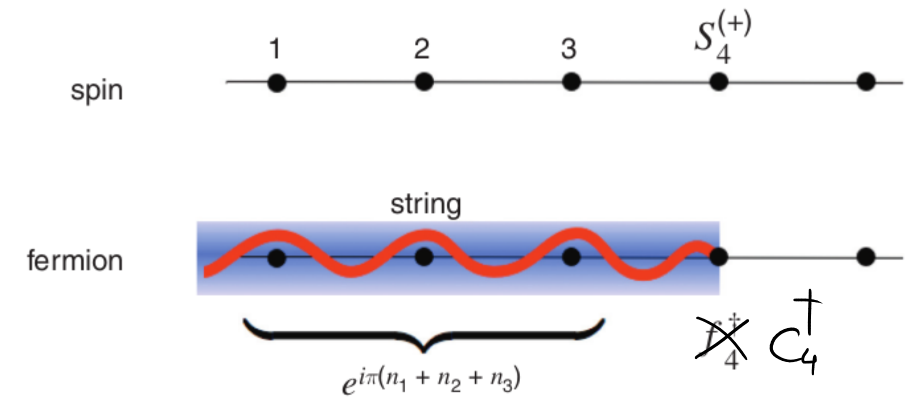
$$c_2^\dagger |100\rangle = (-1)^1 |110\rangle = -|110\rangle$$

$$c_3^\dagger |100\rangle = (-1)^{1+0} |101\rangle = -|101\rangle$$

$$c_3^\dagger |110\rangle = (-1)^{1+1} |111\rangle = |111\rangle$$

$\Rightarrow$ Agregamos el operador Θ_k

$$S_k^+ = c_k^\dagger \Theta_k$$



$= (-1)^{n_1+n_2+n_3}$

Algunas propiedades de Θ_k (string operator)

1. $\Theta_k = (-1)^{\sum_{i<k} n_i} = e^{i\pi \sum_{i<k} n_i}$

2. Anticonmuta con $c_i^\dagger, c_i$ para $i<k$

   Conmuta con $c_i^\dagger, c_i$ para $i \geq k$

Para verlo analizamos $\{e^{i\pi m_j}, c_j\}$

Asumo $m_j = 1$ porque el caso $m_j = 0$ es trivial

si aplico primero $c_j \Rightarrow m_j = 1 \longrightarrow m_j = 0$

luego cuando aplico $e^{i\pi m_j}$ obtengo $+1$

$$\Rightarrow e^{i\pi m_j} c_j = +c_j$$

si aplico primero $e^{i\pi m_j}$ obtengo -1

luego aplico c_j

$$\Rightarrow c_j e^{i\pi m_j} = -c_j$$

Entonces $\{e^{i\pi m_j}, c_j\} = c_j - c_j = 0$

Por otro lado, $[e^{i\pi m_l}, c_j] \underset{l \neq j}{=} 0$

$$\Rightarrow \{e^{i\pi \sum_{j<k} m_j}, c_l\} = 0 \quad \text{si } l < k$$

$$[e^{i\pi \sum_{j<k} m_j}, c_l] = 0 \quad \text{si } l \geq k$$

Esto implica que $[s_j^{\pm}, s_k^{\pm}] = 0$ para $j < k$

Ejemplo: cadena de Heisenberg XY anisotrópica en z

$$H = -J\sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y\right) - J_z \sum_j S_j^z S_{j+1}^z$$

reescribimos usando

$$S_j^x S_{j+1}^x + S_j^y S_{j+1}^y = \frac{1}{2}\left(S_{j+1}^+ S_j^- + S_j^+ S_{j+1}^-\right) \quad \text{(Apéndice)}$$

$$\Rightarrow H = -\frac{J}{2}\sum_j \left(S_{j+1}^+ S_j^- + S_j^+ S_{j+1}^-\right) - J_z \sum_j S_j^z S_{j+1}^z$$

"Fermionicemos" usando Jordan-Wigner

Analicemos término a término

a) $S_{j+1}^+ S_j^- = c_{j+1}^\dagger e^{i\pi\sum_{\ell<j+1} n_\ell} \, c_j \, e^{i\pi\sum_{\ell<j} n_\ell} = c_{j+1}^\dagger \underbrace{\left(e^{i\pi\sum_{\ell<j} n_\ell}\right)^2}_{=1} \underbrace{e^{i\pi n_j} c_j}_{=+c_j}$

separo el término j

$[\theta_j, c_j] = 0$

$$\Rightarrow S_{j+1}^+ S_j^- = c_{j+1}^\dagger c_j$$

Análogamente $S_j^+ S_{j+1}^- = c_j^\dagger c_{j+1}$

b) $S_{j+1}^z S_j^z = \left(n_{j+1} - \frac{1}{2}\right)\left(n_j - \frac{1}{2}\right) = \underbrace{-\frac{n_{j+1}}{2} - \frac{n_j}{2}}_{\text{con la suma } \circledast} + n_{j+1} n_j \underbrace{- \frac{1}{4}}_{\text{cte. lo tiro}}$

$\circledast$ $\sum_j \frac{n_{j+1}}{2} + \frac{n_j}{2} \overset{\text{cambio } j+1 \to j \text{ en la primer suma}}{=} \sum_j n_j$

$$H = -\frac{J}{2}\sum_j \left(S^+_{j+1} S^-_j + S^+_j S^-_{j+1}\right) - J_z \sum_j S^z_j S^z_{j+1}$$

$$H = -\frac{J}{2}\sum_j \left(c^\dagger_{j+1} c_j + c^\dagger_j c_{j+1}\right) + J_z \sum_j n_j - J_z \sum_j n_j n_{j+1} + \underbrace{\frac{1}{4}}_{\text{lo tiro}}$$

Transformo Fourier para diagonalizar

posición sitio

$$c_j = \frac{1}{\sqrt{N}} \sum_k f_k \, e^{ikR_j}$$

$$c^\dagger_j = \frac{1}{\sqrt{N}} \sum_k f^\dagger_k \, e^{-ikR_j}$$

Analizo término por término

a) $$\sum_j n_j = \sum_j \frac{1}{N} \sum_{k,k'} f^\dagger_k f_{k'} e^{-i(k-k')R_j} = \sum_{k,k'} f^\dagger_k f_{k'} \underbrace{\frac{1}{N}\sum_j e^{-i(k-k')R_j}}_{=\delta_{kk'}}$$

$$= \sum_k f^\dagger_k f_k$$

b) $$\frac{1}{2}\sum_j \left(c^\dagger_{j+1} c_j + c^\dagger_j c_{j+1}\right) = \frac{1}{2}\sum_{k,k'} f^\dagger_k f_{k'} \left(e^{-ika} + e^{ika}\right) \underbrace{\frac{1}{N}\sum_j e^{-i(k-k')R_j}}_{=\delta_{kk'}} = \dots$$

$$c^\dagger_{j+1} c_j = \frac{1}{N}\sum_{k,k'} f^\dagger_k f_{k'} \, e^{-ik(R_j+a)} e^{+ik'R_j} = \frac{1}{N}\sum_{k,k'} f^\dagger_k f_{k'} \, e^{-i(k-k')R_j} e^{-ika}$$

$$\dots = \sum_k f^\dagger_k f_k \cos(ka)$$

Hasta ahora

$$H = \sum_k \underbrace{\left[J_z - J\cos(ka)\right]}_{\omega_k} f_k^\dagger f_k \quad - J_z \underbrace{\sum_j n_j n_{j+1}}_{\text{sin transformar}}$$

magnon excitation energy

c) $$\sum_j c_j^\dagger c_j c_{j+1}^\dagger c_{j+1} = \frac{1}{N} \sum_{k,k',k'',k'''} f_k^\dagger f_{k'} f_{k''}^\dagger f_{k'''} \; \frac{1}{N}\sum_j e^{-i(k-k')R_j} \, e^{-i(k''-k''')(R_j+a)}$$

$$= \frac{1}{N} \sum_{k,k',k'',k'''} f_k^\dagger f_{k'} f_{k''}^\dagger f_{k'''} \underbrace{\frac{1}{N}\sum_j e^{-i\left[(k+k'')-(k'+k''')\right]R_j}}_{\delta_{k+k'',k'+k'''}} \; e^{-i(k''-k''')a}$$

$k = k_1 + q$

$k' = k_1$

$k'' = k_2 - q$

$k''' = k_2$

$k + k'' = k_1 + k_2$

$k' + k''' = k_1 + k_2$

$$= \frac{1}{N} \sum_{k_1 k_2 q} f_{k_1+q}^\dagger f_{k_1} f_{k_2-q}^\dagger f_{k_2} \; e^{+iqa}$$

Considero el término

$$c_{j-1}^\dagger c_{j-1} c_j^\dagger c_j = \frac{1}{N} \sum_{k_1 k_2 q} f_{k_1+q}^\dagger f_{k_1} f_{k_2-q}^\dagger f_{k_2} \; e^{-iqa}$$

$$\Rightarrow \sum_j c_j^\dagger c_j c_{j+1}^\dagger c_{j+1} = \frac{1}{2}\sum_j \left(c_{j-1}^\dagger c_{j-1} c_j^\dagger c_j + c_j^\dagger c_j c_{j+1}^\dagger c_{j+1} \right)$$

$$= \frac{1}{N}\sum_{k_1 k_2 q} f_{k_1+q}^\dagger f_{k_1} f_{k_2-q}^\dagger f_{k_2} \cos(qa)$$

$$= \frac{1}{N}\sum_{k_1 k_2 q} f_{k_2-q}^\dagger f_{k_1+q}^\dagger f_{k_1} f_{k_2} \cos(qa)$$

desprecio estos términos para

$$k_1, k_2 \to k, k'$$

$$\Rightarrow H = \sum_k \Big[\underbrace{J_z - J\cos(ka)}_{\omega_k} \Big] f_k^\dagger f_k - \frac{J_z}{N}\sum_{k k' q} f_{k'-q}^\dagger f_{k+q}^\dagger f_k f_{k'} \cos(qa)$$

$\boxed{\text{Caso } J_z = J}$: Heisenberg Ferro

$$\omega_k = 2J\sin^2\left(\frac{ka}{2}\right)$$

$$M = -\frac{N}{2}$$

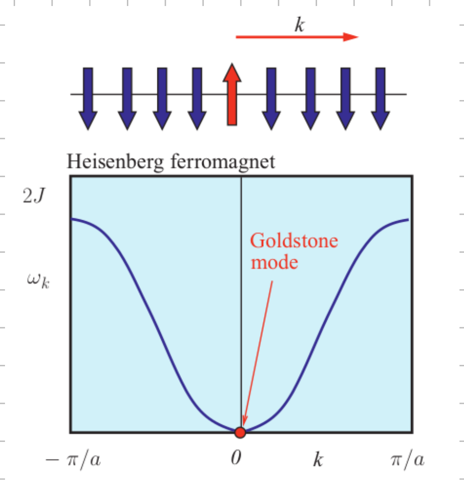
$\uparrow$ magnetización espontánea

$$|GS\rangle = |\downarrow\downarrow\ldots\downarrow\rangle$$

(en cualquier dirección)

$\omega_0 = 0 \Rightarrow$ no tengo costo energético en agregar un "magnon" con $k=0$

con $\lambda \sim \underbrace{\frac{1}{k}}_{k=0} \to +\infty$ (modo de Goldstone)



Roto alrededor de x

$$|\psi\rangle = e^{i d\theta S_x} |\downarrow\downarrow\dots\rangle$$

$$= \mathbb{1}|\downarrow\downarrow\dots\rangle + i\, d\theta \sum_j \underbrace{S_j^x |\downarrow\downarrow\dots\rangle} + O(d\theta^2)$$

para un j fijo tengo que

$$S_j^x |\overset{1}{\downarrow}\dots\overset{j}{\downarrow}\dots\rangle = |\downarrow\dots\uparrow\dots\rangle$$

$$S_j^x |\downarrow\dots\uparrow\dots\rangle = |\downarrow\dots\downarrow\dots\rangle$$

pensar para 1 spin

$$\sigma^x |\downarrow\rangle = |\uparrow\rangle \qquad\qquad \sigma^x |\uparrow\rangle = |\downarrow\rangle$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

entonces $S_j^x |\downarrow\downarrow\dots\rangle = \frac{S_j^+}{2} |\downarrow\downarrow\dots\rangle$
en este caso

$$\Rightarrow |\psi\rangle = |\downarrow\downarrow\dots\rangle + i\frac{d\theta}{2} \underbrace{\sum_j S_j^+ |\downarrow\downarrow\dots\rangle}_{\sum_j c_j^\dagger e^{i\pi\sum_{l<j} n_l}|0\rangle} + O(d\theta^2)$$

$$\sum_j c_j^\dagger e^{i\pi\sum_{l<j} n_l}|0\rangle = \sum_j c_j^\dagger |0\rangle \underset{\text{mult. } \sqrt{N}\cdot\frac{1}{\sqrt{N}}}{=} \sum_k f_k^\dagger \overbrace{\frac{1}{N}\sum_j e^{-ikR_j}}^{\delta_{k0}} \sqrt{N}\,|0\rangle$$

$= \sqrt{N}\, f_0^{\dagger} |0\rangle$

Agregar un magnon con $k=0$ equivale a rotar infinitesimalmente la magnetización (no tiene costo energético)

Caso $J_z = 0$: XY Ferro

$\omega_k = -J\cos(ka)$

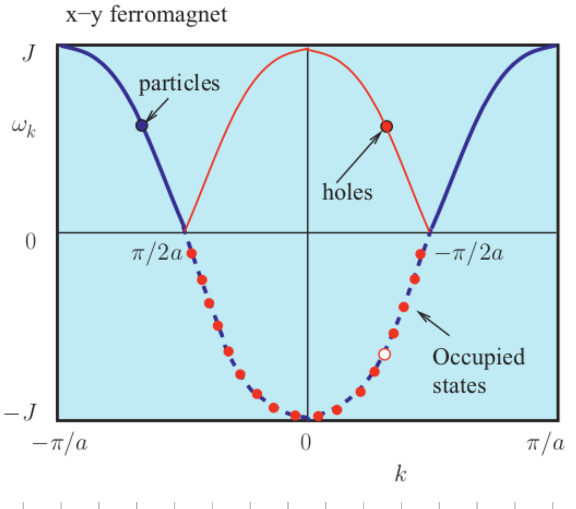
banda medio llena

$|GS\rangle = \prod_{|k|<\frac{\pi}{2a}} f_k^{\dagger}|0\rangle$

$\langle S_z \rangle = \langle n_f - \frac{1}{2} \rangle = 0$

(no hay magnetización espontánea)

para estudiar los estados excitados puedo hacer una transformación partícula-agujero



$$\tilde{f}_k = \begin{cases} f_k & \text{si } |k| < \frac{\pi}{2a} \quad \text{(destruyo part.)} \\ f^\dagger_{-k} & \text{si } |k| > \frac{\pi}{2a} \quad \text{(creo agujero)} \end{cases}$$

$$\Rightarrow H = \sum_k J\,|\cos(ka)| \left(\tilde{f}_k^\dagger \tilde{f}_k - \frac{1}{2} \right)$$

$$E_{GS} = -\frac{1}{2} \sum_k J\,|\cos(ka)| = -a \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \frac{dk}{2\pi} J \cos(ka) = -\frac{J}{\pi}$$

Falta el apéndice →

Apéndice

QVQ

$$S_j^x S_{j+1}^x + S_j^y S_{j+1}^y = \frac{1}{2}\left(S_{j+1}^+ S_j^- + S_j^+ S_{j+1}^- \right)$$

DEM:

$$S_j^+ = S_j^x + i S_j^y$$

$$S_j^- = S_j^x - i S_j^y$$

$$S_{j+1}^+ S_j^- = (S_{j+1}^x + i S_{j+1}^y)(S_j^x - i S_j^y)$$

$$= S_{j+1}^x S_j^x \underline{- i S_{j+1}^x S_j^y} \underline{+ i S_{j+1}^y S_j^x} + S_{j+1}^y S_j^y$$

$$S_j^+ S_{j+1}^- = (S_j^x + i S_j^y)(S_{j+1}^x - i S_{j+1}^y)$$

$$= S_j^x S_{j+1}^x \underline{- i S_j^x S_{j+1}^y} \underline{+ i S_j^y S_{j+1}^x} + S_j^y S_{j+1}^y$$

- $i S_{j+1}^y S_j^x - i S_j^x S_{j+1}^y = i \left[S_{j+1}^y, S_j^x \right] = 0$
- $i S_j^y S_{j+1}^x - i S_{j+1}^x S_j^y = i \left[S_j^y, S_{j+1}^x \right] = 0$

$$\Rightarrow S_{j+1}^+ S_j^- + S_j^+ S_{j+1}^- = 2\left(S_{j+1}^x S_j^x + S_{j+1}^y S_j^y \right)$$