More on Bessel functions

Infinite domain, δ-function normalization

Consider Bessel's equation on the domain $0 < \rho <$ as $R \to \infty$. Bessel's equation, (3.75) or (3.93), says

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(k\rho)}{d\rho}\right) + \left(k^2 - \frac{\nu^2}{\rho^2}\right) J_\nu(k\rho) = 0.$$

As in class, multiply this equation by $\rho\, J_\nu(k'\rho)$ and integrate from $\rho = 0$ to R:

$$\int_0^R \rho\, d\rho\, J_\nu(k'\rho)\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(k\rho)}{d\rho}\right) + \left(k^2 - \frac{\nu^2}{\rho^2}\right) J_\nu(k\rho)\right] = 0.$$

Integrate the first term by parts,

$$\int_0^R \rho\, d\rho\left[-\frac{dJ_\nu(k'\rho)}{d\rho}\,\frac{dJ_\nu(k\rho)}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) J_\nu(k'\rho)J_\nu(k\rho)\right] = -\left[\rho\, J_\nu(k'\rho)\,\frac{dJ_\nu(k\rho)}{d\rho}\right]_0^R.$$

Exchanging the roles of k and k', and subtracting leads to

$$(k^2 - k'^2)\int_0^R \rho\, d\rho\, J_\nu(k'\rho)J_\nu(k\rho) = \left[\rho\, J_\nu(k\rho)\,\frac{dJ_\nu(k'\rho)}{d\rho} - \rho\, J_\nu(k'\rho)\,\frac{dJ_\nu(k\rho)}{d\rho}\right]_0^R.$$

As $\rho \to 0$, both terms on the right-hand side have the same leading behavior, $(kk'\rho^2)^\nu/\Gamma(\nu)$, and so cancel for any value of ν. The next-leading terms go as $(k^2 - k'^2)r^2(kk'\rho^2)^\nu$, and so do not cancel but vanish as $\rho \to 0$ for $\nu > -1$. The surface term on the right-hand side then contains only the contribution from $\rho = R$,

$$\int_0^R \rho\, d\rho\, J_\nu(k'\rho)J_\nu(k\rho) = \frac{k'R\, J_\nu(kR)\, J'_\nu(k'R) - kR\, J_\nu(k'R)\, J'_\nu(kR)}{k^2 - k'^2}\,. \qquad \textbf{(1a)}$$

Using recursion relations, this can also be written

$$\int_0^R \rho\, d\rho\, J_\nu(k'\rho)J_\nu(k\rho) = \frac{k'R\ J_\nu(kR)\ J_{\nu-1}(k'R) - kR\ J_\nu(k'R)\ J_{\nu-1}(kR)}{k^2 - k'^2}\,. \qquad \textbf{(1b)}$$

Although $J_\nu(kR)$ vanishes as $R^{-1/2}$ for large R, the presence of the factor of R means we must investigate in detail the behavior of the Bessel functions at large argument.

The Bessel functions for large argument are given in (3.91),

$$J_\nu(k\rho) \to \sqrt{\frac{2}{\pi kR}}\cos\!\left(kR - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

Making use of the trig identity $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$, either **(1a)** or **(1b)** then leads to

$$\int_0^R \rho\, d\rho\, J_\nu(k'\rho)J_\nu(k\rho) \to \frac{\sin[(k-k')R]}{\pi\sqrt{kk'}\,(k-k')} - \frac{\cos[(k+k')R - \nu\pi]}{\pi\sqrt{kk'}\,(k+k')}. \qquad \textbf{(2)}$$

The first term "oscillates" as $R \to \infty$ and so "averages" to zero, unless $k = k'$, when it becomes large, behavior we expect for a δ-function; the second term "averages" to zero for all k, k'. In more detail, the function

$$\delta_\epsilon(x) = \frac{\sin(x/\epsilon)}{\pi x}$$

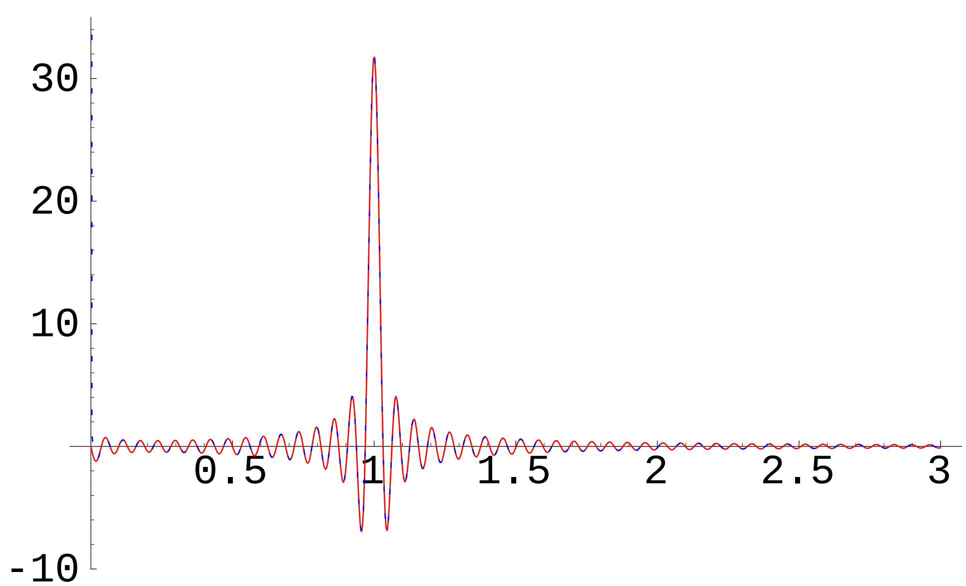
has value $\delta_\epsilon = 1/\epsilon\pi$ at $x = 0$, oscillates rapidly outside the interval $x = \pm\epsilon\pi$, and has integral

$$\int_{-\infty}^{\infty} dx\, \delta_\epsilon(x) = 1.$$

Thus, the limit $\epsilon \to 0$ is a representation of the Dirac δ-function (cf. entry [34] in the *MathWorld* page on δ-functions, `http://mathworld.wolfram.com/DeltaFunction.html`). This functional form also appears in time-dependent perturbation theory in quantum mechanics, where it gives the "energy-conserving" δ-function. The limit in **(2)** then becomes

$$\int_0^\infty \rho\, d\rho\, J_\nu(k'\rho) J_\nu(k\rho) = \frac{1}{\sqrt{kk'}} \lim_{R\to\infty} \Big[\delta_{1/R}(k-k') - \delta_{1/R}\Big(k + k' - \tfrac{1}{R}(\nu - \tfrac{1}{2})\pi\Big)\Big] = \frac{\delta(k-k')}{\sqrt{kk'}} = \frac{\delta(k-k')}{k}. \quad \blacksquare$$

The second term does not contribute, because the argument never vanishes. I was assigned this problem when I was a graduate student, and I have written solutions to it at various times in the past, but this version is more correct than any of those earlier attempts. The figure shows both the exact J_ν result of **(1a/b)** (red) and the cosine approximation **(2)** for large argument (dotted, blue) plotted as a function of k for $k' = 1$, $R = 100$, and $\nu = 1$.



Green's function

Green's function constructions always follow a similar pattern. The Green's function is a solution to $\nabla^2 G = \nabla'^2 G = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x}')$, with the boundary condition that $G \to 0$ as $r \to \infty$ or $r' \to \infty$. Since the δ-function vanishes almost everywhere, we can expand $G(\boldsymbol{x})$ in the modes which are solutions to $\nabla^2 G = 0$,

$$G(\boldsymbol{x}) = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, A_m(k)\, J_m(k\rho)\, e^{im\phi}\, e^{\pm kz},$$

where the coefficients $A_m(k)$ will depend on ρ', ϕ', z'. We could guess more about them, but the result will follow systematically from this starting point. The Bessel function satisfies the boundary condition $G \to 0$ as $\rho \to \infty$, but for the z-dependence we need different expressions for the two regions $z > z'$ and $z < z'$:

$$G^<(\boldsymbol{x}) = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, A_m^<(k)\, J_m(k\rho)\, e^{im\phi}\, e^{+kz} \qquad (z < z'),$$
$$G^>(\boldsymbol{x}) = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, A_m^>(k)\, J_m(k\rho)\, e^{im\phi}\, e^{-kz} \qquad (z > z').$$

Except at $\rho = \rho'$, $\phi = \phi'$, the Green's function must be continuous at $z = z'$, and so we must have $A_m^<(k)\, e^{+kz'} = A_m^>(k)\, e^{-kz'} = C_m(k)$. This leads to the single expression

$$G(\boldsymbol{x}) = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, C_m(k)\, J_m(k\rho)\, e^{im\phi}\, e^{-k(z_> - z_<)}\,,$$

where $z_<$ and $z_>$ are the smaller and larger of z, z'. The remainder of the construction uses the δ-function information, $\nabla^2 G = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x}')$. Integrate this equation over z from $z = z' - \epsilon$ to $z = z' + \epsilon$ for $\epsilon \to 0$ to obtain

$$\int_{z'-\epsilon}^{z'+\epsilon} dz\, \nabla^2 G = \left[\frac{\partial G}{\partial z}\right]_{z=z'-\epsilon}^{z=z'+\epsilon} = -4\pi\, \frac{\delta(\rho - \rho')}{\rho}\, \delta(\phi - \phi') \qquad (*)$$

(the ρ and ϕ derivatives produce factors of k and m that are finite term by term, and so those contributions vanish as $\epsilon \to 0$). Exchanging the roles of k and ρ in part (a), we have an expansion of $\delta(\rho - \rho')$ in Bessel functions, and the representation of $\delta(\phi - \phi')$ in $e^{im\phi}$ is elementary. Writing both sides of (*) as series expansions gives

$$\sum_{m=-\infty}^{\infty} \int_0^\infty dk\, C_m(k)\, J_m(k\rho)\, e^{im\phi}\, \frac{\partial}{\partial z} \left[e^{-k(z_> - z_<)}\right]_{z=z'-\epsilon}^{z=z'+\epsilon}$$
$$= -4\pi \int_0^\infty k\, dk\, J_m(k\rho)\, J_m(k\rho') \sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\, e^{im(\phi-\phi')}.$$

Writing the derivative explicitly and identifying coefficients of $J_m(k\rho)e^{im\phi}$ on left and right hand sides we obtain

$$C_m(k)\left[\frac{\partial}{\partial z}e^{-k(z-z')} - \frac{\partial}{\partial z}e^{-k(z'-z)}\right]_{z=z'} = -2k\,C_m(k)$$
$$= -4\pi\,k\,J_m(k\rho')\,\frac{1}{2\pi}\,e^{-im\phi'} = -2k\,J_m(k\rho')\,e^{-im\phi'}.$$

Thus, $C_m(k) = J_m(k\rho')\,e^{-im\phi'}$, and we have completed the Bessel function representation of the Green's function for free space,

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \sum_{m=-\infty}^{\infty}\int_0^\infty dk\,J_m(k\rho)J_m(k\rho')\,e^{im(\phi-\phi')}\,e^{-k(z_>-z_<)}\,. \qquad \blacksquare$$

In class we constructed Green's functions for the square (see also JDJ Problem 2.15) and for the volume between two spherical surfaces (see also JDJ Section 3.9); and Jackson does a different Bessel function construction in Section 3.11.

Some Bessel function relations

Take the Green's function and evaluate for $\boldsymbol{x}' \to 0$. On the left-hand side

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}} \to \frac{1}{\sqrt{\rho^2+z^2}};$$

while, since $J_m(x) \sim (x/2)^m$, only $m=0$ contributes to the series on the right-hand side as $\rho' \to 0$. Thus,

$$\int_0^\infty dk\,J_0(k\rho)\,e^{-k|z|} = \frac{1}{\sqrt{\rho^2+z^2}}. \qquad \blacksquare$$

Now, evaluate this first result at $\rho = \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}$,

$$\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2}} = \int_0^\infty dk\,J_0[k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\,]\,e^{-k|z|}.$$

But, this is also just the Green's function evaluated at $z'=0$, but arbitrary ρ',

$$\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2}} = \sum_{m=-\infty}^{\infty}\int_0^\infty dk\,J_m(k\rho)J_m(k\rho')\,e^{im(\phi-\phi')}\,e^{-k|z|}.$$

The integral over k amounts to a Laplace transform, and the theory of Laplace transforms assures us that the transformation is unique and invertible, and so the integrands on the right-hand side must be equal:

$$J_0[k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\,] = \sum_{m=-\infty}^{\infty} J_m(k\rho)J_m(k\rho')\,e^{im(\phi-\phi')}. \qquad \blacksquare$$

Evaluate the relation at the bottom of the previous page at $k\rho = k\rho' = x$, $\phi - \phi' = 0$; recall that $J_{-m}(x) = (-1)^m J_m(x)$, and

$$\sum_{m=-\infty}^{\infty} J_m^2(x) = [J_0(x)]^2 + 2\sum_{k=1}^{\infty}[J_k(x)]^2 = J_0(0) = 1. \qquad \blacksquare$$

Take the result and evaluate for $\phi' = 0$ in the limit ρ' becomes large. In this limit the square root becomes

$$\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos\phi + z^2} = \rho' - \rho\,\cos\phi + \mathcal{O}(\frac{\rho^2}{\rho'}),$$

and the large argument limit of the Bessel functions then gives

$$\sqrt{\frac{2}{\pi k\rho'}}\,\cos\Big(k\rho' - k\rho\cos\phi - \frac{\pi}{4}\Big) = \sum_{m=-\infty}^{\infty} J_m(k\rho)\sqrt{\frac{2}{\pi k\rho'}}\,\cos\Big(k\rho' - \frac{m\pi}{2} - \frac{\pi}{4}\Big)\,e^{im\phi},$$

correct up to terms of order $1/\rho'$. Use that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, and equate separately the coefficients of $\cos(k\rho' - \frac{\pi}{4})$ and $\sin(k\rho' - \frac{\pi}{4})$ on each side (equality must hold for all values of ρ'),

$$\cos(k\rho\cos\phi) = \sum_{m=-\infty}^{\infty} J_m(k\rho)\,\cos(\frac{m\pi}{2})\,e^{im\phi},$$

$$\sin(k\rho\cos\phi) = \sum_{m=-\infty}^{\infty} J_m(k\rho)\,\sin(\frac{m\pi}{2})\,e^{im\phi}.$$

Finally, add i times the second to the first,

$$e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} J_m(k\rho)\,e^{im\frac{\pi}{2}}\,e^{im\phi}. \qquad \blacksquare$$

Take the real and imaginary parts of the complex exponential evaluated at $k\rho = x$, $\phi = 0$:

$$\mathrm{Re}\left[\sum_{m=-\infty}^{\infty} i^m\,J_m(x)\right] = \sum_{k=-\infty}^{\infty}(-1)^k\,J_{2k}(x) = J_0(x) + 2\sum_{k=1}^{\infty}(-1)^k\,J_{2k}(x) = \cos x, \qquad \blacksquare$$

$$\mathrm{Im}\left[\sum_{m=-\infty}^{\infty} i^m\,J_m(x)\right] = 2\sum_{k=0}^{\infty}(-1)^{k+1}\,J_{2k+1}(x) = \sin x. \qquad \blacksquare$$

Integral representation

Take the complex exponential, evaluate at $k\rho = x$, multiply by $e^{-im\phi}$, and integrate over ϕ:

$$\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{-im\phi}\left[\sum_{m'=-\infty}^{\infty} i^{m'}\,J_{m'}(x)\,e^{im'\phi}\right] = i^m\,J_m(x) = \int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{ix\cos\phi - im\phi}\,. \qquad \blacksquare$$