Symmetry, Uniqueness, and the Coulomb Law of Force

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The Poynting vector (8) has a magnitude given by

$$|\mathbf{S}| = |\mathbf{v}_n| (|\mathbf{B}|^2/\mu_0) = 1/\mu_0 (3\times 10^5)(5\times 10^{-9})^2 \approx 6 \text{ W/m}^2 \quad (26)$$

and is directed at an angle of 225° with respect to the earth–sun direction. This Poynting vector (26) accounts for approximately 4×10^{-9} of the solar radiation represented by a radiation constant[8] of 14×10^2 W/m^2.

[8] L. Larmore, "Solar Physics and Solar Radiation" in *Space Physics*, edited by D. P. LeGalley and A. Rosen (John Wiley & Sons, Inc., New York, 1964), Chap. 4.

Symmetry, Uniqueness, and the Coulomb Law of Force

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Some arguments from symmetry, frequently occurring in elementary electricity, are criticized, and the importance of the uniqueness theorem in electrostatics is stressed. It is conjectured that for a $1/r^n$ law of force a uniqueness theorem only holds in the case $n=2$. The problem is posed of finding a solution to any electrostatic problem whatsoever (involving conductors) if the law of force is not the Coulomb law.

I. SYMMETRY AND UNIQUENESS

IN view of the important role played by symmetry principles in present-day fundamental physics, it would seem desirable even in more elementary contexts for arguments based upon symmetry considerations to be presented in a clear and logical fashion. My own experience in teaching electrostatics suggests that most students are liable to be convinced by quite invalid symmetry arguments—for example, given a set of collinear point charges it does *not* follow "by symmetry" that any neutral points must lie also on the same line. Presumably not unconnected with this uncritical acceptance of arguments based on symmetry is the fact that false, or at best incomplete, arguments of this type are quite common in elementary textbooks on electricity.

For example, when treating the problem of an isolated charged conducting sphere many textbooks conclude from the spherical symmetry that the charge is distributed uniformly over the sphere. Now this is certainly not valid reasoning—from the spherical symmetry of a problem one cannot deduce spherical symmetry of a solution. All that the spherical symmetry tells one is that from any one equilibrium distribution of charge other equilibrium distributions can be obtained by rotation and/or reflection. More generally, if a problem possesses any kind of symmetry, then a particular solution will not in general be symmetric, but the whole *family of solutions* will of necessity exhibit the particular symmetry in question.

An extremely simple example may be of help in illustrating the above point. The equation $x^2=1$ is symmetric under the operation $x \to -x$. If one argues as a result that any solution $x=a$ must have this symmetry then one concludes that $a=-a$, and hence that the solution is $x=0$! In this example $x=1$ is one solution, and so, by applying the symmetry operation, it follows that $x=-1$ must be another solution. The family of solutions here contain just two members, $x=1$ and $x=-1$, and of course exhibits the symmetry in question.

The *invalid* arguments discussed above were of the type

Symmetry of problem ⇒
symmetry of solution (A)

which should be contrasted with the *valid* deductions

Symmetry of problem ⇒
symmetry of the whole set of solutions (B)

and [an immediate consequence of (B)]

$$\begin{array}{ccc}\text{Symmetry of problem} & & \text{Symmetry of}\\ + & \Rightarrow & \\ \text{Uniqueness of solution} & & \text{the solution}\end{array} \qquad \text{(C)}$$

In applying (B) and (C)[1] to any particular case one should of course be precise in stating exactly what is the problem, what is classed as a solution and what is the relevant group of symmetries. In practice, students are often presented with symmetry arguments so condensed that even the brighter ones may not be able to fill in the missing steps.

II. A SIMPLE PROBLEM

In this connection it may be of some interest to give a fairly full treatment of a simple problem involving symmetry—say that of the restrictions imposed by symmetry considerations on the form taken by the magnetic field produced by steady current flowing along a thin, straight, infinitely long, conducting wire. I take as the symmetry group of the problem that generated by the following three types of transformation:

(i) translations $\mathbf{x}'=\mathbf{x}+\mathbf{a}$ in a direction parallel to the wire (taken to lie along the z axis),

(ii) rotations $\mathbf{x}'=R\mathbf{x}$ about the z axis, and

(iii) reflection in a plane containing the wire, say $x'=x$, $y'=-y$, $z'=z$.

[Of course in accepting (i)–(iii) as the relevant symmetries one is implicitly appealing quite a lot to one's knowledge of what constitutes an electric current; for example, if the electron motions producing the current were microscopically helical rather than linear, then one would not be justified in assuming (iii) as a symmetry operation.]

If $\mathbf{H}$ is one solution of the problem, that is $\mathbf{H}=\mathbf{H}(\mathbf{x})$ is a vector field defined everywhere except possibly on the z axis, then it follows that the transform $\mathbf{H}'$ of $\mathbf{H}$ under a general transformation of the above symmetry group, must be another solution. *If sufficient conditions (consistent with the above symmetry!) are imposed so that there is a unique solution*, then the vector fields $\mathbf{H}$ and $\mathbf{H}'$ must coincide, so that

$$\mathbf{H}'(\mathbf{x})=\mathbf{H}(\mathbf{x}) \qquad (1)$$

for all transformations of the symmetry group.

Neglecting the reflection symmetry (iii) for the moment, the transformation law for a vector field $\mathbf{H}$ under $\mathbf{x}'=R\mathbf{x}+\mathbf{a}$ is, in terms of Cartesian components H_i ($i=1,2,3$),

$$H_i'(\mathbf{x}')=\sum_{j=1}^{3} R_{ij}H_j(\mathbf{x}). \qquad (2)$$

It follows from (1) and (2) that the magnetic field is restricted to satisfy

$$\sum_{j=1}^{3} R_{ij}H_j(\mathbf{x})=H_i(R\mathbf{x}+\mathbf{a}), \qquad (3)$$

where $R=R(\alpha)$ denotes a rotation about Oz through an arbitrary angle α and $\mathbf{a}=(0,0,a)$ is an arbitrary vector along Oz.

In terms of cylindrical polar coordinates (r,θ,z) one quickly deduces that the most general vector field consistent with (3) is

$$\mathbf{H}(r,\theta,z)=f(r)\hat{\mathbf{r}}+g(r)\hat{\boldsymbol{\theta}}+h(r)\hat{\mathbf{z}}, \qquad (4)$$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\mathbf{z}}$ are the unit triad of vectors appropriate to cylindrical polars. The functions f, g, and h are independent of θ and z, but are otherwise undetermined, so that there exists a rather large class of vector fields symmetric under translations along, and rotations about, the z axis.

The consequences of symmetry under the reflection (iii), which in cylindrical polar coordinates reads $r'=r$, $\theta'=-\theta$, $z'=z$, depend on whether we are dealing with a polar or an axial vector field. Given that the magnetic field is an axial vector, then corresponding to the restriction (3) we have

$$\begin{aligned} H_r(r,\theta,z)&=-H_r(r,-\theta,z),\\ H_\theta(r,\theta,z)&=+H_\theta(r,-\theta,z),\\ H_z(r,\theta,z)&=-H_z(r,-\theta,z).\end{aligned} \qquad (5)$$

Hence the most general vector field consistent with (3) and (5) is

$$\mathbf{H}(r,\theta,z)=g(r)\hat{\boldsymbol{\theta}}. \qquad (6)$$

[1] Another point worth making is that the implications (B) and (C) are vacuous if, as may well be the case, the problem in question admits of *no* solution. Usually the argument (C) is used in the form: *if* a solution exists, then it must be such and such; one then checks if "such and such" is indeed a solution.

It should be noted how restrictive the requirement of reflection symmetry is—it rules out, for example, helical lines of force which were allowed by (4).

[For a polar vector there is a change of sign throughout Eq. (5), with the corresponding conclusion

$$\mathbf{H}(r,\theta,z)=f(r)\hat{\mathbf{r}}+h(r)\hat{\mathbf{z}}. \qquad (6')$$

In case of the electric field due to a uniform line of charge, the problem has the additional symmetry transformation

$$x'=x,\ y'=y,\ z'=-z \qquad (7)$$

which, for the polar electric field, implies that, in (6′), $h(r)\equiv 0$.]

If we now introduce the fact that the vector field with which we are dealing is irrotational, so that, at any rate locally, it is derivable from a potential function ϕ:

$$H_r=-\frac{\partial\phi}{\partial r},\quad H_\theta=-\frac{1}{r}\frac{\partial\phi}{\partial\theta},\quad H_z=-\frac{\partial\phi}{\partial z}; \qquad (8)$$

then it follows from (4) that the most general field of this kind which is consistent with (3) is given by the potential function

$$\phi(r,\theta,z)=F(r)+B\theta+Cz, \qquad (9)$$

where $F(r)$ is an undetermined function of r, and B and C are constants. Given also that the field is solenoidal, then (9) becomes

$$\phi(r,\theta,z)=A\log r+B\theta+Cz+D. \qquad (10)$$

It follows finally from (6) that for an axial vector the most general irrotational, solenoidal field consistent with the above symmetry group (and with uniqueness) is given by a potential

$$\phi(r,\theta,z)=B\theta+D, \qquad (11)$$

so that

$$\mathbf{H}(r,\theta,z)=(B/r)\hat{\boldsymbol{\theta}}. \qquad (12)$$

Incidentally, I do not wish to suggest that one *should* use symmetry considerations in the above problem; if one's sole interest is in deriving the magnetic field then, for example, a straightforward use of the Biot–Savart law is preferable. But *if* a symmetry argument is going to be used, then it is clear from the preceding discussion that something more is called for than a remark "by symmetry, the lines of force must be circles."

Given at the outset that one is dealing with an irrotational field, then one expects that the above discussion would be simplified by dealing throughout with the transformation properties of the scalar potential ϕ rather than the more complicated transformation properties of the vector field $\mathbf{H}$. This is indeed the case—through there is one trap for the unwary! It is tempting to proceed as follows: corresponding to the condition (3) we have the simpler equation

$$\phi(\mathbf{x})=\phi(R\mathbf{x}+\mathbf{a}) \qquad (13)$$

so that, in cylindrical polar coordinates,

$$\phi(r,\theta,z)=\phi(r,\ \theta+\alpha,\ z+a); \qquad (14)$$

hence we deduce that ϕ is a function of r alone, in contradiction to our previous conclusion (9). The error in this reasoning was to assume, instead of (1), the corresponding uniqueness condition for ϕ; however the condition (1) on $\mathbf{H}$ only implies that

$$\phi'(\mathbf{x})=\phi(\mathbf{x})+\text{const}, \qquad (15)$$

where the constant may depend upon the particular symmetry transformation. Thus instead of (14) the correct condition on ϕ is

$$\phi(r,\theta,z)=\phi(r,\ \theta+\alpha,\ z+a)+\text{const}, \qquad (16)$$

where the constant may depend upon α and a. From (16) we obtain

$$\phi(r,\theta,z)=F(r)+B\theta+Cz \qquad (9)$$

in agreement with our previous result.

The consequences of symmetry under the reflection $r'=r$, $\theta'=-\theta$, $z'=z$ are also immediate in terms of the potential ϕ. For an axial vector field, ϕ must be a pseudoscalar, so that

$$\phi(r,\ -\theta,\ z)=-\phi(r,\theta,z)+\text{const}. \qquad (17)$$

On applying this condition to (9) we see that $C=0$ and $F(r)=\text{const}$, and hence we re-obtain (11) and (12).

III. A CONJECTURE: DOES UNIQUENESS OCCUR ONLY FOR THE COULOMB LAW OF FORCE?

The remainder of this article consists of a discussion of the relationship between the inverse-square law and uniqueness of solution in electrostatics. Now there is a well-known uniqueness theorem in electrostatics which states that, given the *total*[2] charge on each conductor and given the conditions at infinity, then the electric field is uniquely determined—and hence, using $E_n = 4\pi\sigma$, so is the distribution of charge over each conductor.

Returning now to the problem of an isolated conducting sphere carrying a given total charge, it is only by appealing to this uniqueness theorem that we are able to deduce, by (C), that the charge density is spherically symmetric. Despite this, text books frequently treat the problem of a spherical conductor *before* a discussion of the uniqueness theorem. It seems to be fairly commonly held that there is, in fact, no real need for the uniqueness theorem—that at any rate in the above problem "uniqueness is obvious on physical grounds." This last assertion is one that I have never understood. On occasion, *faute de mieux*, it is reasonable to believe in the *existence* of a solution to a given mathematical problem "on physical grounds"—or, better, in the existence of a solution to some problem differing but "slightly" from the original. Also on occasion, when dealing with a physical system set up by a uniquely prescribed series of operations, it may be reasonable to assume that the future behavior of the system is uniquely determined (although in fact this is not the case if quantum effects are involved!). But the problem of the spherical conductor is not of this type, for one can charge a conductor in infinitely many different ways, and it seems to me to be remarkable, and not at all obvious, that the charge distribution is the same no matter how, and at what points, the given total charge is applied. Contrast the above electrostatic problem with, for example, that of the magnetic field produced by an isolated sphere of homogeneous ferromagnetic material. Here the particular previous circumstances are of crucial importance, and it would be rather a fluke if the magnetic field turned out to be spherically symmetric.[3]

[2] Unfortunately, several textbooks state a theorem only for the case when the charge density σ is known—in which case the conclusion of uniqueness is obvious but of no help in the solving of the usual electrostatic problems (met later in the same book!) where one has to find both the electric field *and* the distribution of charge on the conductors.

[3] For the axial vector magnetic field, such a field must in fact be zero.

The argument against the above belief, that in electrostatics uniqueness of solution is physically obvious, would be even more compelling if one could produce a simple counter-example where, under some law of force other than the Coulomb law, uniqueness does not hold. For I *suspect* that to assert that uniqueness is "obvious" in electrostatics, is tantamount to asserting that the inverse-square law is "obvious."

Certainly the inverse-square law is exceptional in giving rise to a second-order field equation (Laplace's equation), and it is on this fact that the usual proof of the uniqueness theorem is based. The only other law for which I know that a uniqueness theorem holds is that given by the potential $\phi = (1/r)\exp(-kr)$, where again there is a second-order field equation, in this case $\nabla^2\phi = k^2\phi$. Excluding this last case, it would seem that theoretical electrostatics would be an extremely difficult discipline if the law of force were other than the inverse-square law. For even if I restrict my attention to a "simple" $1/r^n$ law, with n differing slightly from 2, I have to admit to being unable to solve any problem whatsoever in "electrostatics," nor can I produce a counter-example against uniqueness (except in the case $n = -1, -3, -5, \cdots$ considered later).

In the case of an isolated charged conducting sphere, for example, I have not succeeded in finding any equilibrium distribution of charge; nor do I know if there is a unique equilibrium distribution, or several such distributions, or perhaps even none. If an equilibrium distribution exists that is not spherically symmetric, then by superposition one can construct an equilibrium distribution that *is* symmetric—perhaps it is only such symmetric distributions which are in stable equilibrium? Or perhaps, there are many stable configurations, not all of which are symmetric? $\cdots$ Almost the only outcome of my attempts to do "unorthodox" electrostatics has been an increased respect for the virtues of the inverse-square law!

The main point of this article is to pose the problems of (1) proving, or disproving, the conjecture of the previous paragraph, that for a $1/r^n$ law of force an electrostatic uniqueness theorem holds only in the case $n = 2$; (2) exhibiting a solution of any conductor problem whatsoever when the law of force is other than the inverse-square law.

IV. THE EQUILIBRIUM DISTRIBUTION OF CHARGE

Of course, for any law of force, one can apply the usual argument to conclude that the electric field must vanish throughout the substance of a conductor in equilibrium.[4] However, there is no longer any reason why the charge density should vanish inside a conductor. If one considers the field $\mathbf{E}$ at a point P on the axis of a uniform plane disk of charge when the force varies at $1/r^n$, then as P approaches the center of the disk one finds that

$$\begin{aligned} E &\to 0 \quad (n<2), \\ &\to \infty \quad (n>2), \end{aligned} \tag{18}$$

and that it is only in the $n=2$ case that a finite but nonzero limit occurs. It follows that if, initially, a conducting sphere has a (positive) uniform surface density of charge, but no volume density, the field at a point P just inside the surface is nonzero and radially outwards or inwards according as $n<2$ or $n>2$. Thus in neither case is there equilibrium—for $n<2$ conduction electrons flow inwards, for $n>2$, outwards. Assuming that a spherically symmetric equilibrium distribution *does* exist, then for $n<2$ a *positively* charged sphere must have a *negative* volume density charge present in the interior; while for $n>2$ it seems likely [in order to avoid the infinity of (18)] that there is no surface density but only a positive volume density. However the situation is somewhat obscured in the latter case for $n \geqslant 3$ because there are convergence difficulties in defining the field at a point inside a volume distribution of charge.

On the assumption of spherical symmetry one can find the condition that the charge density $\rho(r)$ must satisfy in order that equilibrium should result. One simply computes the potential $\phi(r)$ at a radial distance r and equates the result to a constant ϕ_0 for all $r<a$, where a is the radius of the sphere. It turns out that ρ must be a solution of a linear integral equation which, except in the cases $n=1$ and $n=3$, reads

$$(n-1)(3-n)\phi_0 r = 2\pi \int_0^a x\rho(x) \times [(x+r)^{3-n} - |x-r|^{3-n}] dx \, (0 \leqslant r \leqslant a), \tag{19}$$

(For $n \geqslant 4$ there are difficulties in defining the potential at a point inside a volume distribution of charge. Even in the case $n=2$ it does not seem possible to handle the problem of an *ideally* thin conducting rod. When such convergence difficulties occur it may be that one should no longer use the usual picture of continuous distribution of charge, but take into account the existence of elementary point charges. We may avoid such difficulties in the above problem by restricting our attention to the cases $n<3$).

In attempting to find solutions of (19) it seems at first sight that a knowledge of Riemann–Liouville fractional integral transforms[5] would be helpful but in fact I did not succeed in finding any solutions thereby. One can of course solve (19) when $n=2$, the solution being

$$\rho(x) = (\phi_0/4\pi a)\delta(x-a), \tag{20}$$

but the only other cases which I could handle were the cases $n=-1, -3, -5, \cdots$. In these cases it is easily proved, for a nonzero total charge, that no solution of (19) exists. In fact it is simpler in these cases to proceed directly, and without restricting one's considerations to the spherically symmetric distributions; a discussion of the case $n=-1$ follows.

V. THE LAW $\mathbf{E}=e\mathbf{r}$—A COUNTER-EXAMPLE TO MOST HYPOTHESES

For such a law the electric field due to a general set of point charges e_i at positions $\mathbf{r}_i$ is $\mathbf{E}(\mathbf{r}) = \sum_i e_i(\mathbf{r}-\mathbf{r}_i)$, and hence

$$\mathbf{E}(\mathbf{r}) = Q\mathbf{r} - \mathbf{P}, \tag{21}$$

where $Q=\sum_i e_i$ = total charge and $\mathbf{P}=\sum_i e_i\mathbf{r}_i$. (These formulas receive the obvious modifications for continuous distributions of charge.) It follows immediately from (21) that *no matter how a total nonzero charge Q is distributed throughout an isolated conductor (of arbitrary shape), the*

[4] Actually, even this could conceivably not be so, as is illustrated in the ensuing discussion of the case $n=-1$.

[5] See Chap. XIII of *Tables of Integral Transforms*, Bateman Manuscript Project, Vol. II (McGraw-Hill Book Company, New York, 1953).

interior of the conductor never is a region of zero electric field.

If, for example, a charge $Q>0$ is initially distributed uniformly over a spherical conductor, than $\mathbf{E}=Q\mathbf{r}$ is radially outwards so that conduction electrons flow towards the center. The end result is that *all* the conduction electrons coalesce at the center of the conductor, leaving a positive volume density at all other points. *The system is in equilibrium*[6] *despite the fact that the electric field, still given by* $\mathbf{E}=Q\mathbf{r}$, *is not zero throughout the conductor*! In the corresponding problem with $Q<0$, all the conduction electrons end up on the surface of the sphere.

The case of zero total charge is equally bizarre; Eq. (21) with $Q=0$ shows immediately that *any distribution of charge with zero dipole moment will be in equilibrium.* There is thus a huge infinity of different equilibrium distributions; in the case of the sphere, "most" of these are not spherically symmetric. *Moreover all the equilibrium distributions have the same energy.* The "electrostatics" arising from the law $\mathbf{E}=e\mathbf{r}$ is surely the *ne plus ultra* of *non*uniqueness![7]

This last assertion concerning the energy is easily proved. Taking the zero-point of the energy to be that of the configuration when all the charge is coalesced together at one point, then the energy of a general configuration is

$$W=-\tfrac{1}{2}\sum_i\sum_j e_ie_j(\mathbf{r}_i-\mathbf{r}_j)^2$$

$$=-Q\sum_i e_ir_i^2+\mathbf{P}^2. \qquad (22)$$

Thus $W=0$ for any charge distribution satisfying $Q=0$ and $\mathbf{P}=0$.

[6] The equilibrium is in fact stable, as can be seen from the expression (22) for the energy of the system.

[7] Note however that for all these equilibrium distributions of charge the electric field is uniquely $\mathbf{E}=0$ at all points of space.

VI. IS THE INVERSE-SQUARE LAW EXACT?

The law of force just discussed is of course highly unphysical[8] and the resulting "wild" electrostatics is of purely academic interest. However it would be of some interest if progress were made in the theory of electrostatics under a $1/r^n$ law with n differing only slightly from 2. For complete confidence in an exact $n=2$ law can only be based on a knowledge of the consequences of laws for which n differs from 2. The best evidence in favor of the inverse-square law appears to be that provided by the experiment of Plimpton and Lawton[9], with the estimate $|n-2|<2\times10^{-9}$. It is conceivable that one could improve upon even this estimate if one had more knowledge of how "wild" were the cases $n\neq2$. It is interesting to note that the Plimpton and Lawton estimate is based on the *assumption* that for $n\neq2$ the charge density on concentric conducting spherical shells is uniform (although they remark that *slight* deviations from uniformity would not appreciably affect the accuracy of their estimate). No doubt in the above case it is very plausible that at any rate the stable equilibrium configuration of charge *is* spherically symmetric—but it would be more satisfactory to be able to *prove* this. A further point in the Plimpton and Lawton experiment is that it seems difficult to allow for the effect of the presence of the detector inside the inner shell.

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It is a pleasure to thank Dr. W. J. Duffin for several interesting discussions.

[8] No doubt we have been very naive in supposing that solid bodies possessing conduction electrons would still exist in such a case.

[9] S. J. Plimpton and W. E. Lawton, Phys. Rev. **50**, 1066 (1936).