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On approximate formulas for the electrostatic force between two conducting spheres

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A series expression for the electrostatic force between two charged conducting spheres having equal radii and charges is derived using the method of electrical images. This expression is a special case of that for two spheres with arbitrary charges and radii, found by Maxwell using zonal harmonics. Keeping in mind the use of approximate formulas for the interpretation of classroom measurements of the electrostatic force between spheres, we comment on two incorrect approximate formulas and examine the contribution of the first few non-Coulomb terms of the correct formula by comparing with values obtained using a computational approach.

I. INTRODUCTION

Coulomb's law is demonstrated in classroom experiments in different ways, using either a kind of torsion balance[1] or some other tool for electrostatic force measurement.[2–4] Although there are still some doubts about the actual experimental determination of the inverse square law by Coulomb himself,[5] nowadays we see classroom measurements that have sufficient precision even to detect deviations from the simple inverse square relationship between force and distance[6–8] valid for point charges.

That such deviations, due to the redistribution of charge caused by the mutual electrostatic influence, must have occurred in Coulomb-like experiments was mentioned by Maxwell.[9] He also admitted that a quantitative account of this effect requires an ''intricate investigation,'' which, for the case of two spheres, was first carried out ''in extremely able manner'' by Poisson and later ''greatly simplified by Sir W. Thomson in his Theory of Electrical Images.'' A possible role of induction effects in Coulomb's original experiment was recently discussed by Soules[10] using numerical methods for the calculation of the force.

Many authors[11–13] give, as an illustration of the applicability of the method of image charges, a general formula for the force between two spheres which takes into account the effects of charge redistribution. Nevertheless, the derivation of an approximate formula, suitable to obtain the theoretical insight needed to deal with situations met in classroom measurements related to Coulomb's law, is commonly left as a homework exercise for interested readers. Maxwell's comment cited above and the examples given below show that this is not a trivial task.

In Sec. II we present such a formula, derived by the method of electrical images. Two incorrect approximate formulas, published in this journal, are discussed in Sec. III. In Sec. IV we consider the accuracy of the correct formula, keeping in mind its application in classroom measurements. Conclusions are given in Sec. V.

II. A CORRECT APPROXIMATE FORMULA FOR THE ELECTROSTATIC FORCE BETWEEN TWO SPHERES

Calculations for the general case of spheres with arbitrary radii and charges are very complicated. They become simpler if the spheres have equal radii and charges. Consider the case of two perfectly conducting spheres with radius a and charge $+q$ whose center-to-center distance is d.

We replace the spheres by two infinite sets of image charges, q_n and q_n', located on the line of centers between the spheres at positions, with respect to the center of one of the spheres, given by x_n and x_n', where $n=1,2,3,\ldots$. Since the spheres are identical in size and charge, we have[6]

$$q_n=q_n', \quad x_n'=d-x_n, \quad n=1,2,3,\ldots, \tag{1}$$

with

$$q_n=-\frac{a q_{n-1}}{d-x_{n-1}}, \quad n>1, \tag{2}$$

and

$$x_n=\frac{a^2}{d-x_{n-1}}, \quad n>1, \tag{3}$$

with $x_1=0$. Using these recurrence relations, we can find expressions for all the charges in terms of q_1. For example, the first five charges are given by

$$\begin{aligned} q_1&=q_1,\\ q_2&=-\beta q_1,\\ q_3&=\beta^2 q_1/(1-\beta^2),\\ q_4&=\beta^3 q_1/(-1+2\beta^2),\\ q_5&=\beta^4 q_1/(1-3\beta^2+\beta^4), \end{aligned} \tag{4}$$

and they are located, respectively, at

$$\begin{aligned} x_1&=0,\\ x_2&=a\beta,\\ x_3&=a\beta/(1-\beta^2),\\ x_4&=a\beta(1-\beta^2)/(1-2\beta^2),\\ x_5&=a\beta(1-2\beta^2)/(1-3\beta^2+\beta^4), \end{aligned} \tag{5}$$

where

$$\beta=a/d. \tag{6}$$

Notice that if we write q_n as a series in powers of β the leading term will be proportional to β^{n-1}. It may be seen from Eq. (4) that the total charge in each sphere,

$$q=q_1+q_2+q_3+\cdots, \tag{7}$$

is proportional to q_1, so that q_1 can be written as q times a series in powers of β.

The magnitude of the force between the spheres is given by the derivative of the potential energy $W=qq_1/(4\pi\epsilon_0 a)$ with respect to d. We substitute for q_1 its series expansion and take the derivative to find the following expression for the repulsive electrostatic force between two conducting spheres:

$$\begin{aligned} F=F_C(&1-4\beta^3-6\beta^5+14\beta^6-8\beta^7+54\beta^8-50\beta^9\\ &+154\beta^{10}-264\beta^{11}+494\beta^{12}-1092\beta^{13}+1830\beta^{14}\\ &-4192\beta^{15}+7140\beta^{16}-15894\beta^{17}+28234\beta^{18}\\ &-60320\beta^{19}+112056\beta^{20}-230032\beta^{21}+\cdots), \end{aligned} \tag{8}$$

where

$$F_C=q^2/(4\pi\epsilon_0 d^2). \tag{9}$$

This formula coincides with one which can be derived from a general formula found by Maxwell using zonal harmonics.[14] We give this series to order 21 because it is the same order as obtained from Maxwell's formula. The result can be extended to any number of terms using a program that can carry out symbolic calculations, like MATHEMATICA or MAPLE.

To compare approximate and ''exact'' numerical computations, notice two things about Eq. (8) as an approximation.

First, Eq. (8) was obtained using a finite number of charges. To find a correct expression to order N in β, it is necessary to take into account the contributions of $N+2$ image charges within each sphere. This is clear from Eq. (2), where it can be seen that the charge q_{N+3} contributes terms of order $N+2$ or higher in β in the series expansion of q_1. After taking the derivative to find the force, these terms will be of order $N+1$ or higher. These considerations can be important also when comparing Eq. (8) to other series expansions found by analytical approaches.

Second, Eq. (8) can be considered a series expression for the total force between two sets of 23 charges located at certain positions. For this case, we would obtain better results by expanding the series to higher order. This may be important when comparing with numerical approaches using a finite number of image charges and adding the forces between every pair of image charges (one charge of the pair within the first spherical surface and the other within the second). No series expansion is needed.

For distances that are large in comparison with the radius, the series expression for the force converges very quickly because $\beta\ll 1$. This is not the case when the spheres are very close. Then, as will be shown later, the required number of charges is very large and analytical approaches turn out to be very impractical. For small distances one must make a numerical calculation.

III. COMMENTS ON TWO INCORRECT APPROXIMATE FORMULAS

Larson and Goss,[6] using an amazingly simple measuring tool, obtained results which could not be fitted using Coulomb's law. Not knowing a formula which could be used for bringing together experiment and theory, they attempted to derive one. Using the method of images, Larson and Goss found the approximate formula:

$$F_{\mathrm{LG}}=F_C(1-4\beta^3+14\beta^6-\cdots). \tag{10}$$

Comparing it with Eq. (8), one can see that the second non-Coulomb term ($-6\beta^5$) is omitted. Nevertheless, taking only the first non-Coulomb term,[15] which corresponds to the induced dipole effect, Larson and Goss found reasonable agreement between their calculated and experimental values.

Soules,[10] who developed a simple computer program for a high precision numerical calculation of the force, also gave an approximate equation:

$$F_S=F_C(1-\beta^2-2\beta^4). \tag{11}$$

He views this equation as likely to represent data from any experiment done to measure the repulsive force between two spheres, but he does not give experimental data to support this. At least some data can be fitted better with the different simple equation

$$F_{\mathrm{LG}}'=F_C(1-4\beta^3), \tag{12}$$

which approximates more closely the values found numerically for the repulsion force[6] and includes correctly the lowest order term in β due to the induced dipole effect.

Ironically, although Soules used the method of electrical images as a basis for the numerical calculation of the force,

Table I. Comparison between numerical and analytical approaches.

d/a	N.C.	F num	F_1	F_2	F_3	F_{18}	F_S
10	5	0.009 959 54	0.009 960 00	0.009 959 40	0.009 959 54	0.009 959 54	0.009 898 00
9	6	0.012 277 0	0.012 277 9	0.012 276 7	0.012 277 0	0.012 277 0	0.012 189 5
8	6	0.015 500 9	0.015 502 9	0.015 500 1	0.015 500 9	0.015 500 9	0.015 373 2
7	7	0.020 165 3	0.020 170 2	0.020 162 9	0.020 165 3	0.020 165 3	0.019 974 7
6	7	0.027 250 3	0.027 263 4	0.027 241 9	0.027 250 3	0.027 250 3	0.026 963 3
5	8	0.038 679 9	0.038 720 0	0.038 643 2	0.038 679 0	0.038 679 9	0.038 272 0
4	10	0.058 456 5	0.058 593 8	0.058 227 5	0.058 441 2	0.058 456 5	0.058 105 5
3	15	0.094 437 0	0.094 650 2	0.091 906 7	0.094 040 6	0.094 436 0	0.096 022 0
2.5	21	0.121 091	0.119 040	0.109 210	0.118 385	0.121 018	0.126 208
2.4	23	0.127 089	0.123 376	0.110 294	0.123 013	0.126 896	0.133 005
2.3	27	0.133 313	0.126 889	0.109 267	0.127 144	0.132 778	0.139 791
2.2	33	0.139 794	0.128 996	0.104 942	0.130 454	0.138 236	0.146 283
2.1	46	0.146 582	0.128 817	0.095 503	0.132 518	0.141 778	0.152 019
2.01	135	0.152 994	0.125 597	0.080 331	0.132 879	0.139 083	0.155 924
2.001	392	0.153 652	0.125 062	0.078 351	0.132 820	0.138 141	0.156 219
2.000 1	1126	0.153 718	0.125 006	0.078 148	0.132 813	0.138 037	0.156 247
2.000 01	3194	0.153 725	0.125 001	0.078 127	0.132 813	0.138 026	0.156 250
2.000 001	9148	0.153 725	0.125 000	0.078 125	0.132 813	0.138 025	0.156 250

he guessed his approximate formula. At least for the distances treated experimentally, a better and simpler one could be derived with the method of images. This point will be discussed later in more detail.

IV. CONTRIBUTIONS OF THE FIRST NON-COULOMB TERMS

To judge the relative importance of non-Coulomb terms, keeping in mind the possible use of approximate formulas of the force for classroom demonstrations of Coulomb's law and induction effects, we will use results of computer calculations as a reference. To do this, we have written a computer program based on the approach of Soules[10] but we examine more closely certain details. The program calculates iteratively the force between two finite sets of image charges whose magnitudes and positions are found from Eqs. (1) to (3). To stop the iteration one may use two different criteria. In the first one, used by Soules, the program stops calculating when the additional charge is less than some small value ϵ times the total charge. In the second, the program adds image charges until the change in the magnitude of the force is less than ϵ times the previous value of the force. Both approaches give basically the same values for the force, the only difference being the number of charges needed (we take $\epsilon = 10^{-6}$)

More attention was paid to the calculation for distances very close to $2a$, a limit which Soules treated lightly. In his own words, when $d=2a$, ''more than 100 charges on each side are needed to compute the force.'' In fact, even when the spheres do not touch, the charges needed on each side may exceed several thousand.

We now turn to the question: How good are the approximate formulas when one employs different numbers of non-Coulomb terms in Eq. (8)? We can answer this question by comparing results from these formulas with those obtained by the computational approach. Such a comparison for approximate formulas with one, two, three and eighteen non-Coulomb terms, for some distances (measured in units of a), is given in Table I (the columns labeled F_1, F_2, F_3 and F_{18}, respectively). The forces are expressed in a special ''force unit'' equal to $F_U = q^2/(4\pi\epsilon_0 a^2)$. In those units, the Coulomb force becomes $F_C = (a/d)^2 = \beta^2$. The second column in Table I, labeled N.C., gives the number of image charges (on each side) used to reach the desired precision. The last column gives the values for the repulsive force found using the simple formula, Eq. (11), proposed by Soules.

For distances down to $2.4a$, the approximate formula with only one non-Coulomb term is simpler and more accurate than the approximate formula guessed by Soules. For distances between $2.4a$ and $2.1a$, to improve upon Soules's formula one must take into account all eighteen non-Coulomb terms. For shorter distances, Soules's formula is a better approximation. As far as we know, such short distances are not reported in classroom experiments on Coulomb's law.[6,7,16]

We thus suggest that to compare theory and experiment pertaining to charge redistribution, distances should not be less than $2.5a$. In that case a simple approximate formula with only one non-Coulomb term, the induced dipole term, is appropriate. A simple derivation of this term, which sheds some light on its physical interpretation, is given in the Appendix.

If shorter distances are considered, the series expression for the force converges very slowly and one has to add more and more terms to the formula. The formula is no longer simple and it may be seen that, even when the number of terms is increased drastically, the results obtained are rather poor (see the values for F_{18}). Keeping this in mind, the analytical result of Kelvin[17]

$$F = q^2 \frac{(\ln 2 - \frac{1}{4})}{6(\ln 2)^2} \tag{13}$$

for the limiting case of $d=2a$ (touching spheres), which gives, in the special force units mentioned before, a limiting value of 0.153 725 (although in Kelvin's Table 2 he used the value 0.153 726), is really an impressive achievement. One can see in Table I that we attain that value numerically for $d=2.000\ 01a$ with 3194 image charges in each side!

For attraction $(+q$ and $-q)$, the approximate formulas are obtained easily from those for repulsion by changing the sign of all the negative non-Coulomb terms and not, as Larson and Goss suggested,[6] by changing the signs of all the non-Coulomb terms. The approximation for attraction is worse than for the corresponding repulsion. For attraction, at a distance $d=2.5a$, even the three-term formula has an error of 8.6%, whereas for repulsion the corresponding one-term formula has an error of 1.7%. At shorter distances, the difference between the values found through approximate formulas and those calculated numerically increases dramatically. This behavior should be expected, since in a formal approach the force grows without any limit.[18] The number of image charges needed for a desired precision is less for attraction than for repulsion, because all image charge pairs increase the net force.

V. CONCLUSIONS

Thanks to computational methods it is now possible to calculate the values of the electrostatic force between two conducting spheres to any desired precision. Nevertheless, strings of numbers say something only to a knowledgeable eye. For students dealing for the first time with the applicability of the point charge model for electrostatic interaction between extended bodies, it might be better to use the expression

$$F=F_C f(\beta), \tag{14}$$

where the Coulomb and non-Coulomb parts are easily distinguished. Here, $f(\beta)$ is a function of the geometrical features of the situation (center-to-center distance, radii of the spheres) and describes the effects of charge redistribution.

We have found that for the repulsive electrostatic force between two conducting spheres for distances down to $2.5a$, the formula with only one non-Coulomb term, $f(\beta)=1-4\beta^3$, can bring together theory and experimental results for the majority of classroom measurements.[6,7] For introductory students, the form $f(\beta)=1-\alpha\beta^3$ could be justified considering the effect of the induced dipole (see the Appendix), with α to be determined from the experiment.

For shorter distances one has to add many more terms to the expression for $f(\beta)$ and the formula is no longer simple. In that case, the use of computational methods to calculate $f(\beta)$ is the only reasonable choice even if some of the physical insight is lost.

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APPENDIX: FORM OF THE FIRST NON-COULOMB TERM

A very simple way to find the form of the first non-Coulomb term in the expression for the force between two spheres is given below. The first non-Coulomb term in the expression for the force comes from the interaction between the total charge in one of the spheres and the induced dipole in the other and vice versa. To find the form of this term let us consider, therefore, a point charge q located at a distance d from the center of an uncharged conducting sphere of radius a. The point charge causes a redistribution of charge on the sphere which, in the first approximation, can be thought of as a dipole with charges $+q_{\rm ind}$ and $-q_{\rm ind}$ located at the opposite extremes of the diameter that goes along the line that joins the point charge and the center of the sphere. These induced charges are thus separated by a distance $2a$.

At the center of the sphere, the electric field strength of the point charge is $q/(4\pi\epsilon_0 d^2)$, while that due to the induced charges is $2q_{\rm ind}/(4\pi\epsilon_0 a^2)$. As the sphere is a conductor, these two fields must cancel. This happens if $q_{\rm ind}=-qa^2/(2d^2)$. Thus the sphere has a dipole moment $p=2aq_{\rm ind}=-qa^3/d^2$ that, if $d\gg a$, produces a field at the position of the point charge with intensity $E=2p/(4\pi\epsilon_0 d^3)=-2qa^3/(4\pi\epsilon_0 d^5)$ and thus a force

$$F=-\frac{2q^2a^3}{4\pi\epsilon_0 d^5}=-\frac{2q^2}{4\pi\epsilon_0 d^2}\left(\frac{a}{d}\right)^3=-2F_C\beta^3, \tag{A1}$$

which indeed suggests that the first correction to the Coulomb force goes as $F_C\beta^3$. The numerical coefficient in Eq. (A1) differs by a factor of 2 from that found in Eq. (8).

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