1 El efecto Cerenkov

Por las ecuaciones de Maxwell

$$\begin{aligned} \mathbf{E} &= -\nabla\phi \\ \mathbf{H} &= -\nabla\psi \end{aligned} \tag{1}$$

por lo tanto

$$\begin{aligned} \mathbf{E}^* = \mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{B} = -\nabla\phi + \frac{1}{c}\mathbf{V}\times\mathbf{B} \\ \mathbf{H}^* = \mathbf{H} - \frac{1}{c}\mathbf{V}\times\mathbf{D} = -\nabla\psi - \frac{1}{c}\mathbf{V}\times\mathbf{D} \end{aligned} \tag{2}$$

y las relaciones constitutivas dan

$$\begin{aligned} \left(1-\frac{V^2}{c^2}\right)\mathbf{D} &= \epsilon\left[\mathbf{E}^* - \frac{\mathbf{V}}{c^2}(\mathbf{V}\cdot\mathbf{E}^*)\right] - \frac{\mathbf{V}}{c}\times\mathbf{H}^* \\ \left(1-\frac{V^2}{c^2}\right)\mathbf{B} &= \mu\left[\mathbf{H}^* - \frac{\mathbf{V}}{c^2}(\mathbf{V}\cdot\mathbf{H}^*)\right] + \frac{\mathbf{V}}{c}\times\mathbf{E}^* \end{aligned} \tag{3}$$

Entonces, llamando

$$\gamma^2 = \frac{1}{(1-\frac{V^2}{c^2})} \tag{4}$$

$$\begin{aligned} \mathbf{E}^* = -\nabla\phi + \gamma^2\frac{\mathbf{V}}{c}\times\left[\mu\mathbf{H}^* + \frac{\mathbf{V}}{c}\times\mathbf{E}^*\right] \\ \mathbf{H}^* = -\nabla\psi - \gamma^2\frac{\mathbf{V}}{c}\times\left[\epsilon\mathbf{E}^* - \frac{\mathbf{V}}{c}\times\mathbf{H}^*\right] \end{aligned} \tag{5}$$

vemos que

$$\begin{aligned} \frac{\mathbf{V}}{c}\cdot\mathbf{E}^* &= -\frac{\mathbf{V}}{c}\cdot\nabla\phi \\ \frac{\mathbf{V}}{c}\times\frac{\mathbf{V}}{c}\times\mathbf{E}^* &= -\frac{\mathbf{V}}{c}\left(\frac{\mathbf{V}}{c}\cdot\nabla\phi\right) - \frac{V^2}{c^2}\mathbf{E}^* \end{aligned} \tag{6}$$

Entonces

$$\mathbf{E}^* = -\nabla\phi + \mu\gamma^2\frac{\mathbf{V}}{c}\times\mathbf{H}^* + \gamma^2\left[-\frac{\mathbf{V}}{c}\left(\frac{\mathbf{V}}{c}\cdot\nabla\phi\right) - \frac{V^2}{c^2}\mathbf{E}^*\right] \tag{7}$$

Ahora

$$1 + \gamma^2\frac{V^2}{c^2} = \gamma^2 \tag{8}$$

de modo que

$$\begin{aligned} \mathbf{E}^* &= -\left(1-\frac{V^2}{c^2}\right)\nabla\phi - \frac{\mathbf{V}}{c}\left(\frac{\mathbf{V}}{c}\cdot\nabla\phi\right) + \mu\frac{\mathbf{V}}{c}\times\mathbf{H}^* \\ &= -\nabla\phi - \frac{\mathbf{V}}{c}\times\left(\frac{\mathbf{V}}{c}\times\nabla\phi\right) + \mu\frac{\mathbf{V}}{c}\times\mathbf{H}^* \end{aligned} \tag{9}$$

Ahora

$$
\begin{aligned}
\mathbf{H}^* &= -\nabla\psi + \gamma^2 \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \mathbf{H}^*\right) - \epsilon\gamma^2 \frac{\mathbf{V}}{c} \times \left[-\nabla\phi - \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right) + \mu \frac{\mathbf{V}}{c} \times \mathbf{H}^*\right] \\
&= -\nabla\psi - (\epsilon\mu - 1)\, \gamma^2 \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \mathbf{H}^*\right) - \epsilon\gamma^2 \frac{\mathbf{V}}{c} \times \left[-\nabla\phi - \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right)\right] \\
&= -\nabla\psi + (\epsilon\mu - 1)\, \gamma^2 \frac{\mathbf{V}}{c} \left(\frac{\mathbf{V}}{c} \cdot \nabla\psi\right) + (\epsilon\mu - 1)\left(\gamma^2 - 1\right) \mathbf{H}^* - \epsilon\gamma^2 \frac{\mathbf{V}}{c} \times \left[-\nabla\phi - \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right)\right] \\
&= -\frac{\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \left[\nabla\psi - (\epsilon\mu - 1)\, \gamma^2 \frac{\mathbf{V}}{c} \left(\frac{\mathbf{V}}{c} \cdot \nabla\psi\right) + \epsilon\gamma^2 \frac{\mathbf{V}}{c} \times \left[-\nabla\phi - \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right)\right]\right] \\
&= -\frac{\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \left[\nabla\psi - (\epsilon\mu - 1)\, \gamma^2 \frac{\mathbf{V}}{c} \left(\frac{\mathbf{V}}{c} \cdot \nabla\psi\right) - \epsilon \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right)\right] \qquad (10)
\end{aligned}
$$

Volvemos a $\mathbf{E}^*$

$$
\begin{aligned}
\mathbf{E}^* &= -\nabla\phi - \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right) \\
&- \mu \frac{\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \frac{\mathbf{V}}{c} \times \left[\nabla\psi - (\epsilon\mu - 1)\, \gamma^2 \frac{\mathbf{V}}{c} \left(\frac{\mathbf{V}}{c} \cdot \nabla\psi\right) - \epsilon \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right)\right] \\
&= -\nabla\phi + \frac{(\epsilon\mu - 1)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right) - \frac{\mu\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \frac{\mathbf{V}}{c} \times \nabla\psi \qquad (11)
\end{aligned}
$$

Ahora

$$
\begin{aligned}
\left(1 - \frac{V^2}{c^2}\right) \mathbf{B} &= \mu \left[\mathbf{H}^* - \frac{\mathbf{V}}{c^2} \left(\mathbf{V} \cdot \mathbf{H}^*\right)\right] + \frac{\mathbf{V}}{c} \times \mathbf{E}^* \\
&= -\mu \frac{\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \left[\nabla\psi - (\epsilon\mu - 1)\, \gamma^2 \frac{\mathbf{V}}{c} \left(\frac{\mathbf{V}}{c} \cdot \nabla\psi\right) - \epsilon \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right)\right] \\
&+ \mu \frac{\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \frac{\mathbf{V}}{c^2} \left(\mathbf{V} \cdot \nabla\psi\right) \left[1 - (\epsilon\mu - 1)\, \gamma^2 \frac{V^2}{c^2}\right] \\
&+ \frac{\mathbf{V}}{c} \times \left[-\nabla\phi + \frac{(\epsilon\mu - 1)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right) - \frac{\mu\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \frac{\mathbf{V}}{c} \times \nabla\psi\right] \qquad (12)
\end{aligned}
$$

El último renglón da

$$
\begin{aligned}
& -\frac{\mathbf{V}}{c} \times \nabla\phi - \frac{(\epsilon\mu - 1)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \frac{V^2}{c^2} \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right) - \frac{\mu\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \nabla\psi\right) \\
= & -\left[1 + \frac{(\epsilon\mu - 1)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \frac{V^2}{c^2}\right] \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right) - \frac{\mu\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \nabla\psi\right) \qquad (13)
\end{aligned}
$$

Se ve que $\mathbf{B}$ depende de ϕ sólo a través de un término lineal en $\mathbf{V} \times \nabla\phi$. Ahora

$$
\nabla \cdot (\mathbf{V} \times \nabla\phi) = \partial_i \epsilon^{ijk} V_j \partial_k \phi = 0 \qquad (14)
$$

Por lo tanto, $\nabla \cdot \mathbf{B}$ es una ecuación homogénea en ψ, de donde deducimos que $\psi = 0$. Ahora

$$
\mathbf{E}^* = -\nabla\phi + \frac{(\epsilon\mu - 1)}{\left(1 - \epsilon\mu \frac{V^2}{c^2}\right)} \frac{\mathbf{V}}{c} \times \left(\frac{\mathbf{V}}{c} \times \nabla\phi\right)
$$

$$
\begin{aligned}
\mathbf{H}^* &= \frac{\epsilon\left(1-\frac{V^2}{c^2}\right)}{\left(1-\epsilon\mu\frac{V^2}{c^2}\right)}\left(\frac{\mathbf{V}}{c}\times\nabla\phi\right)\\
\left(1-\frac{V^2}{c^2}\right)\mathbf{B} &= -\mu\frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\epsilon\mu\frac{V^2}{c^2}\right)}\left[-\epsilon\left(\frac{\mathbf{V}}{c}\times\nabla\phi\right)\right]-\left[1+\frac{(\epsilon\mu-1)}{\left(1-\epsilon\mu\frac{V^2}{c^2}\right)}\frac{V^2}{c^2}\right]\left(\frac{\mathbf{V}}{c}\times\nabla\phi\right) \qquad (15)
\end{aligned}
$$

Finalmente

$$
\mathbf{B}=\frac{(\epsilon\mu-1)}{\left(1-\epsilon\mu\frac{V^2}{c^2}\right)}\frac{\mathbf{V}}{c}\times\nabla\phi \qquad (16)
$$

Ahora vamos por **D**

$$
\left(1-\frac{V^2}{c^2}\right)\mathbf{D} = \epsilon\left[-\nabla\phi+\frac{(\epsilon\mu-1)}{\left(1-\epsilon\mu\frac{V^2}{c^2}\right)}\frac{\mathbf{V}}{c}\times\left(\frac{\mathbf{V}}{c}\times\nabla\phi\right)+\frac{\mathbf{V}}{c^2}\left(\mathbf{V}\cdot\nabla\phi\right)\right]-\frac{\mathbf{V}}{c}\times\frac{\epsilon\left(1-\frac{V^2}{c^2}\right)}{\left(1-\epsilon\mu\frac{V^2}{c^2}\right)}\left(\frac{\mathbf{V}}{c}\times\nabla\phi\right) \qquad (17)
$$

Observamos que

$$
\frac{\mathbf{V}}{c^2}\left(\mathbf{V}\cdot\nabla\phi\right)=\frac{\mathbf{V}}{c}\times\left(\frac{\mathbf{V}}{c}\times\nabla\phi\right)+\frac{V^2}{c^2}\nabla\phi \qquad (18)
$$

y finalmente

$$
\mathbf{D}=\epsilon\left[-\nabla\phi+\frac{(\epsilon\mu-1)}{\left(1-\epsilon\mu\frac{V^2}{c^2}\right)}\frac{\mathbf{V}}{c}\times\left(\frac{\mathbf{V}}{c}\times\nabla\phi\right)\right] \qquad (19)
$$

Ahora

$$
\frac{\mathbf{V}}{c}\times\left(\frac{\mathbf{V}}{c}\times\nabla\phi\right)=\frac{\mathbf{V}}{c}\left(\frac{\mathbf{V}}{c}\cdot\nabla\phi\right)-\frac{V^2}{c^2}\nabla\phi \qquad (20)
$$

Por lo tanto

$$
\begin{aligned}
\mathbf{D} &= \epsilon\left[-\left(1+\frac{(\epsilon\mu-1)\frac{V^2}{c^2}}{\left(1-\epsilon\mu\frac{V^2}{c^2}\right)}\right)\nabla\phi+\frac{(\epsilon\mu-1)}{\left(1-\epsilon\mu\frac{V^2}{c^2}\right)}\frac{\mathbf{V}}{c}\left(\frac{\mathbf{V}}{c}\cdot\nabla\phi\right)\right]\\
&= \frac{-\epsilon}{\left(1-\epsilon\mu\frac{V^2}{c^2}\right)}\left[\left(1-\frac{V^2}{c^2}\right)\nabla\phi-(\epsilon\mu-1)\frac{\mathbf{V}}{c}\left(\frac{\mathbf{V}}{c}\cdot\nabla\phi\right)\right] \qquad (21)
\end{aligned}
$$

Si $\mathbf{V}=V\mathbf{I}$, entonces

$$
\mathbf{D}=-\epsilon\left\{\partial_x\phi\,\mathbf{I}-\frac{\left(1-\frac{V^2}{c^2}\right)}{\left(\epsilon\mu\frac{V^2}{c^2}-1\right)}\left[\partial_y\phi\,\mathbf{J}+\partial_z\phi\,\mathbf{K}\right]\right\} \qquad (22)
$$

y por lo tanto la Ley de Gauss

$$
\partial_x^2\phi-\eta^2\left[\partial_y^2\phi+\partial_z^2\phi\right]=-\frac{4\pi}{\epsilon}q\delta\left(\mathbf{x}\right) \qquad (23)
$$

$$
\eta^2=\frac{\left(1-\frac{V^2}{c^2}\right)}{\left(\epsilon\mu\frac{V^2}{c^2}-1\right)} \qquad (24)
$$

El caso que nos interesa es cuando $\eta^2>0$. Entonces

$$
\phi=-\frac{4\pi q}{\eta\epsilon}\int\frac{d^2k_\perp}{(2\pi)^2}\,e^{i\mathbf{k}_\perp\mathbf{x}_\perp}\,\frac{\sin\left(\eta k_\perp x\right)}{k_\perp}\theta\left(x\right) \qquad (25)
$$

$\mathbf{x}_\perp = (y,z)$. En coordenadas cilíndricas

$$\phi = -\frac{q}{\pi\eta\epsilon}\int dk_\perp\, d\varphi\, e^{ik_\perp r\cos\varphi}\sin(\eta k_\perp x)\theta(x) \tag{26}$$

Haciendo la integral en $k_\perp$

$$\phi = -\frac{q}{2\pi\eta\epsilon}\int_0^{2\pi} d\varphi\left[\frac{1}{r\cos\varphi+\eta x+i\epsilon}-\frac{1}{r\cos\varphi-\eta x+i\epsilon}\right]\theta(x) \tag{27}$$

Si $r<\eta x$, la integral es inmediata

$$\phi = -\frac{q}{\eta\epsilon}\frac{1}{\sqrt{\eta^2x^2-r^2}} \tag{28}$$

La integral se anula para $r>\eta x$ por "causalidad".

Por otro lado, el potencial vector

$$\mathbf{a} = \eta'^2\phi\frac{\mathbf{V}}{c} \tag{29}$$

$$\eta'^2 = \frac{(\epsilon\mu-1)}{\left(\epsilon\mu\frac{V^2}{c^2}-1\right)} \tag{30}$$

En el referencial del laboratorio,

$$\begin{aligned}\Phi &= \gamma\left(\phi-\frac{V}{c}a\right)\\ A_x &= \gamma\left(-\frac{V}{c}\phi+a\right)\\ \mathbf{A}_\perp &= 0\end{aligned} \tag{31}$$

donde el lado derecho es evaluado en

$$\begin{aligned}t &= \gamma\left(T+\frac{V}{c^2}X\right)\\ x &= \gamma(X+VT)\\ \mathbf{x}_\perp &= \mathbf{X}_\perp\end{aligned} \tag{32}$$

Entonces

$$\begin{aligned}\Phi &= \gamma\left(1-\eta'^2\frac{V^2}{c^2}\right)\phi = -\gamma\eta^2\phi\\ A_x &= \gamma\frac{V}{c}\left(\eta'^2-1\right)\phi = \gamma\epsilon\mu\frac{V}{c}\eta^2\phi\\ \mathbf{A}_\perp &= 0\end{aligned} \tag{33}$$

donde

$$\phi = -\frac{4\pi q}{\eta\epsilon}\int\frac{d^2k_\perp}{(2\pi)^2}\,e^{i\mathbf{k}_\perp\mathbf{X}_\perp}\,\frac{\sin(\gamma\eta k_\perp(X+VT))}{k_\perp}\theta(x) \tag{34}$$

Por lo tanto tenemos una superposición de ondas que se propagan en la dirección

$$\mathbf{K} = (-\gamma\eta k_\perp, \pm\mathbf{k}_\perp) \tag{35}$$

con

$$K = \sqrt{1+\gamma^2\eta^2}k_\perp = \sqrt{\frac{\epsilon\mu\frac{V^2}{c^2}}{\left(\epsilon\mu\frac{V^2}{c^2}-1\right)}}\; k_\perp \tag{36}$$

y frecuencia

$$\Omega = V\gamma\eta k_\perp = \frac{V}{\sqrt{\epsilon\mu\frac{V^2}{c^2}-1}} k_\perp \tag{37}$$

Por lo tanto, la velocidad de propagación es

$$C = \frac{\Omega}{K} = \frac{c}{\sqrt{\epsilon\mu}} \tag{38}$$

La dirección de propagación forma con el eje Y el mismo ángulo que el frente de onda forma con el eje X, es decir

$$\tan\theta = \frac{-K_x}{K_y} = \frac{1}{\sqrt{\epsilon\mu\frac{V^2}{c^2}-1}} \tag{39}$$

Entonces

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{c}{\sqrt{\epsilon\mu}V} \tag{40}$$

como esperábamos.

2 Arrastre parcial del éter

En 1818, Fresnel predijo que la velocidad de una onda en un medio de índice de refracción $n = \sqrt{\epsilon\mu}$ que a su vez se mueve con velocidad V es

$$\bar{c} = \frac{c}{n} + \left(1 - \frac{1}{n^2}\right) V \tag{41}$$

resultado que sería debido al "arrastre parcial" del éter por el medio en movimiento. Esto fue confirmado experimentalmente en 1851 por Fizeau. De hecho, el resultado de Fresnel se puede explicar simplemente si la velocidad de la onda respecto al medio es c/n, y sumando las dos velocidades a la manera relativista

$$\begin{aligned} \bar{c} &= \frac{V + \frac{c}{n}}{1 + \frac{V}{nc}} \\ &= \frac{c}{n}\frac{1 + \frac{nV}{c}}{1 + \frac{V}{nc}} \\ &\approx \frac{c}{n}\left[1 + \left(n - \frac{1}{n}\right)\frac{V}{c} + \dots\right] \end{aligned} \tag{42}$$

Es interesante deducir esta fórmula de la electrodinámica de medios en movimiento. Consideramos una onda de frecuencia ω y número de onda $\mathbf{k}$ paralelo a la velocidad $\mathbf{V}$ del medio. Las ecuaciones de Maxwell son

$$\begin{aligned} \mathbf{k}\cdot\mathbf{D} = \mathbf{k}\cdot\mathbf{B} &= 0 \\ \mathbf{k}\times\mathbf{E}^* - \frac{1}{c}\left(\omega - \mathbf{k}\cdot\mathbf{V}\right)\mathbf{B} &= 0 \\ \mathbf{k}\times\mathbf{H}^* + \frac{1}{c}\left(\omega - \mathbf{k}\cdot\mathbf{V}\right)\mathbf{D} &= 0 \end{aligned} \tag{43}$$

Las relaciones constitutivas son

$$\begin{aligned} \left(1-\frac{V^2}{c^2}\right)\mathbf{D} &= \epsilon\left[\mathbf{E}^*-\frac{\mathbf{V}}{c^2}\left(\mathbf{V}\cdot\mathbf{E}^*\right)\right]-\frac{\mathbf{V}}{c}\times\mathbf{H}^* \\ \left(1-\frac{V^2}{c^2}\right)\mathbf{B} &= \mu\left[\mathbf{H}^*-\frac{\mathbf{V}}{c^2}\left(\mathbf{V}\cdot\mathbf{H}^*\right)\right]+\frac{\mathbf{V}}{c}\times\mathbf{E}^* \end{aligned} \tag{44}$$

implican que $\mathbf{k}\cdot\mathbf{E}^*=\mathbf{k}\cdot\mathbf{H}^*=0$. Entonces

$$\mathbf{k}\times\left(\mathbf{k}\times\mathbf{E}^*\right)-\frac{\gamma^2}{c}\left(\omega-\mathbf{k}\cdot\mathbf{V}\right)\left[\mu\mathbf{k}\times\mathbf{H}^*-\mathbf{E}^*\left(\frac{\mathbf{k}\cdot\mathbf{V}}{c}\right)\right]=0 \tag{45}$$

Ahora

$$\mathbf{k}\times\mathbf{H}^*=-\frac{\gamma^2}{c}\left(\omega-\mathbf{k}\cdot\mathbf{V}\right)\left[\epsilon\mathbf{E}^*-\frac{\mathbf{V}}{c}\times\mathbf{H}^*\right] \tag{46}$$

y

$$\frac{\mathbf{V}}{c}\times\mathbf{H}^*=\frac{\mathbf{k}\cdot\mathbf{V}}{ck^2}\,\mathbf{k}\times\mathbf{H}^* \tag{47}$$

de modo que

$$\mathbf{k}\times\mathbf{H}^*=-\frac{\frac{\gamma^2}{c}\left(\omega-\mathbf{k}\cdot\mathbf{V}\right)\epsilon\mathbf{E}^*}{1-\frac{\gamma^2}{c^2k^2}\left(\omega-\mathbf{k}\cdot\mathbf{V}\right)\mathbf{k}\cdot\mathbf{V}} \tag{48}$$

Finalmente, la relación de dispersión es

$$-k^2\left(1-\frac{\gamma^2}{c^2k^2}\left(\omega-\mathbf{k}\cdot\mathbf{V}\right)\mathbf{k}\cdot\mathbf{V}\right)^2+\left(\frac{n\gamma^2}{c}\left(\omega-\mathbf{k}\cdot\mathbf{V}\right)\right)^2=0 \tag{49}$$

que efectivamente se reduce a

$$\omega=\bar{c}k \tag{50}$$