FÍSICA TEÓRICA 1 – 2do. Cuatrimestre de 2022

APUNTE DE FÓRMULAS PARA SEPARACIÓN DE VARIABLES

Coordenadas cartesianas

Ortogonalidad:

$$\int_0^a dx \sin(k_{n'}x)\sin(k_n x) = \frac{a}{2}\delta_{nn'}\,, \quad k_n = n\pi/a$$

$$\int_0^{+\infty} dx \sin(k'x)\sin(kx) = \frac{\pi}{2}\,\delta(k-k')\,, \quad k>0$$

$$\int_{-\infty}^{+\infty} dx \sin(k'x)\sin(kx) = \pi\,\delta(k-k')\,, \quad k>0$$

$$\int_{-\infty}^{+\infty} dx \cos(k'x)\cos(kx) = \pi\,\delta(k-k')\,, \quad k>0$$

$$\int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = 2\pi\delta(k-k')\,, \quad k\in\mathbb{R}$$

Coordenadas esféricas

Coordenadas y etiquetas: $\theta\in[0,\pi]$ $\varphi\in[0,2\pi]$ $l\in\mathbb{N}_0\,;-l\le m\le l\,, m\in\mathbb{Z}$

Armónicos esféricos: $Y_{lm}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}$

Funciones asociadas de Legendre: $P_l^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$ $(m\geqslant 0)$

Polinomios de Legendre: $P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$

Ortogonalidad: $(x=\cos\theta)$

$\int_0^{2\pi} d\varphi \int_0^{\pi}\sin(\theta)\,d\theta\, Y^*_{l'm'}(\theta,\varphi)Y_{lm}(\theta,\varphi)=\delta_{l'l}\delta_{m'm}$ $\int_{-1}^1 dx\, P_l(x)P_{l'}(x)=\frac{2}{2l+1}\delta_{l'l}$

$\int_{-1}^1 dx\, P_{l'}^m(x)P_l^m(x)=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l'l}$ $\int_0^{2\pi} d\varphi\, e^{im\varphi}e^{-im'\varphi}=2\pi\delta_{m'm}$

Completitud:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)=\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta')$$

Teorema de adición de armónicos esféricos: $[\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')\,]$

$$P_l(\cos\gamma)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

Propiedades:

$Y_{l,-m}(\theta,\varphi) = (-1)^m Y^*_{lm}(\theta,\varphi)$ $\qquad$ $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$

Recurrencia: $\frac{dP_{l+1}}{dx}(x) - \frac{dP_{l-1}}{dx}(x) = (2l+1)P_l(x)$ $\qquad$ $P_l(-x) = (-1)^l P_l(x)$

$$P_l(0) = \begin{cases} 0 & \text{si} \quad l \quad \text{es impar} \\ \frac{(-1)^{l/2}(l-1)!!}{2^{l/2}\left(\frac{l}{2}\right)!} & \text{si} \quad l \quad \text{es par} \end{cases} \qquad P_l(1) = 1$$

Primeros polinomios de Legendre:

$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x) \qquad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

Primeros armónicos esféricos:

$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$

$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\, cos\theta \qquad Y_{11}(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{i\varphi}$

$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{4\pi}}\, \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \qquad Y_{21}(\theta,\varphi) = -\sqrt{\frac{15}{8\pi}}\, \sin\theta\, \cos\theta\, e^{i\varphi} \qquad Y_{22}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\, \sin^2\theta\, e^{2i\varphi}$

Coordenadas cilíndricas

Coordenadas y etiquetas: $\qquad \varphi \in [0, 2\pi] \quad \nu \in \mathbb{N}_0.$

De la separación de variables: $\Phi \sim Q(\varphi)Z(z)R(\rho)$

$$\begin{aligned} Q''(\varphi) &= -\beta Q(\varphi) \\ Z''(z) &= \lambda Z(z) \\ R''(\rho) + R'(\rho)/\rho &= \left(\beta/\rho^2 - \lambda\right) R(\rho) \end{aligned} \quad .$$

λ	β	$Q(\varphi)$	$Z(z)$	$R(\rho)$
k^2	0	$1, \varphi$	$e^{\pm kz}$	$J_\nu(k\rho), N_\nu(k\rho)$
	ν^2	$e^{\pm i\nu\varphi}$		
0	0	$1, \varphi$	$1, z$	$1, \ln(\rho)$
	ν^2	$e^{\pm i\nu\varphi}$		$\rho^{\pm\nu}$
$-k^2$	0	$1, \varphi$	$e^{\pm ikz}$	$I_\nu(k\rho), K_\nu(k\rho)$
	ν^2	$e^{\pm i\nu\varphi}$		

$(k > 0)$

Para cada ν, las funciones $\{J_\nu(k\rho), N_\nu(k\rho)\}$ son independientes y forman base si:

intervalo finito: $\quad \rho \in [0; a] \;\rightarrow\; k \equiv k_{\nu n} = \chi_{\nu n}/a$

intervalo infinito: $\quad \rho \in [0; \infty) \rightarrow\; k \in [0; \infty)$

con $\{\chi_{\nu n}\}_{n\in\mathbb{N}}$ las infinitas raíces de $J_\nu(x)$.

Ortogonalidad de las funciones de Bessel J_ν:

$$\int_0^a d\rho\, \rho\, J_\nu(x_{\nu n'}\rho/a) J_\nu(x_{\nu n}\rho/a) = \frac{a^2}{2}\left[J_{|\nu|+1}(x_{\nu n})\right]^2 \delta_{nn'}$$

$$\int_0^\infty d\rho\,\rho\,J_\nu(k\rho)J_\nu(k'\rho)=\frac{\delta(k-k')}{k}$$

Propiedades: $(x>0)$

$$J_0(0)=1\,,\quad J_{\nu\,\neq 0}(0)=0\,,\quad x\ll 1:\quad J_\nu(x)\sim\tfrac{1}{\Gamma(\nu+1)}\left(\tfrac{x}{2}\right)^\nu\quad x\gg\nu^2:\quad J_\nu(x)\approx\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)$$

$$N_\nu(x\to 0)\to\infty\qquad \nu\in\mathbb{N}:\quad J_{-\nu}(x)=(-1)^\nu J_\nu(x)\,,\qquad \textstyle\int dx\,x^\nu J_{\nu-1}(x)=x^\nu J_\nu(x)$$

Fórmulas de recurrencia:

$$\Omega_{\nu-1}(x)+\Omega_{\nu+1}(x)=\frac{2\nu}{x}\Omega_\nu(x)$$

$$\Omega_{\nu-1}(x)-\Omega_{\nu+1}(x)=2\frac{d\Omega_\nu}{dx}(x)$$

Donde Ω_ν representa a cualquiera de las funciones J_ν, N_ν, $H_\nu^{(1)}=J_\nu+iN_\nu$ y $H_\nu^{(2)}=J_\nu-iN_\nu$.

Para las modificadas:

$$I_0(0)=1\,,\ I_{\nu>0}(0)=0\,,\ I_\nu(x\to\infty)\to\infty\qquad K_\nu(x\to 0)\to\infty\,,\ K_\nu(x\to\infty)=0$$

$$I_{-\nu}=I_\nu\,,\qquad K_{-\nu}=K_\nu$$

Wronskiano: $W[K_\nu,I_\nu](x)\equiv K_\nu(x)I'_\nu(x)-K'_\nu(x)I_\nu(x)=\frac{1}{x}$

Sobre la delta de Dirac $\delta(\rho-\rho')$ en cilíndricas, notar que si $\rho'=0$, vale:

$$\int_0^{\epsilon>0}d\rho\,\delta(\rho)=1$$

Funciones trigonométricas: definiciones y propiedades

$$\tan(x)=\tfrac{\sin(x)}{\cos(x)}\qquad \sec(x)=\tfrac{1}{\cos(x)}\qquad \coth(x)=\tfrac{\cos(x)}{\sin(x)}\qquad \text{cosec}(x)=\tfrac{1}{\sin(x)}$$

$$\sin^2(x)+\cos^2(x)=1\qquad \sin^2(x)=\tfrac{1}{2}(1-\cos(2x))\qquad \cos^2(x)=\tfrac{1}{2}(1+\cos(2x))$$

$$2\sin(x)\sin(y)=\cos(x-y)-\cos(x+y)\qquad \sin(x\pm y)=\sin(x)\cos(y)\pm\sin(y)\cos(x)$$

$$2\cos(x)\cos(y)=\cos(x-y)+\cos(x+y)\qquad \cos(x\pm y)=\cos(x)\cos(y)\mp\sin(x)\sin(y)$$

$$2\sin(x)\sin(y)=\sin(x-y)+\sin(x+y)\qquad \tan(x\pm y)=\tfrac{\tan(x)\pm\tan(y)}{1\mp\tan(x)\tan(y)}$$

Funciones hiperbólicas: definiciones y propiedades

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\text{cosech}(x) = \frac{1}{\sinh(x)} \qquad \text{sech}(x) = \frac{1}{\cosh(x)} \qquad \text{cotanh}(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad \sinh(x \pm y) = \sinh(x)\cosh(y) \pm \sinh(y)\cosh(x)$$

$$\tanh(x \pm y) = \frac{\tanh(x) \pm \tanh(y)}{1 \pm \tanh(x)\tanh(y)} \qquad \cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$