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Radiation of the electromagnetic field beyond the dipole approximation

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An expression for the intensity of electromagnetic radiation is derived up to the order after the dipole approximation. Our approach is based on the fundamental equations taught in an introductory course in classical electrodynamics, and the derivation is carried out using straightforward mathematical transformations.
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I. INTRODUCTION

Multipole expansion is a well-known technique in calculations of electromagnetic[1,2] and gravitational fields[3,4] at large distances from sources. In stationary cases, such as electrostatic problems, this approach is mainly linked to simple series expansions and certain symmetrization procedures.[5,6] Treatment of magnetostatic problems requires more elaborate techniques beyond the lowest order.[7] It becomes even trickier in the case of the electromagnetic radiation field. Moreover, in textbooks on classical electrodynamics one of the terms beyond the dipole approximation is typically omitted in expressions for the radiated power even if the detailed derivations are given (see, e.g., Refs. 8–10; for an exception see Ref. 11).

The goal of this article is to derive an expression for the radiated power in the approximation an order beyond the dipole approximation, using the basic equations of electrodynamics and the techniques from vector and tensor calculus. From the methodological point of view, our approach is advantageous compared to the derivations based on gauge symmetries[12] or on solutions to the scattering problem.[13] Indeed, our derivation only requires knowledge of the fundamental relations from an introductory course in classical electrodynamics and involves straightforward mathematical transformations. In addition, the simplicity of our approach allows one to obtain a correction to the dipole radiation sufficient for any practical purposes; more general derivations might be found in Refs. 7 and 14–17.

The rest of the paper is organized as follows. An expression for the radiated power via the Poynting vector and magnetic field is obtained in Sec. II. The multipole expansion of the radiative part of the vector potential is given in Sec. III. The detailed derivations of each contribution to the radiated power are presented in Sec. IV. A brief discussion of the results is given in Sec. V.

II. POYNTING VECTOR AND RADIATED POWER

In a region of space without charges or currents, the energy conservation law for the electromagnetic field in differential form is

$$\frac{\partial w}{\partial t} + \operatorname{div}\mathbf{S} = 0, \tag{1}$$

where w is the energy density (we use Gaussian units)

$$w = \frac{E^2 + B^2}{8\pi}, \tag{2}$$

and $\mathbf{S}$ is the energy flux density (the Poynting vector)

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}. \tag{3}$$

Applying Gauss's theorem to Eq. (1), the radiated power

$$I = -\frac{dW}{dt} \equiv \frac{d}{dt}\int w\,dV$$

can be reduced to the surface integral of the Poynting vector

$$I = \oint_\Sigma \mathbf{S}\cdot d\boldsymbol{\Sigma} = \int_{\Omega=4\pi} |\mathbf{S}| r^2\, d\Omega, \tag{4}$$

where on the r.h.s. the integration is performed over the complete solid angle. From this expression, one can conclude that only those fields contribute to the radiation, which ensure the dependence $|\mathbf{S}| \propto 1/r^2$ as $r\to\infty$.

In the far zone, the magnetic and electric fields can be written as the sums

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \tag{5}$$

where the second terms ($\mathbf{E}_1$ and $\mathbf{B}_1$) are proportional to $1/r$ and thus correspond to the radiation part of the field. Since the electric ($\mathbf{E}_1$) and magnetic ($\mathbf{B}_1$) field vectors are orthogonal and equal by magnitude, it is sufficient to consider the norm of the Poynting vector for the radiation fields

$$|\mathbf{S}_1| = \frac{c}{4\pi}|\mathbf{E}_1\times\mathbf{B}_1| = \frac{c}{4\pi}|\mathbf{B}_1|^2. \tag{6}$$

Then the radiated power is given by

$$I = \frac{c}{4\pi}\int_{\Omega=4\pi} |\mathbf{B}_1|^2 r^2\, d\Omega. \tag{7}$$

The magnetic field is

$$\mathbf{B}(\mathbf{r},t) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r},t), \tag{8}$$

where $\mathbf{A}(\mathbf{r},t)$ is the vector potential, which we calculate in Sec. III.

III. VECTOR POTENTIAL

We consider a system of electric charges in a volume V near the origin and assume that the observation point $\mathbf{r}$ is far enough that: $|\mathbf{r}| \equiv r \gg V^{1/3}$, see Fig. 1.

The vector potential of such a system in the Lorenz gauge is given by (cf. Ref. 8, p. 408)

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{c}\int_V dV'\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\mathbf{j}\left(\mathbf{r}',t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right), \tag{9}$$

where $dV'\equiv dx'\,dy'\,dz'$, and where retardation effects have been taken into account.

We write for $r'\ll r$ the following approximations over r'/r:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}\simeq\frac{1}{r}, \tag{10}$$

$$|\mathbf{r}-\mathbf{r}'|\simeq r-\mathbf{r}'\cdot\nabla r=r-\frac{\mathbf{r}'\cdot\mathbf{r}}{r}, \tag{11}$$

or, using the unit vector $\mathbf{n}=\mathbf{r}/r$,

$$|\mathbf{r}-\mathbf{r}'|\simeq r-\mathbf{n}\cdot\mathbf{r}'. \tag{12}$$

Further terms bringing higher powers of r in the denominators can be neglected when considering the radiation part of the field.

Within this approximation, the vector potential yields

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{cr}\int_V dV'\,\mathbf{j}\left(\mathbf{r}',t-\frac{r}{c}+\frac{1}{c}\mathbf{n}\cdot\mathbf{r}'\right). \tag{13}$$

This can be expanded as a Taylor series in $(\mathbf{n}\cdot\mathbf{r}')/c$ as follows:

$$\begin{aligned}\mathbf{A}(\mathbf{r},t)=&\frac{1}{cr}\int_V dV'\,\mathbf{j}\left(\mathbf{r}',t-\frac{r}{c}\right)\\&+\frac{d}{dt}\frac{1}{c^2r}\int_V dV'\,(\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}\left(\mathbf{r}',t-\frac{r}{c}\right)\\&+\frac{d^2}{dt^2}\frac{1}{2c^3r}\int_V dV'\,(\mathbf{n}\cdot\mathbf{r}')^2\,\mathbf{j}\left(\mathbf{r}',t-\frac{r}{c}\right)+\cdots.\end{aligned} \tag{14}$$

The current density for a system of point changes e_i at $\mathbf{r}_i$ moving with velocities $\mathbf{v}_i=\dot{\mathbf{r}}_i$ can be written using Dirac delta-functions as follows:

$$\mathbf{j}(\mathbf{r},\tau)=\sum_i e_i\mathbf{v}_i(\tau)\delta(\mathbf{r}-\mathbf{r}_i(\tau)), \tag{15}$$

where retarded time $\tau=t-r/c$ was introduced for convenience.

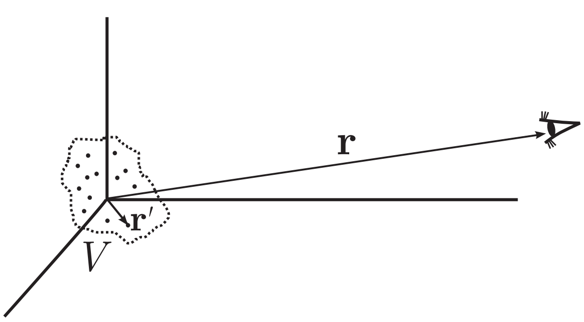

Fig. 1. The system of charges is located near the origin and a distant observer sits at point $\mathbf{r}$.

The first term in expansion (14) easily simplifies to the time derivative of the dipole moment

$$\begin{aligned}\int_V dV'\,\mathbf{j}\left(\mathbf{r}',t-\frac{r}{c}\right)&=\sum_i e_i\mathbf{v}_i(\tau)=\frac{d}{d\tau}\sum_i e_i\mathbf{r}_i(\tau)\\&=\frac{d}{d\tau}\mathbf{d}(\tau)=\dot{\mathbf{d}}(\tau).\end{aligned} \tag{16}$$

Thus, we have in the leading order

$$\mathbf{A}(\mathbf{r},t)=\frac{\dot{\mathbf{d}}\left(t-\frac{r}{c}\right)}{cr}+\cdots. \tag{17}$$

To obtain the next-order correction, we make the following transformations:

$$\begin{aligned}\int_V dV'(\mathbf{n}\cdot\mathbf{r}')\mathbf{j}\left(\mathbf{r}',t-\frac{r}{c}\right)&=\sum_i e_i\mathbf{v}_i(\tau)\int_V dV'\mathbf{n}\cdot\mathbf{r}'\delta\left(\mathbf{r}'-\mathbf{r}_i(\tau)\right)\\&=\sum_i e_i\mathbf{v}_i(\mathbf{n}\cdot\mathbf{r}_i)\\&=\sum_i e_i\frac{d\mathbf{r}_i}{d\tau}(\mathbf{n}\cdot\mathbf{r}_i)\\&=\frac{1}{2}\frac{d}{d\tau}\sum_i e_i\mathbf{r}_i(\mathbf{n}\cdot\mathbf{r}_i)\\&\quad+\frac{1}{2}\sum_i e_i\{\mathbf{v}_i(\mathbf{n}\cdot\mathbf{r}_i)-\mathbf{r}_i(\mathbf{n}\cdot\mathbf{v}_i)\}\\&=\frac{1}{2}\frac{d}{d\tau}\sum_i e_i\mathbf{r}_i(\mathbf{n}\cdot\mathbf{r}_i)\\&\quad+\frac{1}{2}\sum_i e_i\mathbf{n}\times(\mathbf{v}_i\times\mathbf{r}_i).\end{aligned} \tag{18}$$

As we will see below, in the radiation parts of the field, the vector potential appears only as cross product with the unit vector $\mathbf{n}$. Thus, one can add to $\mathbf{A}$ an arbitrary vector proportional to $\mathbf{n}$ without changing the results. It is convenient to add to the first term of (18)

$$\frac{1}{2}\frac{d}{d\tau}\sum_i e_i\mathbf{r}_i(\mathbf{n}\cdot\mathbf{r}_i)\to\frac{1}{2}\frac{d}{d\tau}\sum_i e_i\left\{\mathbf{r}_i(\mathbf{n}\cdot\mathbf{r}_i)-\mathbf{n}\frac{r_i^2}{3}\right\}.$$

We will work with the vector potential $\mathbf{A}$ shifted as described above. The sum over i is a contraction of the electric quadrupole moment tensor

$$Q_{jk}=\sum_i e_i\left\{x_j^{(i)}x_k^{(i)}-\frac{r_i^2}{3}\delta_{jk}\right\}$$

with the unit vector $\mathbf{n}$, yielding some vector $\boldsymbol{\mathcal{D}}$ with components

$$\mathcal{D}_j=\sum_k Q_{jk}n_k. \tag{19}$$

The remaining part of this correction involves the cross product of the magnetic dipole moment

$$\mathfrak{m} = \frac{1}{2c}\sum_i e_i\,\mathbf{r}_i \times \mathbf{v}_i \tag{20}$$

with the unit vector $\mathbf{n}$.

The last term, which is written explicitly in Eq. (14), can be transformed as follows:

$$\begin{aligned}
&\frac{d^2}{dt^2}\frac{1}{2c^3 r}\int_V dV'\,(\mathbf{n}\cdot\mathbf{r}')^2\,\mathbf{j}\left(\mathbf{r}', t-\frac{r}{c}\right)\\
&= \frac{1}{2c^3 r}\frac{d^2}{dt^2}\int_V dV'\,\mathbf{e}_i j_i\left(\mathbf{r}', t-\frac{r}{c}\right) n_k x'_k n_l x'_l\\
&= \frac{1}{2c^2 r}\ddot{\mathcal{M}}_{ikl}\left(\mathbf{r}', t-\frac{r}{c}\right) n_k n_l \mathbf{e}_i,
\end{aligned} \tag{21}$$

where the $\mathbf{e}_i$ are the unit vectors of the Cartesian coordinate system and the summation over repeating indices is implied. The third-rank current quadrupole tensor is defined as

$$\mathcal{M}_{ikl} = \frac{1}{c}\int_V dV'\, j_i x'_k x'_l. \tag{22}$$

Note that we do not attempt to make this tensor traceless and just retain the main $x'_k x'_l$ term, which is sufficient for the purposes of further derivations.

Collecting all contributions in Eq. (14), we arrive at the vector potential in the following form:

$$\mathbf{A}(\mathbf{r},t) = \frac{\dot{\mathbf{d}}}{cr} + \frac{\dot{\mathfrak{m}}\times\mathbf{n}}{cr} + \frac{1}{2c^2 r}\ddot{\mathcal{D}} + \frac{1}{2c^2 r}\ddot{\mathcal{M}}_{ikl} n_k n_l \mathbf{e}_i, \tag{23}$$

where all the quantities are evaluated at the retarded time $\tau = t - r/c$. The resulting expression might be compared, e.g., to the radiation part of the vector potential in Ref. 18.

IV. BRINGING IT ALL TOGETHER

The next step is to calculate the magnetic field $\mathbf{B}$ using expression (23) for the vector potential. We have

$$\begin{aligned}
\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times\Bigg[&\frac{1}{cr}\dot{\mathbf{d}}\left(t-\frac{r}{c}\right) + \frac{1}{cr}\dot{\mathfrak{m}}\left(t-\frac{r}{c}\right)\\
&\times\mathbf{n} + \frac{1}{2c^2 r}\ddot{\mathcal{D}}\left(t-\frac{r}{c}\right) + \frac{1}{2c^2 r}\ddot{\mathcal{M}}_{ikl}\left(t-\frac{r}{c}\right) n_k n_l \mathbf{e}_i\Bigg],
\end{aligned}$$

where, for convenience, we have written the time argument explicitly.

In order to retain the radiation part $\mathbf{B}_1 \propto 1/r$, it is sufficient to keep solely the terms where the nabla operator acts only on the time argument

$$\begin{aligned}
\nabla\times\mathbf{f}\left(t-\frac{r}{c}\right) &= \nabla\left(t-\frac{r}{c}\right)\times\dot{\mathbf{f}}\left(t-\frac{r}{c}\right)\\
&= \frac{1}{c}\,\dot{\mathbf{f}}\left(t-\frac{r}{c}\right)\times\mathbf{n}.
\end{aligned}$$

This yields

$$\begin{aligned}
\mathbf{B}_1 &= \frac{\ddot{\mathbf{d}}\times\mathbf{n}}{c^2 r} + \frac{(\ddot{\mathfrak{m}}\times\mathbf{n})\times\mathbf{n}}{c^2 r} + \frac{\dddot{\mathcal{D}}\times\mathbf{n}}{2c^3 r}\\
&\quad + \frac{1}{2c^3 r}\dddot{\mathcal{M}}_{ikl} n_k n_l\,\varepsilon_{irs} n_r \mathbf{e}_s.
\end{aligned} \tag{24}$$

The four terms in Eq. (24) correspond to the electric dipole, magnetic dipole, electric quadrupole, and current quadrupole term, respectively. Explicitly this reads

$$\mathbf{B}_1 = \mathbf{B}_\mathrm{d} + \mathbf{B}_\mathrm{m} + \mathbf{B}_\mathrm{Q} + \mathbf{B}_\mathcal{M}, \tag{25}$$

where (note the $1/c$ in the definitions of the magnetic moments!)

$$\mathbf{B}_\mathrm{d} = \frac{\ddot{\mathbf{d}}\times\mathbf{n}}{c^2 r} \propto \frac{1}{c^2}, \tag{26}$$

$$\mathbf{B}_\mathrm{m} = \frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathfrak{m}})}{c^2 r} \propto \frac{1}{c^3}, \tag{27}$$

$$\mathbf{B}_\mathrm{Q} = \frac{\dddot{\mathcal{D}}\times\mathbf{n}}{2c^3 r} \propto \frac{1}{c^3}, \tag{28}$$

$$\mathbf{B}_\mathcal{M} = \frac{1}{2c^3 r}\dddot{\mathcal{M}}_{ikl} n_k n_l\,\varepsilon_{irs} n_r \mathbf{e}_s \propto \frac{1}{c^4}. \tag{29}$$

Thus, the magnitude of the Poynting vector contains the following terms:

$$\begin{aligned}
|\mathbf{S}_1| = \frac{1}{4\pi}\Bigg(&\underbrace{c|\mathbf{B}_\mathrm{d}|^2}_{\propto 1/c^3} + \underbrace{2c\,\mathbf{B}_\mathrm{d}\cdot\mathbf{B}_\mathrm{m}}_{\propto 1/c^4} + \underbrace{2c\,\mathbf{B}_\mathrm{d}\cdot\mathbf{B}_\mathrm{Q}}_{\propto 1/c^4}\\
&+ \underbrace{c|\mathbf{B}_\mathrm{m}|^2}_{\propto 1/c^5} + \underbrace{c|\mathbf{B}_\mathrm{Q}|^2}_{\propto 1/c^5} + \underbrace{2c\,\mathbf{B}_\mathrm{m}\cdot\mathbf{B}_\mathrm{Q}}_{\propto 1/c^5} + \underbrace{2c\,\mathbf{B}_\mathrm{d}\cdot\mathbf{B}_\mathcal{M}}_{\propto 1/c^5}\Bigg)\\
&+ \mathcal{O}(c^{-6}).
\end{aligned} \tag{30}$$

A. Electric and magnetic dipole radiation

The leading contribution to the radiated power is given by the electric dipole term. It reads

$$I_\mathrm{d} = \frac{c}{4\pi}\int_{\Omega=4\pi}|\mathbf{B}_\mathrm{d}|^2 r^2\,d\Omega = \frac{1}{4\pi c^3}\int_{\Omega=4\pi}|\ddot{\mathbf{d}}\times\mathbf{n}|^2\,d\Omega. \tag{31}$$

Assuming that $\ddot{\mathbf{d}}$ is directed along the Oz axis and that the θ angle of the spherical coordinate system is that between $\ddot{\mathbf{d}}$ and $\mathbf{n}$, we obtain

$$\begin{aligned}
I_\mathrm{d} &= \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\int_{\Omega=4\pi}\sin^2\theta\,d\Omega = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin^3\theta\\
&= \frac{2\ddot{\mathbf{d}}^2}{3c^3} \propto \frac{1}{c^3}.
\end{aligned} \tag{32}$$

In a similar fashion we can calculate the magnetic dipole contribution

$$I_{\mathrm{m}} = \frac{c}{4\pi}\int_{\Omega=4\pi} |\mathbf{B}_{\mathrm{m}}|^2 r^2\, d\Omega = \frac{1}{4\pi c^3}\int_{\Omega=4\pi} |(\ddot{\mathfrak{m}}\times\mathbf{n})\times\mathbf{n}|^2\, d\Omega \tag{33}$$

considering θ as the angle between $\ddot{\mathfrak{m}}$ and $\mathbf{n}$. We thus see that

$$I_{\mathrm{m}} = \frac{2\ddot{\mathfrak{m}}^2}{3c^3} \propto \frac{1}{c^5}. \tag{34}$$

B. Terms with zero contributions

The product of the electric dipole and magnetic dipole terms $(\mathbf{B}_{\mathrm{d}}\cdot\mathbf{B}_{\mathrm{m}})$ contains the scalar product

$$(\ddot{\mathbf{d}}\times\mathbf{n})\cdot[(\ddot{\mathfrak{m}}\times\mathbf{n})\times\mathbf{n}] = (\ddot{\mathbf{d}}\times\mathbf{n})\cdot[\mathbf{n}(\mathbf{n}\cdot\ddot{\mathfrak{m}})-\ddot{\mathfrak{m}}]$$
$$= (\ddot{\mathbf{d}}\cdot\underbrace{(\mathbf{n}\times\mathbf{n})}_{=0})(\mathbf{n}\cdot\ddot{\mathfrak{m}}) - (\ddot{\mathbf{d}}\times\mathbf{n})\cdot\ddot{\mathfrak{m}} = \mathbf{n}\cdot(\ddot{\mathbf{d}}\times\ddot{\mathfrak{m}}).$$

Integration of the respective term in Eq. (4) using spherical coordinates with θ corresponding to the angle between $(\ddot{\mathbf{d}}\times\ddot{\mathfrak{m}})$ and $\mathbf{n}$ yields zero due to

$$\int_0^{\pi} \sin\theta\cos\theta\, d\theta = 0.$$

Similar considerations apply to the product of the electric dipole and the electric quadrupole terms $(\mathbf{B}_{\mathrm{d}}\cdot\mathbf{B}_{\mathrm{Q}})$. This product contains $(\ddot{\mathbf{d}}\times\mathbf{n})\cdot(\dddot{\boldsymbol{\mathcal{D}}}\times\mathbf{n})$, which can be transformed as follows:

$$(\ddot{\mathbf{d}}\times\mathbf{n})\cdot(\dddot{\boldsymbol{\mathcal{D}}}\times\mathbf{n}) = \ddot{\mathbf{d}}\cdot\left[\mathbf{n}\times(\dddot{\boldsymbol{\mathcal{D}}}\times\mathbf{n})\right]$$
$$= \ddot{\mathbf{d}}\cdot\dddot{\boldsymbol{\mathcal{D}}} - (\mathbf{n}\cdot\ddot{\mathbf{d}})(\mathbf{n}\cdot\dddot{\boldsymbol{\mathcal{D}}}).$$

Bearing in mind the definition of $\boldsymbol{\mathcal{D}}$, we obtain

$$\ddot{\mathbf{d}}\cdot\dddot{\boldsymbol{\mathcal{D}}} \equiv \ddot{d}_i\dddot{\mathcal{D}}_i = \ddot{d}_i\,\dddot{Q}_{ij}n_j,$$
$$(\mathbf{n}\cdot\ddot{\mathbf{d}})(\mathbf{n}\cdot\dddot{\boldsymbol{\mathcal{D}}}) \equiv \ddot{d}_i n_i\dddot{\mathcal{D}}_j n_j = \ddot{d}_i\,\dddot{Q}_{jk}n_i n_j n_k,$$

where the summation over repeated indices is implied, as before. It is easy to show that the integration of the n_i components over the complete solid angle gives zero, e.g.,

$$\int_{\Omega=4\pi} n_x\, d\Omega = \int_0^{2\pi} d\psi \int_0^{\pi} d\theta\,\sin\theta\cdot\underbrace{\sin\theta\cos\psi}_{n_x=x/r} = 0.$$

The same holds true for triple products

$$\int_{\Omega=4\pi} n_i n_j n_k\, d\Omega = 0.$$

This means that there is no contribution into the radiated power from the product of the electric dipole and quadrupole terms.

In fact, there is a common reason for the above two contributions being zero: they contain products of an odd number of n_i components. Such expressions always yield zero upon integration over the complete solid angle. This is a consequence of independence of such an integral of the choice of axes orientation. The product of an odd number of n_i factors is an odd-rank symmetric tensor, and the only tensor of this type invariant under axes rotation is zero. The same rationale would apply, e.g., to the products of $(\mathbf{B}_{\mathrm{m}}\cdot\mathbf{B}_{\mathcal{M}})$ and $(\mathbf{B}_{\mathrm{Q}}\cdot\mathbf{B}_{\mathcal{M}})$, which appear in higher orders of expansion of the radiated power.

Consider now the product of the magnetic dipole and the electric quadrupole terms $(\mathbf{B}_{\mathrm{m}}\cdot\mathbf{B}_{\mathrm{Q}})$. The expression $(\dddot{\boldsymbol{\mathcal{D}}}\times\mathbf{n})\cdot[(\ddot{\mathfrak{m}}\times\mathbf{n})\times\mathbf{n}]$ can be transformed as

$$(\dddot{\boldsymbol{\mathcal{D}}}\times\mathbf{n})\cdot[(\ddot{\mathfrak{m}}\times\mathbf{n})\times\mathbf{n}] = \mathbf{n}\cdot(\dddot{\boldsymbol{\mathcal{D}}}\times\ddot{\mathfrak{m}}).$$

Since $\boldsymbol{\mathcal{D}}$ is a vector obtained as a contraction of the electric quadrupole moment tensor Q_{ij} with the unit vector $\mathbf{n}=\mathbf{r}/r$, we obtain

$$\mathbf{n}\cdot(\dddot{\boldsymbol{\mathcal{D}}}\times\ddot{\mathfrak{m}}) = n_i\varepsilon_{ijk}\dddot{\mathcal{D}}_j\ddot{\mathfrak{m}}_k = \varepsilon_{ijk}\,\dddot{Q}_{jl}n_l n_i\ddot{\mathfrak{m}}_k.$$

The integrals of two unit vectors over the solid angle can be shown to equal

$$\int_{\Omega=4\pi} n_l n_i\, d\Omega = \frac{4\pi}{3}\,\delta_{li}.$$

The Kronecker delta contracts with $\dddot{Q}_{jl}$, hence the above term is proportional to

$$\varepsilon_{ijk}\,\dddot{Q}_{jl}\delta_{li}\ddot{\mathfrak{m}}_k = \varepsilon_{ijk}\,\dddot{Q}_{ji}\ddot{\mathfrak{m}}_k = 0$$

since this is the double contraction of the antisymmetric Levi-Civita symbol with symmetric quadrupole tensor derivatives.

Such a double contraction of the Levi-Civita symbol with symmetric tensors will also yield zero contributions for terms of a similar nature in higher orders of expansion.

C. Electric quadrupole radiation

The square of the electric quadrupole term $|\dddot{\boldsymbol{\mathcal{D}}}\times\mathbf{n}|^2$ can be transformed as follows:

$$(\dddot{\boldsymbol{\mathcal{D}}}\times\mathbf{n})\cdot(\dddot{\boldsymbol{\mathcal{D}}}\times\mathbf{n}) = \dddot{\boldsymbol{\mathcal{D}}}\cdot\dddot{\boldsymbol{\mathcal{D}}} - (\mathbf{n}\cdot\dddot{\boldsymbol{\mathcal{D}}})(\mathbf{n}\cdot\dddot{\boldsymbol{\mathcal{D}}})$$
$$= \dddot{\mathcal{D}}_i\dddot{\mathcal{D}}_i - n_i\dddot{\mathcal{D}}_i n_j\dddot{\mathcal{D}}_j$$
$$= \dddot{Q}_{ij}n_j\,\dddot{Q}_{ik}n_k - n_i\,\dddot{Q}_{ik}n_k n_j\,\dddot{Q}_{jl}n_l.$$

Thus, the power of the electric quadrupole radiation is equal to

$$I_{\mathrm{Q}} = \frac{c}{4\pi}\frac{1}{4c^6}\int_{\Omega=4\pi}\left\{\dddot{Q}_{ij}\,\dddot{Q}_{ik}n_j n_k - \dddot{Q}_{ik}\,\dddot{Q}_{jl}n_i n_j n_k n_l\right\} d\Omega. \tag{35}$$

The integral of four unit vectors is (see Ref. 8, p. 415)

$$\int_{\Omega=4\pi} n_i n_j n_k n_l \, d\Omega = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{36}$$

Taking into account that the electric quadrupole moment tensor is traceless, $Q_{ii} = 0$, Eq. (35) reduces to

$$I_{\mathrm{Q}} = \frac{1}{20c^5} \dddot{Q}_{ij} \dddot{Q}_{ij}\,, \tag{37}$$

where summation over indices i, j is implied.

It is worth noting that definitions of the quadruple moment tensor differ in the literature. For instance, using the definition

$$\tilde{Q}_{jk} = \sum_i e_i \{3x_j^{(i)} x_k^{(i)} - r_{(i)}^2 \delta_{jk}\}$$

we obtain a different multiplier in the expression for the quadrupole radiation power, viz.,

$$I_{\mathrm{Q}} = \frac{1}{180c^5} \dddot{\tilde{Q}}_{ij} \dddot{\tilde{Q}}_{ij}. \tag{38}$$

D. Anapole radiation

The product of the electric dipole term

$$\mathbf{B}_{\mathrm{d}} = \frac{1}{c^2 r} \ddot{\mathbf{d}} \times \mathbf{n} = \frac{1}{c^2 r} \varepsilon_{pqt} \ddot{d}_p n_q \mathbf{e}_t$$

with the current quadrupole term (29) yields

$$\begin{aligned}
2\,\mathbf{B}_{\mathrm{d}} \cdot \mathbf{B}_{\mathcal{M}} &= 2\frac{1}{c^2 r} \varepsilon_{pqt} \ddot{d}_p n_q \mathbf{e}_t \cdot \frac{1}{2c^3 r} \dddot{\mathcal{M}}_{ikl} n_k n_l \,\varepsilon_{irs} n_r \mathbf{e}_s \\
&= \frac{1}{c^5 r^2} \underbrace{\mathbf{e}_t \cdot \mathbf{e}_s}_{=\delta_{ts}} \varepsilon_{pqt} \varepsilon_{irs} n_k n_l n_q n_r \ddot{d}_p \dddot{\mathcal{M}}_{ikl} \\
&= \frac{1}{c^5 r^2} \ddot{d}_p \dddot{\mathcal{M}}_{ikl} \varepsilon_{pqt} \varepsilon_{irt} n_k n_l n_q n_r.
\end{aligned}$$

It involves the contraction of the Levi-Civita symbols

$$\varepsilon_{pqt}\varepsilon_{irt} = \delta_{pi}\delta_{qr} - \delta_{pr}\delta_{qi}$$

resulting in

$$2\,\mathbf{B}_{\mathrm{d}} \cdot \mathbf{B}_{\mathcal{M}} = \frac{1}{c^5 r^2}\left(\ddot{d}_i \dddot{\mathcal{M}}_{ikl} n_k n_l - \ddot{d}_q \dddot{\mathcal{M}}_{ikl} n_k n_l n_q n_i\right),$$

where the square of a unit vector $n_r n_r = 1$.

We will denote the contribution to the radiated power originating from this term as I_{A}. It equals

$$I_{\mathrm{A}} = \frac{c}{4\pi} \frac{1}{c^5} \int_{\Omega=4\pi} \left(\ddot{d}_i \dddot{\mathcal{M}}_{ikl} n_k n_l - \ddot{d}_q \dddot{\mathcal{M}}_{ikl} n_k n_l n_q n_i\right) d\Omega.$$

Performing the transformations in a manner similar to that in Subsection IV C when dealing with the electric quadrupole radiation, we arrive at

$$\begin{aligned}
I_{\mathrm{A}} &= \frac{1}{c^4}\left(\frac{4}{15} \dddot{\mathcal{M}}_{ikk} \ddot{d}_i - \frac{2}{15} \dddot{\mathcal{M}}_{iik} \ddot{d}_k\right) \\
&= -\frac{2}{15c^4}\left(\dddot{\mathcal{M}}_{kki} - 2\,\dddot{\mathcal{M}}_{ikk}\right) \ddot{d}_i.
\end{aligned}$$

This result can be presented in a more convenient form introducing a vector

$$\begin{aligned}
\mathbf{T}(\tau) &= \frac{1}{10}(\mathcal{M}_{kki} - 2\mathcal{M}_{ikk})\mathbf{e}_i \\
&= \frac{1}{10c} \int_V dV' \left\{ \left(\mathbf{j}(\mathbf{r}', \tau) \cdot \mathbf{r}'\right)\mathbf{r}' - 2r'^2 \mathbf{j}(\mathbf{r}', \tau) \right\}
\end{aligned} \tag{39}$$

known as the *anapole* [The term was proposed by Zel'dovich following the suggestion by Kompaneets (Ref. 19).] *moment* or *toroidicity*. The former name is due to the fact that this expression has no correspondence in the multipole expansion of static electric and magnetic fields. On the other hand, a toroidal solenoid produces a field, which can be described by $\mathbf{T}$. Note that terms corresponding to the toroidal moments are sometimes included in the definitions of the dynamic electric multipole moments, and corresponding terms given for the radiated power.[11,13,17]

The power of the anapole radiation is thus

$$I_{\mathrm{A}} = -\frac{4}{3c^4} \dddot{\mathbf{T}} \cdot \ddot{\mathbf{d}}. \tag{40}$$

In summary, the radiated power up to $1/c^5$ is given by

$$\begin{aligned}
I &= I_{\mathrm{d}} + I_{\mathrm{m}} + I_{\mathrm{Q}} + I_{\mathrm{A}} \\
&= \frac{2\ddot{\mathbf{d}}^2}{3c^3} + \frac{2\ddot{\mathbf{m}}^2}{3c^3} + \frac{1}{20c^5} \dddot{Q}_{ij} \dddot{Q}_{ij} - \frac{4}{3c^4} \dddot{\mathbf{T}} \cdot \ddot{\mathbf{d}}.
\end{aligned} \tag{41}$$

Since the $1/c$ factor enters both the magnetic moment and the toroidicity, each of the last three terms is proportional to $1/c^5$, and is therefore a $1/c^2$ lower correction to the leading term, i.e., to the (electric) dipole approximation. Any subsequent terms in the expansion of the vector potential would yield corrections of the order $1/c^7$ and higher. From the considerations of symmetric properties of tensor products, it can be shown that even powers of $1/c$ would be missing from the expansion, just as the $1/c^4$ terms are missing in Eq. (30).

V. DISCUSSION

It is not difficult to see why the anapole term is often neglected when considering the radiated power. The reason is that I_{A} contains the second time derivative of the electric dipole moment $\ddot{\mathbf{d}}$, which also defines the electric dipole radiated power I_{d}. Usually, the latter is sufficient as the principal approximation and the calculation of the corrections is required only if $I_{\mathrm{d}} = 0$. But in this case $\ddot{\mathbf{d}} = 0$ and hence $I_{\mathrm{A}} = 0$ as well. This means that the anapole term becomes relevant only if the radiated power requires a higher precision than the dipole term alone.

In the static case, the electromagnetic field of the torus is zero outside the system but time-dependent distributions of charges and currents generate the electric field, in particular, with the radiation pattern of a dipole.[20] Such a moment is known as toroidicity, toroidal dipole or anapole moment. In higher approximations, other toroidal multipoles appear as well.[7,14,15,17]

Applying the same strategy as in derivations of the electric and magnetic dipole radiation in Sec. IV A, it is straightforward to show that the radiation of the torus is given by

$$I_{\rm torus} = \frac{2\,\dddot{\mathbf{T}}^2}{3c^5} \propto \frac{1}{c^7}. \tag{42}$$

This contribution, together with the electric dipole (32) and anapole (40) radiation, completes the square

$$I_{\rm d+t} = \frac{2}{3c^3}\left(\ddot{\mathbf{d}} - \frac{1}{c}\,\dddot{\mathbf{T}}\right)^2 = \frac{2\ddot{\mathbf{d}}^2}{3c^3} - \frac{4}{3c^4}\,\dddot{\mathbf{T}}\cdot\ddot{\mathbf{d}} + \frac{2\,\dddot{\mathbf{T}}^2}{3c^5}, \tag{43}$$

cf. also Ref. 14.

In summary, we have obtained an expression for the power of electromagnetic radiation in the approximation beyond the dipole approximation, i.e., with terms proportional to $1/c^5$, which is an accuracy sufficient for comparisons with most present-day measurements.

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[1]J. Frenkel, *Allgemeine Mechanik der Elektrizität* (Springer-Verlag, Wien, 1926).

[2]J. B. French and Y. Shimamoto, "Theory of multipole radiation," Phys. Rev. **91**(4), 898–899 (1953).

[3]Kip S. Thorne, "Multipole expansions of gravitational radiation," Rev. Mod. Phys. **52**(2), 299–339 (1980).

[4]Luc Blanchet, "On the multipole expansion of the gravitational field," Class. Quantum Grav. **15**(7), 1971–1999 (1998).

[5]Carl A. Kocher, "Point-multipole expansions for charge and current distributions," Am. J. Phys. **46**(5), 578–579 (1978).

[6]M. Bezerra, W. J. M. Kort-Kamp, M. V. Cougo-Pinto, and C. Farina, "How to introduce the magnetic dipole moment," Eur. J. Phys. **33**(5), 1313–1320 (2012).

[7]C. G. Gray, G. Karl, and V. A. Novikov, "Magnetic multipolar contact fields: The anapole and related moments," Am. J. Phys. **78**(9), 936–948 (2010).

[8]John David Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley & Sons, Inc., New York, 1999).

[9]David J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice Hall, Upper Saddle River, New Jersey, 1999).

[10]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 3rd revised English edition (Pergamon Press, Oxford–New York–Toronto–Sydney–Braunschweig, 1971).

[11]C. G. Gray, "Multipole expansions of electromagnetic fields using Debye potentials," Am. J. Phys. **46**(2), 169–179 (1978).

[12]V. M. Dubovik and L. A. Tosunyan, "Toroidal moments in the physics of electromagnetic and weak interactions," Sov. J. Part. Nucl. **14**(5), 504–519 (1983).

[13]Rasoul Alaee, Carsten Rockstuhl, and I. Fernandez-Corbaton, "An electromagnetic multipole expansion beyond the long-wavelength approximation," Opt. Commun. **407**, 17–21 (2018).

[14]V. M. Dubovik and V. V. Tugushev, "Toroid moments in electrodynamics and solid-state physics," Phys. Rep. **187**(4), 145–202 (1990).

[15]C. Vrejoiu and R. Zus, "Singular behavior of the multipole electromagnetic field," J. Phys. A **43**(40), 405208 (2010).

[16]E. Radescu, Jr. and G. Vaman, "Cartesian multipole expansions and tensorial identities," Prog. Electromagn. Res. B **36**, 89–111 (2012).

[17]Ivan Fernandez-Corbaton, Stefan Nanz, and Carsten Rockstuhl, "On the dynamic toroidal multipoles from localized electric current distributions," Sci. Rep. **7**(1), 7527 (2017).

[18]R. E. Raab and O. L. de Lange, *Multipole Theory in Electromagnetism: Classical, Quantum: Symmetry Aspects, with Applications* (Clarendon Press, Oxford, 2005), p. 17.

[19]Ia. B. Zel'dovich, "Electromagnetic interaction with parity violation," J. Exp. Theor. Phys. **6**(6), 1184–1186 (1957).

[20]N. J. Carron, "On the fields of a torus and the role of the vector potential," Am. J. Phys. **63**(8), 717–729 (1995).

Newton's Rings

Isaac Newton decided that a particle model of light did a better job of describing optical phenomena that the wave theory of Christiaan Huygens. It is thus ironic that Newton's rings, which demonstrate interference effects, are one of the proofs of the wave theory of light. In the demonstration, a flat glass plate is pressed up against the curved surface of a plano-convex lens. Interference effects in the resulting air gap produces a series of light and dark fringes when viewed in monochromatic light. The tiny device, only 5 cm across, was made by the McIntosh Battery and Optical Co. of Chicago and is in the Greenslade Collection. (Picture and Notes by Thomas B. Greenslade, Jr., Kenyon College)