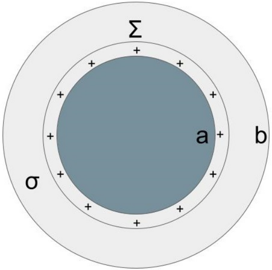
Física Teórica 1 - Práctica

Tensor de Maxwell. Fenómenos dependientes del tiempo.

Problema 3



Intuición física.

¿Qué va a pasar?

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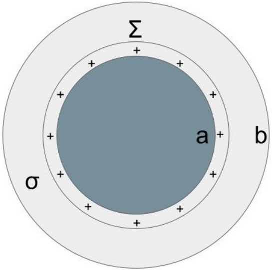

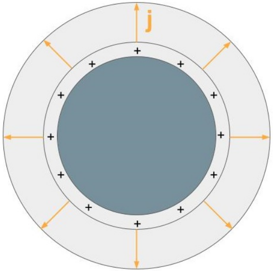

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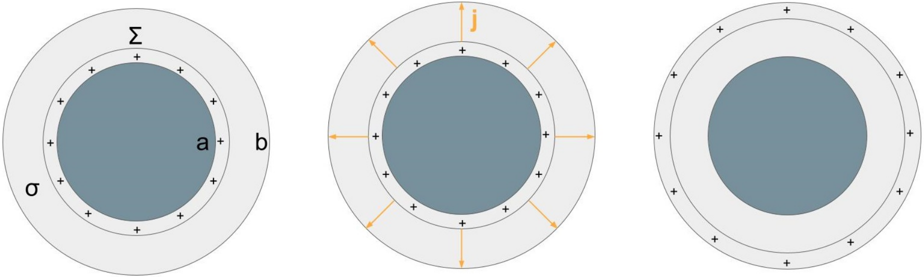

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Problema 3

a) Dado un punto $\mathbf{r}$ veamos cuanto vale $\mathbf{B}(\mathbf{r}, t = 0)$. Por simetría de rotación no puede tener componente tangencial. Podemos ver que la componente radial también se anula. Para eso tomemos un plano que pase por, el origen dado por el centro de la esfera y que contenga a ese punto. Si hacemos una reflexión R respecto de ese plano el problema no cambia con lo cual el campo magnético nuevo $\mathbf{B}'$debería conicidir con el anterior $\mathbf{B}' = \mathbf{B}$, pero el campo magnético cambia de signo ante una reflexión

$$\mathbf{B} = -\mathbf{B} \implies \mathbf{B} = 0.$$

Problema 3

b) Dentro del conductor

$$\nabla \cdot \mathbf{D} = 4\pi\rho$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

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$$0 = \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

$$\implies \nabla \cdot \mathbf{E} = \frac{4\pi}{\epsilon}\rho, \quad \nabla \times \mathbf{E} = 0, \quad \frac{\epsilon}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}\mathbf{J} = 0$$

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Usando la ley de Ohm $\mathbf{J} = \sigma\mathbf{E}$ tenemos

$$\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi\sigma}{\epsilon}\mathbf{E} = 0$$

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$$\boxed{\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, 0)e^{-t/\tau}}$$

con $\tau = \frac{\epsilon}{4\pi\sigma}$.

Problema 3

Usando la ley de Gauss

$$\int\int_S \mathbf{D}\cdot d\mathbf{A} = 4\pi Q_L^{\text{encerrada}}$$

tenemos

$$\mathbf{D}(\mathbf{r},0) = \begin{cases} 0 & r < a \\ \frac{Q}{r^2}\hat{r} & a < r \end{cases},$$

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Luego usando $\mathbf{E} = \mathbf{D}/\epsilon$ tenemos

$$\mathbf{E}(\mathbf{r}, 0) = \begin{cases} 0, & r < a \\ \frac{Q}{\epsilon r^2}\hat{r}, & a < r < b. \\ \frac{Q}{r^2}\hat{r}, & b < r \end{cases}$$

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Escribamos las ecuaciones de Maxwell fuera del conductor

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Entonces afuera del conductor $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(r, 0)$ y se tiene

$$\mathbf{E}(\mathbf{r}, t) = \begin{cases} 0, & r < a \\ e^{-t/\tau}\frac{Q}{\epsilon r^2}\hat{r}, & a < r < b. \\ \frac{Q}{r^2}\hat{r}, & b < r \end{cases}$$

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Así la corriente resulta

$$\mathbf{J}(\mathbf{r}, t) = \sigma \mathbf{E}(\mathbf{r}, t) = \begin{cases} 0, & r < a \\ \frac{\sigma Q}{\epsilon r^2} e^{-t/\tau}\hat{r}, & a < r < b. \\ 0, & b < r \end{cases}$$

Problema 3

La densidad de carga en volumen viene dada por

$$\rho(\mathbf{r}) = \frac{1}{4\pi} \nabla \cdot \mathbf{E}(\mathbf{r}) = 0$$

porque el campo eléctrico es el de una carga puntual en el origen (pero dentro del radio interior no hay carga).

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$$\Sigma(r,t) = \frac{1}{4\pi} \left[\mathbf{D}(\mathbf{r},t)|_{r=r^+} - \mathbf{D}(\mathbf{r},t)|_{r=r^-} \right] \cdot \hat{r}$$

como $\mathbf{D}(\mathbf{r},t)$ es continuo excepto en las superficies $r = a$ y $r = b$ tenemos

$$\Sigma(r,t) = 0, \quad \forall r \neq a, b.$$

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Mientras que

$$\boxed{\Sigma(a,t)} = \frac{1}{4\pi}\left[\epsilon\mathbf{E}(\mathbf{r},t)|_{r=r^+} - 0\right] \cdot \hat{r} \boxed{= \Sigma e^{-t/\tau}}$$

$$\boxed{\Sigma(b,t)} = \frac{1}{4\pi}\left[\mathbf{E}(\mathbf{r},t)|_{r=r^+} - \epsilon\mathbf{E}(\mathbf{r},t)|_{r=r^-}\right] \cdot \hat{r} \boxed{= \frac{Q}{4\pi b^2}(1 - e^{-t/\tau}).}$$

Problema 3

Recuerdo el flujo de energía a través de la superficie S de un volumen V es

$$\frac{d}{dt}U + \int_V d^3\mathbf{r}\,\mathbf{j}\cdot\mathbf{E} = -\int\int_S d^2r\,\mathbf{S}\cdot\mathbf{n}$$

$$U = \frac{1}{8\pi}\int_V d^3r\,(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\,, \quad \mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}.$$

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Como $\mathbf{H} = 0$ entonces $\mathbf{S} = 0$ (no hay pérdida de energía por ondas electromagnéticas) y

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entonces la variación de energía es

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Queremos verificarlo para $t_i = 0$ y $t_f = +\infty$.

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Necesitamos

$$\mathbf{E}\cdot\mathbf{D} = \begin{cases} 0, & r < a \\ \left(e^{-t/\tau}\frac{Q}{\epsilon r^2}\hat{r}\right)\left(\epsilon e^{-t/\tau}\frac{Q}{\epsilon r^2}\hat{r}\right), & a < r < b. \\ \left(\frac{Q}{r^2}\hat{r}\right)\left(\frac{Q}{r^2}\hat{r}\right), & b < r \end{cases} = \begin{cases} 0, & r < a \\ \frac{1}{\epsilon}\left(\frac{Q}{r^2}\right)^2 e^{-2t/\tau}, & a < r < b. \\ \left(\frac{Q}{r^2}\right)^2, & b < r \end{cases}$$

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Entonces

$$\begin{aligned} U(t) &= \frac{1}{8\pi}\int_V d^3r\,(\mathbf{E}\cdot\mathbf{D}) \\ &= \frac{1}{8\pi}\int d\Omega\left[\int_a^b dr\ r^2\frac{1}{\epsilon}\left(\frac{Q}{r^2}\right)^2 e^{-2t/\tau} + \int_b^\infty dr\ r^2\left(\frac{Q}{r^2}\right)^2\right] \\ &= \frac{Q^2}{2\epsilon}\left(\frac{1}{a}-\frac{1}{b}\right)e^{-2t/\tau} + \frac{Q^2}{2b}. \end{aligned}$$

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Luego

$$\boxed{U(0) - U(t) = \frac{Q^2}{2\epsilon}\left(\frac{1}{a}-\frac{1}{b}\right)\left(1 - e^{-2t/\tau}\right)}$$

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Mientras que la energía por efecto Joule es

$$\boxed{\int_0^t \left[\int_V d^3\mathbf{r}\ \mathbf{j}\cdot\mathbf{E}\right]} = \int_0^t dt' \left[\int_a^b r^2 dr d\Omega \sigma |\mathbf{E}(\mathbf{r},t')|^2\right]$$

$$= \int_0^t dt' \left[\int_a^b r^2 dr d\Omega \sigma e^{-2t'/\tau} \left(\frac{Q}{\epsilon r^2}\right)^2\right]$$

$$\boxed{= \frac{Q^2}{2\epsilon}\left(\frac{1}{a}-\frac{1}{b}\right)\left(1-e^{-2t/\tau}\right).}$$

Obtenemos el mismo resultado que para la variación de U verificando la ley de conservación.

Repaso

¿Cómo calculo la fuerza electrostática?

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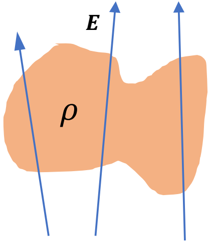

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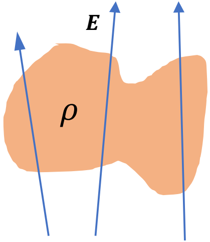

$$\mathbf{F} = \int_V d^3\mathbf{r}\rho(\mathbf{r})\mathbf{E}(\mathbf{r})$$

Repaso

$$\frac{d\mathbf{P}_{\mathsf{mec}}}{dt} = \mathbf{F}_0 + \mathbf{F}_{\mathsf{EM}} = \mathbf{F}_0 + \int_V d^3\mathbf{r} \left[\rho(\mathbf{r})\mathbf{E}(\mathbf{r}) + \frac{1}{c}\mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) \right]$$

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Reemplazando: $\rho = \frac{1}{4\pi}\nabla \cdot \mathbf{E}$, $\frac{1}{c}\mathbf{j} = \frac{1}{4\pi}\left(\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right)$ $\uparrow$

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Tensor de Maxwell: $T_{ij} := E_iE_j + B_iB_j - \frac{\delta_{ij}}{2}(E^2 + B^2)$

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Caso estático: $dP/dt = 0 \implies \quad \mathbf{F}_0 = -\frac{1}{4\pi}\oint_S d^2r\,\mathbf{T} \cdot \mathbf{n}$

Repaso

Fuerza electro-magnética estática

$$\mathbf{F} = \frac{1}{4\pi} \oint_S d^2r \; \mathbf{T} \cdot \mathbf{n}$$

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Fuerza electro-magnética estática

$$\boxed{\mathbf{F} = \frac{1}{4\pi} \oint_S d^2r \; \mathbf{T} \cdot \mathbf{n}}$$

$\mathbf{T} \cdot \mathbf{n} = (\mathbf{E} \cdot \mathbf{n})\mathbf{E} + (\mathbf{B} \cdot \mathbf{n})\mathbf{B} - \frac{1}{2}(E^2 + B^2)\mathbf{n}$ y S es cualquier superficie que encierre la distribución

Electrostática

- Campo perpendicular a la superficie ($\mathbf{E} = E\mathbf{n}$)

$$\mathbf{T} \cdot \mathbf{n} = (\mathbf{E} \cdot \mathbf{n})\mathbf{E} - \frac{1}{2}E^2\mathbf{n} = \frac{1}{2}E^2\mathbf{n} \tag{1}$$

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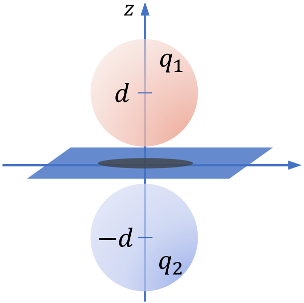
- Campo paralelo a la superficie ($\mathbf{E} \cdot \mathbf{n} = 0$)

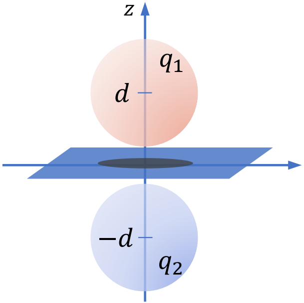
$$\mathbf{T} \cdot \mathbf{n} = (\mathbf{E} \cdot \mathbf{n})\mathbf{E} - \frac{1}{2}E^2\mathbf{n} = -\frac{1}{2}E^2\mathbf{n} \tag{2}$$

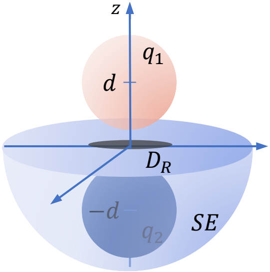
Enunciado y superficie



Problema 4a

Enunciado y superficie





$$\mathbf{F}_2 = \text{lím}_{R\to\infty} \frac{1}{4\pi} \left[\int_{D_R} d^2r \mathbf{T}\cdot\mathbf{n} + \int_{SE} d^2r \mathbf{T}\cdot\mathbf{n} \right]$$

Problema 4a

$$\left|\int_{SE} d^2r\,\mathbf{T}\cdot\mathbf{n}\right| \leq \int_{SE} d^2r\,|\mathbf{T}\cdot\mathbf{n}| \leq \int_{SE} d^2r\,\frac{3}{2}E^2$$

$$|\mathbf{T}\cdot\mathbf{n}| = \left|(\mathbf{E}\cdot\mathbf{n})\mathbf{E} - \tfrac{1}{2}E^2\mathbf{n}\right| \leq |\mathbf{E}\cdot\mathbf{n}||\mathbf{E}| + \tfrac{1}{2}E^2|\mathbf{n}| \leq E^2 + \tfrac{1}{2}E^2 = \tfrac{3}{2}E^2 \qquad \uparrow$$

Problema 4a

$$\left|\int_{SE} d^2r \mathbf{T}\cdot\mathbf{n}\right| \leq \int_{SE} d^2r |\mathbf{T}\cdot\mathbf{n}| \leq \int_{SE} d^2r \frac{3}{2}E^2$$

$|\mathbf{T}\cdot\mathbf{n}| = \left|(\mathbf{E}\cdot\mathbf{n})\mathbf{E} - \frac{1}{2}E^2\mathbf{n}\right| \leq |\mathbf{E}\cdot\mathbf{n}||\mathbf{E}| + \frac{1}{2}E^2|\mathbf{n}| \leq E^2 + \frac{1}{2}E^2 = \frac{3}{2}E^2 \quad \uparrow$

$$\left|\int_{SE} d^2r \mathbf{T}\cdot\mathbf{n}\right| \leq \frac{3}{2}\left(Q\frac{1}{R^2} + ...\right)^2 \int_{SE} d^2r \leq \frac{3}{2}\left(Q\frac{1}{R^2} + ...\right)^2 \frac{1}{2} 4\pi R^2 \to_{R\to\infty} 0$$

$E = |\mathbf{E}_{SE}| \leq Q\frac{1}{R^2} + |3\mathbf{p}\cdot\hat{R} - \mathbf{p}|\frac{1}{R^3} + ... \quad \uparrow$

¿Cuando puede fallar esto?

Problema 4a

Tomando el límite de $R \to \infty$ nos queda solo la integral sobre el disco infinito D (plano $z = 0$)

$$\mathbf{F}_2 = \frac{1}{4\pi} \int_D d^2r \mathbf{T} \cdot \mathbf{n} = \frac{1}{4\pi} \int_D d^2r \left[(\mathbf{E}_D \cdot \mathbf{n})\mathbf{E}_D - \frac{1}{2} E_D^2 \mathbf{n} \right] = \frac{1}{4\pi} \left[A - \frac{1}{2} B \right]$$

Problema 4a

Tomando el límite de $R \to \infty$ nos queda solo la integral sobre el disco infinito D (plano $z = 0$)

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$$\mathbf{E}_D(\rho) = \frac{q_1}{(d^2+\rho^2)^{3/2}} (-d\hat{z} + \rho\hat{\rho}(\varphi)) + \frac{q_2}{(d^2+\rho^2)^{3/2}} (d\hat{z} + \rho\hat{\rho}(\varphi)) = \frac{1}{(d^2+\rho^2)^{3/2}} (td\hat{z} + s\rho\hat{\rho}(\varphi))$$

$$t := q_2 - q_1, \quad s := q_1 + q_2$$

Problema 4a

$$\boxed{\mathbf{E}_D(\rho) = \frac{1}{(d^2+\rho^2)^{3/2}}(td\hat{z} + s\rho\hat{\rho}(\varphi))}$$

Problema 4a

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$$E_D^2 = \left[\frac{td}{(d^2+\rho^2)^{3/2}}\right]^2 + \left[\frac{s\rho}{(d^2+\rho^2)^{3/2}}\right]^2 = t^2 d^2 \frac{1}{(d^2+\rho^2)^3} + s^2 \frac{\rho^2}{(d^2+\rho^2)^3}$$

Problema 4a

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$$\boxed{B = \int_D d^2r E_D^2 =} \int_0^{2\pi}\int_0^{\infty} \rho d\varphi d\rho E_D^2 = t^2d^2 2\pi I_1 + s^2 2\pi I_2 = \boxed{(s^2+t^2)\frac{\pi}{2d^2}}$$

$$I_1 := \int_0^{\infty} \frac{1}{(d^2+\rho^2)^3}\rho d\rho = \frac{1}{4}\frac{1}{d^4}, \qquad I_2 := \int_0^{\infty} \frac{\rho^2}{(d^2+\rho^2)^3}\rho d\rho = \frac{1}{4}\frac{1}{d^2}$$

Problema 4a

$$\boxed{B = (s^2 + t^2)\frac{\pi}{2d^2}}$$

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Problema 4a

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Problema 4a

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$$= \boxed{2\pi t^2 d^2 \hat{z} I_1} \quad \longleftarrow \quad \int_0^{2\pi} \hat{\rho}(\varphi) d\varphi = 0$$

Problema 4a

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Problema 4a

$$\boxed{B = (s^2 + t^2)\frac{\pi}{2d^2}}$$

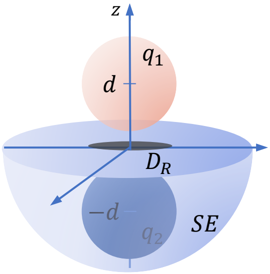
$$\boxed{A = 2\pi t^2 d^2 \hat{z} I_1}$$

$$\boxed{\mathbf{F}_2 =} \frac{1}{4\pi}\left[A - \frac{1}{2}B\right]$$

$$= \frac{1}{4\pi}\left[2\pi t^2 d^2 \hat{z}\frac{1}{4}\frac{1}{d^4} - \frac{1}{2}(s^2 + t^2)\frac{\pi}{2d^2}\hat{z}\right] = \boxed{-q_1 q_2 \frac{1}{4d^2}\hat{z}}$$

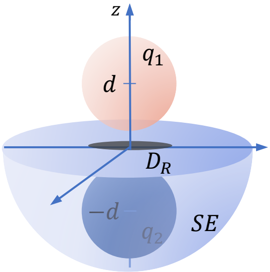
Problema 4a

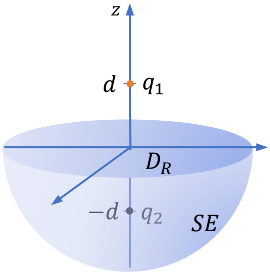
Equivalencia



Problema 4a

Equivalencia

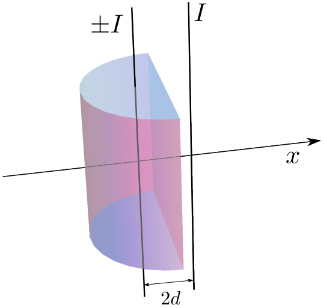




$$\mathbf{F}_2 = \frac{1}{4\pi}\left[\int_{D_R} d^2r\,\mathbf{T}\cdot\mathbf{n} + \int_{SE} d^2r\,\mathbf{T}\cdot\mathbf{n}\right] = -q_1 q_2 \frac{1}{4d^2}\hat{z}$$

Problema 4b

Enunciado y superficie



$$\mathbf{F}_2 = \frac{1}{4\pi}\left[\int_P d^2r\mathbf{T}\cdot\mathbf{n} + \int_{SC} d^2r\mathbf{T}\cdot\mathbf{n}\right]$$

Problema 4b

$$\left|\int_{SC} d^2r \mathbf{T}\cdot\mathbf{n}\right| \leq \int_{SC} d^2r |\mathbf{T}\cdot\mathbf{n}| \leq \int_{SC} d^2r \frac{1}{2}B^2 \sim 2\pi RL\frac{1}{R^2} \to_{R\to\infty} 0$$

$$|\mathbf{T}\cdot\mathbf{n}| = \left|-\frac{1}{2}B^2\mathbf{n}\right| \leq \frac{1}{2}B^2 \sim \frac{1}{R^2}$$

Problema 4b

$$\mathbf{F}_2 = \frac{1}{4\pi}\int_P d^2r\mathbf{T}\cdot\mathbf{n} = \frac{1}{4\pi}\int_P d^2r\left[(\mathbf{B}_P\cdot\mathbf{n})\mathbf{B}_P - \frac{1}{2}B_P^2\mathbf{n}\right]$$

Problema 4b

$$\mathbf{F}_2 = \frac{1}{4\pi}\int_P d^2r\mathbf{T}\cdot\mathbf{n} = \frac{1}{4\pi}\int_P d^2r\left[(\mathbf{B}_P\cdot\mathbf{n})\mathbf{B}_P - \frac{1}{2}B_P^2\mathbf{n}\right]$$

Recordando que el campo magnético de un hilo conductor viene dado por

$$\mathbf{B}(\rho) = \frac{2I}{c\rho}\hat{z}\times\hat{\rho} = \frac{2I}{c\rho}\hat{\theta}$$

tenemos que sobre el plano $x = 0$ viene dado por

$$\begin{aligned}\mathbf{B}_P(y) &= \frac{2I_1}{c(d^2+y^2)}\hat{z}\times(-d\hat{x}+y\hat{y}) + \frac{2I_2}{c(d^2+y^2)}\hat{z}\times(d\hat{x}+y\hat{y})\\ &= \frac{2I_1}{c(d^2+y^2)}(-d\hat{y}-y\hat{x}) + \frac{2I_2}{c(d^2+y^2)}(d\hat{y}-y\hat{x}) = -\frac{2}{c(d^2+y^2)}(td\hat{y}+sy\hat{x})\end{aligned}$$

donde definimos

$$t := (I_1 - I_2), \quad s := I_2 + I_1$$

Problema 4b

$$\boxed{\mathbf{B}_P(\rho) = -\frac{2}{c(d^2+y^2)}(td\hat{y} + sy\hat{x})}$$

Problema 4b

$$\boxed{\mathbf{B}_P(\rho) = -\frac{2}{c(d^2+y^2)}(td\hat{y} + sy\hat{x})}$$

$$B_P^2 = \left[\frac{-2td}{c(d^2+y^2)}\right]^2 + \left[\frac{-2sy}{c(d^2+y^2)}\right]^2 = \frac{4t^2d^2}{c^2}\frac{1}{(d^2+y^2)^2} + \frac{4s^2}{c^2}\frac{y^2}{(d^2+y^2)^2}$$

Problema 4b

$$\boxed{\mathbf{B}_P(\rho) = -\frac{2}{c(d^2+y^2)}(td\hat{y} + sy\hat{x})}$$

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$$\boxed{\int_P d^2r B_P^2 =} L\left[\frac{4t^2d^2}{c^2}J_1 + \frac{4s^2}{c^2}J_2\right] = \boxed{L\left(t^2+s^2\right)\frac{2\pi}{c^2 d}}$$

$$J_1 := \int_{-\infty}^{\infty} dy \frac{1}{(d^2+y^2)^2} = \frac{\pi}{2d^3}, \qquad J_2 := \int_{-\infty}^{\infty} dy \frac{y^2}{(d^2+y^2)^2} = \frac{\pi}{2d}$$

Problema 4b

$$\boxed{\int_P d^2 r B_P^2 = L\left(t^2 + s^2\right)\frac{2\pi}{c^2 d}}$$

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$$\boxed{\int_P d^2r(\mathbf{B}_P \cdot \mathbf{n})\mathbf{B_P} =} = L\frac{4}{c^2}\left[st\int_{-\infty}^{\infty} dy\frac{y}{(d^2+y^2)^2}d\hat{y} + \int_{-\infty}^{\infty} dy\frac{s^2}{(d^2+y^2)^2}y^2\hat{x}\right]$$

Problema 4b

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$$= \boxed{L\frac{4s^2}{c^2}J_2\hat{x}}$$

Problema 4b

$$\boxed{\int_P d^2 r B_P^2 = L\left(t^2+s^2\right)\frac{2\pi}{c^2 d}}$$

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$$= \boxed{L\frac{4s^2}{c^2} J_2 \hat{x}}$$

Problema 4b

$$\boxed{\int_P d^2 r B_P^2 = L\left(t^2 + s^2\right)\frac{2\pi}{c^2 d}}$$

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Problema 4b

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$$\mathbf{B}_P \cdot \hat{x} = -\frac{2}{c(d^2 + y^2)} s y$$

$$\boxed{\int_P d^2 r (\mathbf{B}_P \cdot \mathbf{n}) \mathbf{B_P} = L \frac{4s^2}{c^2} J_2 \hat{x}}$$

Problema 4b

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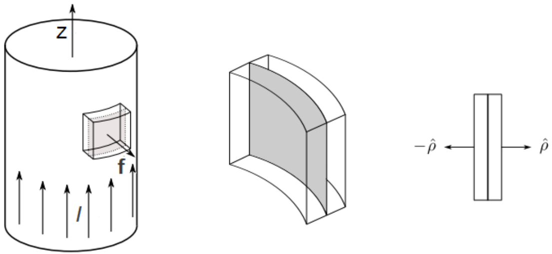
$$\mathbf{B}_P \cdot \hat{x} = -\frac{2}{c(d^2+y^2)} sy$$

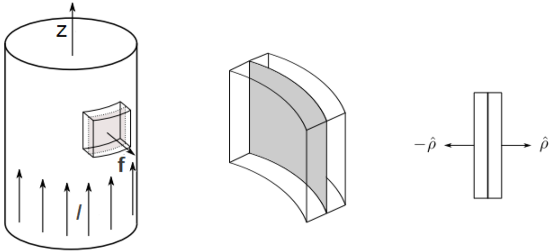
$$\boxed{\int_P d^2r (\mathbf{B}_P \cdot \mathbf{n})\mathbf{B_P} = L\frac{4s^2}{c^2} J_2 \hat{x}}$$

$$\boxed{\frac{\mathbf{F}_2}{L}} = \frac{1}{4\pi L}\int_P d^2r \left[(\mathbf{B}_P \cdot \mathbf{n})\mathbf{B_P} - \frac{1}{2}B_P^2 \mathbf{n}\right] = \frac{1}{4dc^2}\left[s^2 - t^2\right]\hat{x} = \boxed{\frac{I_1 I_2}{c^2 d}\hat{x}}$$

Problema 7

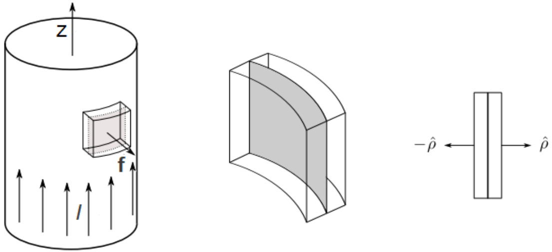


Problema 7



$B(\mathbf{r}) = \frac{2I}{c\rho}\hat{\varphi}$, campo paralelo ec. (2)

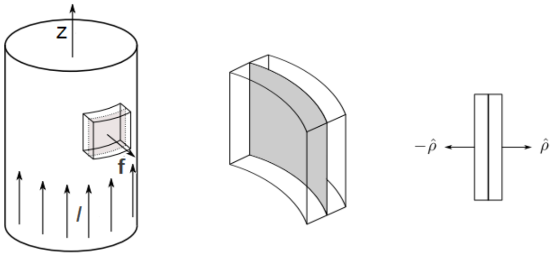
Problema 7



$B(\mathbf{r}) = \frac{2I}{c\rho}\hat{\varphi}$, campo paralelo ec. (2)

$$\mathbf{F} = \int_S d^2r \frac{1}{4\pi}\mathbf{T}\cdot\mathbf{n}$$

Problema 7



$B(\mathbf{r}) = \frac{2I}{c\rho}\hat{\varphi}$, campo paralelo ec. (2)

$$\mathbf{F} = \int_S d^2r \frac{1}{4\pi}\mathbf{T}\cdot\mathbf{n}$$

$$\boxed{\mathbf{f} =} \frac{1}{4\pi}\mathbf{T}\cdot\mathbf{n} = -\frac{1}{8\pi}B(\rho = a)^2\mathbf{n} = \boxed{-\frac{1}{2\pi}\left(\frac{I}{ca}\right)^2\hat{\rho}}$$