

FÍSICA TEÓRICA 1 – 2do. Cuatrimestre de 2024

APUNTE DE FÓRMULAS PARA SEPARACIÓN DE VARIABLES

Coordenadas cartesianas

Ortogonalidad:

$$\int_0^a dx \sin(k_n'x) \sin(k_nx) = \frac{a}{2} \delta_{nn'}, \quad k_n = n\pi/a$$

$$\int_0^{+\infty} dx \sin(k'x) \sin(kx) = \frac{\pi}{2} \delta(k - k'), \quad k > 0$$

$$\int_{-\infty}^{+\infty} dx \sin(k'x) \sin(kx) = \pi \delta(k - k'), \quad k > 0$$

$$\int_{-\infty}^{+\infty} dx \cos(k'x) \cos(kx) = \pi \delta(k - k'), \quad k > 0$$

$$\int_{-\infty}^{\infty} dx e^{i(k-k')x} = 2\pi \delta(k - k'), \quad k \in \mathbb{R}$$

Coordenadas esféricas

Coordenadas y etiquetas: $\theta \in [0, \pi]$ $\varphi \in [0, 2\pi]$ $l \in \mathbb{N}_0$; $-l \leq m \leq l, m \in \mathbb{Z}$

Armónicos esféricos: $Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\varphi}$

Funciones asociadas de Legendre: $P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$ ($m \geq 0$)

Polinomios de Legendre: $P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$

Ortogonalidad: ($x = \cos \theta$)

$$\int_0^{2\pi} d\varphi \int_0^\pi \sin(\theta) d\theta Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) = \delta_{l'l} \delta_{m'm}$$

$$\int_{-1}^1 dx P_l(x) P_{l'}(x) = \frac{2}{2l+1} \delta_{l'l}$$

$$\int_{-1}^1 dx P_{l'}^m(x) P_l^m(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{l'l}$$

$$\int_0^{2\pi} d\varphi e^{im\varphi} e^{-im'\varphi} = 2\pi \delta_{m'm}$$

Completitud:

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) = \delta(\varphi - \varphi') \delta(\cos \theta - \cos \theta')$$

Teorema de adición de armónicos esféricos: $[\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi')]]$

$$P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

Propiedades:

$$Y_{l,-m}(\theta, \varphi) = (-1)^m Y_{lm}^*(\theta, \varphi) \qquad P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

Recurrencia: $\frac{dP_{l+1}}{dx}(x) - \frac{dP_{l-1}}{dx}(x) = (2l+1)P_l(x)$ $P_l(-x) = (-1)^l P_l(x)$

$$P_l(0) = \begin{cases} 0 & \text{si } l \text{ es impar} \\ \frac{(-1)^{l/2}(l-1)!!}{2^{l/2}(\frac{l}{2})!} & \text{si } l \text{ es par} \end{cases} \qquad P_l(1) = 1$$

Primeros polinomios de Legendre:

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}(3x^2 - 1) \quad P_3(x) = \frac{1}{2}(5x^3 - 3x) \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Primeros armónicos esféricos:

$$\begin{aligned} Y_{00}(\theta, \varphi) &= \frac{1}{\sqrt{4\pi}} \\ Y_{10}(\theta, \varphi) &= \sqrt{\frac{3}{4\pi}} \cos \theta & Y_{11}(\theta, \varphi) &= -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi} \\ Y_{20}(\theta, \varphi) &= \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) & Y_{21}(\theta, \varphi) &= -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\varphi} & Y_{22}(\theta, \varphi) &= \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\varphi} \end{aligned}$$

Coordenadas cilíndricas

Coordenadas y etiquetas:

$$\varphi \in [0, 2\pi] \quad \nu \in \mathbb{N}_0 \\ k \geq 0$$

De la separación de variables: $\Phi \sim Q(\varphi)Z(z)R(\rho)$

$$Q''(\varphi) = -\beta Q(\varphi)$$

$$Z''(z) = \lambda Z(z)$$

$$R''(\rho) + R'(\rho)/\rho = (\beta/\rho^2 - \lambda) R(\rho)$$

λ	β	$Q(\varphi)$	$Z(z)$	$R(\rho)$
k^2	$\nu^2 = 0$	$1, \varphi$	$e^{\pm kz}$	$J_\nu(k\rho), N_\nu(k\rho)$
	ν^2	$e^{\pm i\nu\varphi}$		
0	0	$1, \varphi$	$1, z$	$1, \ln(\rho)$
	ν^2	$e^{\pm i\nu\varphi}$		$\rho^{\pm\nu}$
$-k^2$	$\nu^2 = 0$	$1, \varphi$	$e^{\pm ikz}$	$I_\nu(k\rho), K_\nu(k\rho)$
	ν^2	$e^{\pm i\nu\varphi}$		

Para cada ν , las funciones $\{J_\nu(k\rho), N_\nu(k\rho)\}$ son independientes y forman base si:

$$\begin{aligned} \text{intervalo finito:} & \quad \rho \in [0; a] \rightarrow k \equiv k_{\nu n} = \chi_{\nu n}/a \\ \text{intervalo infinito:} & \quad \rho \in [0; \infty) \rightarrow k \in [0; \infty) \end{aligned}$$

con $\{\chi_{\nu n}\}_{n \in \mathbb{N}}$ las infinitas raíces de $J_\nu(x)$.

Ortogonalidad de las funciones de Bessel J_ν :

$$\int_0^a d\rho \rho J_\nu(x_{\nu n'} \rho/a) J_\nu(x_{\nu n} \rho/a) = \frac{a^2}{2} [J_{|\nu|+1}(x_{\nu n})]^2 \delta_{nn'}$$

$$\int_0^\infty d\rho \rho J_\nu(k\rho) J_\nu(k'\rho) = \frac{\delta(k-k')}{k}$$

Propiedades: ($x > 0$)

$$J_0(0) = 1, \quad J_{\nu \neq 0}(0) = 0, \quad x \ll 1: \quad J_\nu(x) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \quad x \gg \nu^2: \quad J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\nu(x \rightarrow 0) \rightarrow \infty \quad \nu \in \mathbb{N}: \quad J_{-\nu}(x) = (-1)^\nu J_\nu(x), \quad \int dx x^\nu J_{\nu-1}(x) = x^\nu J_\nu(x)$$

Fórmulas de recurrencia:

$$\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) = \frac{2\nu}{x} \Omega_\nu(x)$$

$$\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) = 2 \frac{d\Omega_\nu}{dx}(x)$$

Donde Ω_ν representa a cualquiera de las funciones $J_\nu, N_\nu, H_\nu^{(1)} = J_\nu + iN_\nu$ y $H_\nu^{(2)} = J_\nu - iN_\nu$.

Para las modificadas:

$$I_0(0) = 1, \quad I_{\nu > 0}(0) = 0, \quad I_\nu(x \rightarrow \infty) \rightarrow \infty \quad K_\nu(x \rightarrow 0) \rightarrow \infty, \quad K_\nu(x \rightarrow \infty) = 0$$

$$I_{-\nu} = I_\nu, \quad K_{-\nu} = K_\nu$$

Wronskiano: $W[K_\nu, I_\nu](x) \equiv K_\nu(x)I'_\nu(x) - K'_\nu(x)I_\nu(x) = \frac{1}{x}$

Sobre la delta de Dirac $\delta(\rho - \rho')$ en cilíndricas, notar que si $\rho' = 0$, vale:

$$\int_0^{\epsilon > 0} d\rho \delta(\rho) = 1$$

Funciones trigonométricas: definiciones y propiedades

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \sec(x) = \frac{1}{\cos(x)} \quad \coth(x) = \frac{\cos(x)}{\sin(x)} \quad \operatorname{cosec}(x) = \frac{1}{\sin(x)}$$

$$\sin^2(x) + \cos^2(x) = 1 \quad \sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$2 \sin(x) \sin(y) = \cos(x-y) - \cos(x+y) \quad \sin(x \pm y) = \sin(x) \cos(y) \pm \sin(y) \cos(x)$$

$$2 \cos(x) \cos(y) = \cos(x-y) + \cos(x+y) \quad \cos(x \pm y) = \cos(x) \cos(y) \mp \sin(x) \sin(y)$$

$$2 \sin(x) \sin(y) = \sin(x-y) + \sin(x+y) \quad \tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x) \tan(y)}$$

Funciones hiperbólicas: definiciones y propiedades

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2} \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\operatorname{cosech}(x) = \frac{1}{\sinh(x)} \quad \operatorname{sech}(x) = \frac{1}{\cosh(x)} \quad \operatorname{cotanh}(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\sinh(x \pm y) = \sinh(x) \cosh(y) \pm \sinh(y) \cosh(x)$$

$$\tanh(x \pm y) = \frac{\tanh(x) \pm \tanh(y)}{1 \pm \tanh(x) \tanh(y)}$$

$$\cosh(x \pm y) = \cosh(x) \cosh(y) \pm \sinh(x) \sinh(y)$$