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$SO(4, 2)$-Formulation of the Symmetry Breaking in Relativistic Kepler Problems with or without Magnetic Charges*

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The relativistic Kepler problems in Dirac and Klein–Gordon forms are solved by dynamical group methods for particles having both electric and magnetic charges (dyons). The explicit forms of the $O(4, 2)$-algebra and two special $O(2, 1)$-algebras (which coincide in the symmetry limit) are given, and a new group-theoretical form of the symmetry breaking is pointed out. The Klein–Gordon $O(2, 1)$-algebra also solves the dynamics in the case of very strong coupling constants (attractive singular potential), if the principal series of representations is used instead of the discrete series.

1. INTRODUCTION

It is well known by now that the nonrelativistic Schrödinger theory for the Kepler problem can be treated completely algebraically in an irreducible unitary representation of the dynamical group $O(4, 2)$. In the Appendix, to which we shall refer frequently, we give a new version of this treatment. It is also clear that the relativistic Kepler problem (Klein–Gordon and Dirac equations) does not have the $O(4)$-symmetry of the nonrelativistic problem. If the two particles forming the atom have both electric and magnetic charges, then the $O(4)$-symmetry is broken even in the nonrelativistic Kepler problem. The main purpose of this paper is to show the remarkable group-theoretical way the $O(4)$-symmetry is broken in the above cases. All the above problems are actually exactly soluble, though some of these solutions have not yet been reported in the literature. We hope also to demonstrate the power of the method of dynamical groups in solving these problems, including the strong coupling case.

For the ordinary relativistic Dirac problem, the correspondence between the bound-state spectrum and an $O(4, 1)$-representation was given by Kiefer and Fradkin[1] and Pratt and Jordan.[2] The spectrum-correspondence is not the complete solution of the problem and the operators given in Ref. 1 are extremely complicated, because at that time the importance of the tilted states (see Appendix) was not recognized. The role of the $O(4)$-symmetry of the relativistic hydrogen atom (no spins) in covariant theories based on the Bethe–Salpeter equation was studied in Refs. 3 and 4. Although the use of the dynamical group $O(2, 1)$ for the radial wave equation of the Klein–Gordon and second-order Dirac equations is also known,[5] the complete dynamical group has not been given before.

Early studies of the group property and solution of the Kepler problems with both electric and magnetic charges are due to Fierz[6] and Banderet.[7] More recently Hurst[8] related the Dirac quantization condition[9] to the condition of integrability of the Lie-algebra to the Lie group. Zwanziger[10] has solved a related nonrelativistic, Kepler problem with magnetic charges plus an extra particular $1/r^2$ potential by using the $O(4)$-symmetry. This case is particularly simple, as we shall observe again. The relativistic Kepler problem with magnetic monopoles, as far as spectrum is concerned, was studied recently by Berrondo and McIntosh.[11] It was then recognized that the Kepler problem with magnetic charges realizes a different representation of the dynamical group $O(4, 2)$ than the ordinary Kepler problem, and a new quantum number μ arises.[12] With this a connection is established to the $O(4, 2)$-models of hadrons and to a theory of electromagnetic origin of strong interactions.[12]

Thus the motivation to complete the study of the Kepler problem with magnetic charges is threefold:

(1) to give the solutions of the Schrödinger, Klein–Gordon, and Dirac forms of the Kepler problem in the case of particles with both electric and magnetic charges and to treat the case of the very large coupling constant;

(2) to exhibit the dynamical group $O(4, 2)$ for these cases and the nature of symmetry breaking, because the type of symmetry breaking may be applicable to other symmetry-breaking processes;

(3) to have results applicable to the theory of strong interaction phenomena based on the concept of magnetic charges.

We should mention that the spin–orbit symmetry breaking of the relativistic atom has also been treated in the context of the covariant infinite-dimensional wave equations. In the spinless case, the relevant infinite component wave equation for the H atom contains also correctly the recoil effects.[13] In the case of spin, one can use the basic $O(4, 2)$-group (enlarged by Dirac matrices to account for the spins), but one adds suitable new terms in the wave equation to describe the spin–orbit interactions.[14]

2. GROUP THEORETICAL SOLUTIONS

A. Hamiltonians

It will be convenient to treat Schrödinger, Klein–Gordon, and Dirac forms in a parallel fashion, as we go along.

We consider a particle with electric charge e and magnetic charge g, or simply with charge $\mathbf{q} = (e, g)$. The electromagnetic field is described by the vector potentials $A_\mu = (A_0, -\mathbf{A}) = (\varphi_E, -\mathbf{A}_B)$ and $\tilde{A}_\mu = (\tilde{A}_0, -\tilde{\mathbf{A}}) = (\varphi_B, +\mathbf{A}_E)$. The relativistic Lagrangian of the spinless particle in the field is

$$L = mc\sqrt{u^2} + (e/c)A_\mu u^\mu + (g/c)\tilde{A}_\mu u^\mu. \tag{2.1}$$

Hence we have the canonical momentum

$$p_\mu = mcu_\mu + (e/c)A_\mu + (g/c)\tilde{A}_\mu \tag{2.2}$$

and, from the Euler–Lagrange equation, the Minkowski force

$$K_\mu = [(e/c)F_{\mu\nu} + (g/c)\tilde{F}_{\mu\nu}]u^\nu, \tag{2.3}$$

where

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu},$$
$$\tilde{F}_{\mu\nu} = \tilde{A}_{\nu;\mu} - \tilde{A}_{\mu;\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}.$$

From the spatial components of K_μ we find, as desired,

$$\mathbf{F} = e\mathbf{E} - (e/c)(\mathbf{B}\times\mathbf{v}) + g\mathbf{B} + (g/c)(\mathbf{E}\times\mathbf{v}). \tag{2.3'}$$

Because $u_\mu u^\mu = 1$, we obtain from (2.2)

$$[p_\mu - (e/c)A_\mu - (g/c)\tilde{A}_\mu]^2 = m^2c^2. \tag{2.4}$$

Consequently,

$$H^{(KG)} \equiv cp_0 = eA_0 + g\tilde{A}_0 + [m^2c^4 + (c\mathbf{p} - e\mathbf{A} - g\tilde{\mathbf{A}})^2]^{\frac{1}{2}}. \tag{2.5}$$

This is the desired Hamiltonian in the Klein–Gordon form. To obtain the Hamiltonian in the Schrödinger form, we expand formally the square root and subtract the rest energy (physically this expansion is meaningful only if $\tilde{\mathbf{A}}$ is small because g is very large!) with the following result:

$$H^{(S)} \equiv eA_0 + g\tilde{A}_0 + (1/2m)[\mathbf{p} - (e/c)\mathbf{A} - (g/c)\tilde{\mathbf{A}}]^2. \tag{2.6}$$

Finally, the Dirac form of the Hamiltonian is obtained by linearizing (2.5) with Dirac matrices:

$$H_I^{(D)} \equiv eA_0 + g\tilde{A}_0 + \boldsymbol{\alpha}\cdot(c\mathbf{p} - e\mathbf{A} - g\tilde{\mathbf{A}}) + \gamma^0mc^2. \tag{2.7}$$

We also give the second-order Dirac Hamiltonian

$$H_{II}^{(D)} \equiv eA_0 + g\tilde{A}_0 + [(c\mathbf{p} - e\mathbf{A} - g\tilde{\mathbf{A}})^2 + m^2c^4 - e\hbar c(\boldsymbol{\sigma}\cdot\mathbf{B} - i\boldsymbol{\alpha}\cdot\mathbf{E}) - g\hbar c(-\boldsymbol{\sigma}\cdot\mathbf{E} - i\boldsymbol{\alpha}\cdot\mathbf{B})]^{\frac{1}{2}}. \tag{2.8}$$

B. Two-Body System

Let the particle of charge $\mathbf{q}_1 = (e_1, g_1)$ move now in the field of another particle of charge $\mathbf{q}_2 = (e_2, g_2)$ situated at the origin and thought to be heavy. In Eqs. (2.5)–(2.8), we replace (e, g) by (e_1, g_1) and (in Gaussian units) let

$$A_0 = \frac{e_2}{r}, \quad \tilde{A}_0 = \frac{g_2}{r}, \quad \mathbf{A} = +g_2\mathbf{D}(\mathbf{r}), \quad \tilde{\mathbf{A}} = -e_2\mathbf{D}(\mathbf{r}),$$

where

$$\mathbf{D}(\mathbf{r}) = \frac{\mathbf{r}\times\hat{\mathbf{n}}(\mathbf{r}\cdot\hat{\mathbf{n}})}{r[r^2 - (\mathbf{r}\cdot\hat{\mathbf{n}})^2]}, \tag{2.9}$$

where $\hat{n}$ is an arbitrary unit vector. $\mathbf{D}(\mathbf{r})$ has the desired property $\nabla\times\mathbf{D}(\mathbf{r}) = \hat{\mathbf{r}}/r^2$. We then obtain, with the abbreviations ($\hbar = c = 1$)

$$\alpha = -(e_1e_2 + g_1g_2), \tag{2.10}$$

$$\mu = (e_1g_2 - g_1e_2), \tag{2.11}$$

$$\boldsymbol{\pi} = \mathbf{p} - \mu\mathbf{D}(\mathbf{r}), \tag{2.12}$$

the Hamiltonians

$$\begin{aligned} H^{(S)} &\equiv (1/2m)\pi^2 - \alpha/r, \\ H^{(KG)} &\equiv [\pi^2 + m^2]^{\frac{1}{2}} - \alpha/r, \\ H_I^{(D)} &\equiv \boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \gamma^0m - \alpha/r, \\ H_{II}^{(D)} &\equiv [\pi^2 + m^2 - (\mu\boldsymbol{\sigma} + i\alpha\boldsymbol{\alpha})\cdot\hat{\mathbf{r}}/r^2]^{\frac{1}{2}} - \alpha/r. \end{aligned} \tag{2.13}$$

C. The Dynamical Group $O(4, 2)$ and the Two $O(2, 1)$-Algebras

Consider the following generalized operators which reduce to those of the usual hydrogen atom, (A1) and (A14), in the special case $\mu = 0$:

$$\begin{aligned} \mathbf{J} &= \mathbf{r}\times\boldsymbol{\pi} - \mu\hat{\mathbf{r}}, \\ \mathbf{A} &= \tfrac{1}{2}\mathbf{r}\pi^2 - \boldsymbol{\pi}(\mathbf{r}\cdot\boldsymbol{\pi}) + (\mu/r)\mathbf{J} + \frac{\mu^2}{2r^2}\mathbf{r} - \tfrac{1}{2}\mathbf{r}, \\ \mathbf{M} &= \tfrac{1}{2}\mathbf{r}\pi^2 - \boldsymbol{\pi}(\mathbf{r}\cdot\boldsymbol{\pi}) + (\mu/r)\mathbf{J} + \frac{\mu^2}{2r^2}\mathbf{r} + \tfrac{1}{2}\mathbf{r}, \\ \boldsymbol{\Gamma} &= r\boldsymbol{\pi}, \\ \Gamma_0 &= \tfrac{1}{2}(r\pi^2 + r + \mu^2/r), \\ \Gamma_4 &= \tfrac{1}{2}(r\pi^2 - r + \mu^2/r), \\ T &= \mathbf{r}\cdot\boldsymbol{\pi} - i. \end{aligned} \tag{2.14}$$

These operators also satisfy the commutation relations of the Lie algebra of $O(4, 2)$ as before, as can be verified by direct, though laborious, calculation. This fact is more remarkable than it appears at first glance, for the generalized momenta $\boldsymbol{\pi} = \mathbf{p} - \mu\mathbf{D}(\mathbf{r})$ [see Eqs. (2.12) and (2.9)] no longer commute among themselves as do the canonical momenta $\mathbf{p}$; rather we find $[\pi_i, \pi_j] = i\mu e_{ijk}x_k/r^3$. The Casimir operator of the $O(2, 1)$-subgroup generated by Γ_0, Γ_4, T is again as in (A3)

$$Q^2 = \Gamma_0^2 - \Gamma_4^2 - T^2 = J^2. \tag{2.15}$$

Thus, if we had an associated Hamiltonian

$$H^a \equiv \pi^2/2m - \alpha/r + \mu^2/2mr^2, \tag{2.16}$$

we would have

$$\begin{aligned}\Theta &= r(H^a - E)\\ &= (1/2m)(\Gamma_0 + \Gamma_4) - E(\Gamma_0 - \Gamma_4) - \alpha,\end{aligned}$$

i.e., precisely the same equation as (A4); hence all the equations up to (A13) would equally apply to this case. Instead of Eq. (A15) the Casimir operators would be[15]

$$Q_2 = 3(\mu^2 - 1),$$
$$Q_3 = 0, \quad Q_4 = \mu^2(1 - \mu^2). \tag{2.17}$$

In particular, we would have $O(4)$ symmetry, the Balmer formula (A8), etc. This is indeed the case studied by Zwanziger,[10] but not the case we want to solve. There is no physical reason to assume the extra scalar potential $\mu^2/2mr^2$ in (2.16). Instead we want to use the Hamiltonians (2.13) which include no extra scalar potential.

We notice that the three operators

$$\begin{aligned}\Gamma_0' &= \tfrac{1}{2}(r\pi^2 + r),\\ \Gamma_4' &= \tfrac{1}{2}(r\pi^2 - r),\\ T' &= T\end{aligned} \tag{2.18}$$

also generate an $O(2, 1)$-algebra with the Casimir operator

$$Q'^2 = J^2 - \mu^2. \tag{2.19}$$

For our Hamiltonian $H^{(S)} \equiv \pi^2/2m - \alpha/r$, we have

$$\begin{aligned}\Theta &= r(H^{(S)} - E)\\ &= (1/2m)(\Gamma_0' + \Gamma_4') - E(\Gamma_0' - \Gamma_4') - \alpha,\end{aligned} \tag{2.20}$$

again an equation of exactly the same type as (A4). Thus, in terms of the spectrum of Γ_0' and Γ_4', we can immediately use the solutions (A8) and (A11). The only thing we do not know *a priori* is the range of $(J^2 - \mu^2)$, Eq. (2.19), that is contained in the spectrum of $H^{(S)}$. For the ordinary atom, a single representation of the full dynamical group $O(4, 2)$ [Eqs. (A14) and (A15)] determines the spectrum of the Casimir operator Q^2 of the $O(2, 1)$-subgroup and hence J^2. Now, however, the primed generators (2.18) cannot be completed to an $O(4, 2)$-algebra as the unprimed ones given in (2.14). Indeed, if they could be, we would still get an $O(4)$-symmetry which we know we do not have. Thus, we have *two* $O(2, 1)$-algebras, each commuting with $\mathbf{J}$, whose Casimir operators, (2.15) and (2.19), are related by

$$Q^2 - Q'^2 = \mu^2. \tag{2.21}$$

We can indeed view μ^2 as the parameter of symmetry breaking; for $\mu^2 = 0$ we get back the results of the Appendix. It is important to note that the "unsymmetrical" case is also exactly soluble; this is because we know the range of J^2 from the $O(4, 2)$-representation (2.14), and we know the spectrum of Γ_0' and Γ_4' from the value of Q'^2. Thus, for $E < 0$, we solve $\Theta\tilde{\Phi} = 0$ (2.20) by analogy with (A7) and (A8), and immediately have

$$[(-2E/m)^{\frac{1}{2}}\Gamma_0' - \alpha]\Phi = 0$$

and

$$E_{n'} = -\tfrac{1}{2}(m\alpha^2/n'^2), \tag{2.22}$$

where n' is the (discrete) spectrum of Γ_0'. From (2.19), letting

$$Q'^2 = j(j + 1) - \mu^2 = \varphi'(\varphi' + 1),$$

we find

$$\varphi' = -\tfrac{1}{2} \pm [(j + \tfrac{1}{2})^2 - \mu^2]^{\frac{1}{2}}. \tag{2.23}$$

Hence in the D_+-representation of $O(2, 1)$—which is bounded below—the spectrum of Γ_0' has the range

$$\begin{aligned}n' &= -\varphi', -\varphi' + 1, -\varphi' + 2, \cdots\\ &= \tfrac{1}{2} + [(j + \tfrac{1}{2})^2 - \mu^2]^{\frac{1}{2}}, \tfrac{3}{2} + [(j + \tfrac{1}{2})^2 - \mu^2]^{\frac{1}{2}}, \cdots.\end{aligned}$$

(For comparison the range of the eigenvalues of Γ_0 is $n = j + 1, j + 2, \cdots$.) Consequently, Eq. (2.22) can be written as

$$E_s = -\tfrac{1}{2}m\alpha^2\{s + \tfrac{1}{2} + [(j + \tfrac{1}{2})^2 - \mu^2]^{\frac{1}{2}}\}^{-2},$$
$$s = 0, 1, 2, 3, \cdots. \tag{2.24}$$

For $\mu = 0$, we recover the Balmer formula. For fixed $\mu \neq 0$, we see from the $O(4, 2)$-representation (2.14)–(2.17) that again, for each $n(\Gamma_0)$, the range of j is

$$j: |\mu|, |\mu| + 1, |\mu| + 2, \cdots n - 1, \tag{2.25}$$

which completes the specification of the spectrum.[15]

In the case of the Klein–Gordon Hamiltonian, the $O(4, 2)$-representation (2.14) remains the same. But instead of (2.18), we see that

$$\begin{aligned}\Gamma_0' &= \tfrac{1}{2}(r\pi^2 + r - \alpha^2/r),\\ \Gamma_4' &= \tfrac{1}{2}(r\pi^2 - r - \alpha^2/r),\\ T' &= T\end{aligned} \tag{2.26}$$

also form the Lie algebra of an $O(2, 1)$-group with the Casimir operator

$$Q'^2 = \Gamma_0'^2 - \Gamma_4'^2 - T^2 = J^2 - \mu^2 - \alpha^2 = \varphi'(\varphi' + 1),$$

so that

$$\varphi' = -\tfrac{1}{2} \pm [(j + \tfrac{1}{2})^2 - \mu^2 - \alpha^2]^{\frac{1}{2}}. \quad (2.27)$$

The Lie algebra (2.26) solves the square of $H^{(KG)}$ given in (2.13) in the sense that

$$\begin{aligned}\Theta &= r[(H^{(KG)} + \alpha/r)^2 - (E + \alpha/r)^2] \\ &= r\pi^2 - (E^2 - m^2)r - 2\alpha E - \alpha^2/r \\ &= \Gamma_0' + \Gamma_4' - (E^2 - m^2)(\Gamma_0' - \Gamma_4') - 2\alpha E, \quad (2.28)\end{aligned}$$

which is again an equation of the type (A4) or (2.20). The equation $\Theta\tilde{\Phi} = 0$ can again easily be solved by putting $\tilde{\Phi} = e^{i\theta T}\Phi$ and by choosing $\tanh\theta = (E^2 - m^2 + 1)/(E^2 - m^2 - 1)$. Then

$$\{[-4(E^2 - m^2)]^{\frac{1}{2}}\Gamma_0' - 2\alpha E\}\Phi = 0. \quad (2.29)$$

From (2.27), the spectrum of Γ_0' is given by

$$n' = s + \tfrac{1}{2} + [(j + \tfrac{1}{2})^2 - \mu^2 - \alpha^2]^{\frac{1}{2}}, \qquad s = 0, 1, 2, \cdots.$$

Hence, the energy spectrum becomes

$$E_s = m(1 + \alpha^2\{s + \tfrac{1}{2} + [(j + \tfrac{1}{2})^2 - \mu^2 - \alpha^2]^{\frac{1}{2}}\}^{-2})^{-\frac{1}{2}} \qquad s = 0, 1, 2, \cdots. \quad (2.30)$$

Finally, in the case of the second-order Dirac equation, we introduce instead of (2.26) the $O(2, 1)$-algebra

$$\begin{aligned}\Gamma_0' &= \tfrac{1}{2}\{r\pi^2 + r + (1/r)[-\alpha^2 - (\mu\boldsymbol{\sigma} + i\alpha\boldsymbol{\alpha})\cdot\hat{\mathbf{r}}]\}, \\ \Gamma_4' &= \tfrac{1}{2}\{r\pi^2 - r + (1/r)[-\alpha^2 - (\mu\boldsymbol{\sigma} + i\alpha\boldsymbol{\alpha})\cdot\hat{\mathbf{r}}]\}, \\ T' &= T, \quad (2.31)\end{aligned}$$

with the Casimir operator

$$Q'^2 = J^2 - \mu^2 - \alpha^2 - (\mu\boldsymbol{\sigma} + i\alpha\boldsymbol{\alpha})\cdot\hat{\mathbf{r}}. \quad (2.32)$$

The operator

$$\Gamma = \boldsymbol{\sigma}\cdot\mathbf{J} + (\mu\boldsymbol{\sigma} + i\alpha\boldsymbol{\alpha})\cdot\hat{\mathbf{r}} + 1 \quad (2.33)$$

has the property that

$$\Gamma^2 = (\mathbf{J} + \tfrac{1}{2}\boldsymbol{\sigma})^2 - \mu^2 - \alpha^2 + \tfrac{1}{4}.$$

Let

$$\mathfrak{J} = \mathbf{J} + \tfrac{1}{2}\boldsymbol{\sigma};$$

then

$$\Gamma^2 = \mathfrak{J}^2 - \mu^2 - \alpha^2 + \tfrac{1}{4} = j(j+1) - \mu^2 - \alpha^2 + \tfrac{1}{4}. \quad (2.34)$$

Note that j now denotes the total angular momentum of the spin-$\frac{1}{2}$ particle in the atom. The eigenvalues of Γ are then

$$\Gamma : \gamma = \pm[(j + \tfrac{1}{2})^2 - \mu^2 - \alpha^2]^{\frac{1}{2}}. \quad (2.35)$$

Now, from (2.32),

$$Q'^2 = \Gamma^2 - \Gamma = \gamma^2 - \gamma = \varphi'(\varphi' + 1)$$

or

$$\varphi' = -\gamma \quad \text{or} \quad \gamma - 1. \quad (2.36)$$

For the Dirac Hamiltonian (2.13), we get then from the second-order equation

$$\begin{aligned}\Theta &= r[(H^{(D)} + \alpha/r)^2 - (E + \alpha/r)^2] \\ &= r\pi^2 - (E^2 - m^2)r - 2\alpha E - (1/r) \\ &\quad \times [\alpha^2 + (\mu\boldsymbol{\sigma} + i\alpha\boldsymbol{\alpha})\cdot\hat{\mathbf{r}}] \\ &= \Gamma_0' + \Gamma_4' - (E^2 - m^2)(\Gamma_0' - \Gamma_4') - 2\alpha E, \quad (2.37)\end{aligned}$$

i.e., the same equation as (2.28). Thus we can immediately write down the energy spectrum in analogy to the previous case:

$$E_s = m[1 + \alpha^2\{s + [(j + \tfrac{1}{2})^2 - \mu^2 - \alpha^2]^{\frac{1}{2}}\}^{-2}]^{-\frac{1}{2}}, \qquad s = 0, 1, 2, 3, \cdots. \quad (2.38)$$

This differs from the Klein–Gordon spectrum (2.30) in the additive term $\frac{1}{2}$ after s and in the eigenvalue j of total angular momentum which here includes spin.

D. The Case of Large Coupling Constants

Equations (2.30) and (2.38) hold only for a small coupling constant

$$\alpha^2 < (j + \tfrac{1}{2})^2 - \mu^2,$$

because then φ' [which is associated with the Casimir operator Q'^2 of $O(2, 1)$] is real [Eq. (2.27)], and we obtain the D^+-representations of the discrete series.

If α^2 is large, however, as is the case for magnetic charges, ($\alpha = 137/4$ instead of $1/137$ for ordinary atoms!)[12] we must use for φ' a value corresponding to the principal series of representations

$$\varphi' = -\tfrac{1}{2} + i\lambda, \quad \lambda \text{ real}. \quad (2.39)$$

Then

$$Q'^2 = \varphi'(\varphi' + 1) = -\lambda^2 - \tfrac{1}{4}.$$

In the case of the Klein–Gordon equation, for example, from Eq. (2.27)

$$Q'^2 = J^2 - \mu^2 - \alpha^2,$$

and we obtain

$$\lambda = \pm[\alpha^2 + \mu^2 - (j + \tfrac{1}{2})^2]^{\frac{1}{2}}. \quad (2.40)$$

Thus we have a particular representation in the principal series.

In the case of the principal series of representations of the Lie algebra of $O(2, 1)$, the spectrum of Γ_0 ranges from $-\infty$ to $+\infty$, i.e., it is not bounded below. Moreover, a new invariant quantum number E_0 occurs in addition to the invariant φ.[16] The spectrum of Γ_0' is then

$$\Gamma_0': E_0 + s, \quad s = 0, \pm 1, \pm 2, \cdots. \tag{2.41}$$

We have then from (2.29)

$$E_s = m[1 + \alpha^2(s + E_0)^{-2}]^{-\frac{1}{2}}. \tag{2.42}$$

The new quantum number is fixed within the $O(2, 1)$-subgroup; it can be determined only within the representation of the big group $O(4, 2)$.

The physical reason for the drastic change in the case of a large coupling constant for relativistic equations is that we now have a large *attractive* singular potential at $r = 0$. In the case of attractive singular potentials we cannot use the usual boundary conditions of the Schrödinger treatment; the solutions form an overcomplete set, and one needs indeed a new quantum number to characterize the problem completely.[17]

APPENDIX

The operators

$$\begin{aligned}\Gamma_0 &= \tfrac{1}{2}(rp^2 + r),\\ \Gamma_4 &= \tfrac{1}{2}(rp^2 - r),\\ T &= \mathbf{r}\cdot\mathbf{p} - i\end{aligned} \tag{A1}$$

satisfy the commutation relations of the Lie algebra of the group $O(2, 1)$:

$$[\Gamma_0, \Gamma_4] = iT, \quad [\Gamma_4, T] = -i\Gamma_0, \quad [T, \Gamma_0] = i\Gamma_4. \tag{A2}$$

The Casimir operator is given by

$$Q^2 = \Gamma_0^2 - \Gamma_4^2 - T^2 = (\mathbf{r} \times \mathbf{p})^2 = J^2. \tag{A3}$$

Consequently, from the Hamiltonian $H = p^2/2m - \alpha/r$, we obtain ($\hbar = c = 1$)

$$\Theta \equiv r(H - E) = 1/2m(\Gamma_0 + \Gamma_4) - E(\Gamma_0 - \Gamma_4) - \alpha. \tag{A4}$$

The equation

$$\Theta\tilde{\Phi} = 0 \tag{A5}$$

can be solved as follows. Let

$$\tilde{\Phi} = e^{i\theta T}\Phi, \tag{A6}$$

and choose $\tanh\theta = (E + 1/2m)/(E - 1/2m)$; then Eq. (A5) reduces to

$$[(-2E/m)^{\frac{1}{2}}\Gamma_0 - \alpha]\Phi = 0. \tag{A7}$$

Thus Φ's are the eigenstates of Γ_0 with discrete eigenvalues n if $E < 0$. Hence,

$$E_n = -\tfrac{1}{2}(m\alpha^2/n^2). \tag{A8}$$

For $E > 0$, Γ_0 cannot be diagonalized; we go back to Eqs. (A4) and (A5), and let

$$\tilde{\Phi} = e^{i\theta' T}\Phi' \tag{A9}$$

and choose $\tanh\theta' = (E - 1/2m)/(E + 1/2m)$. Then again, from (A5),

$$[(2E/m)^{\frac{1}{2}}\Gamma_4 - \alpha]\Phi' = 0. \tag{A10}$$

Now Γ_4 has a continuous real spectrum λ. Hence

$$E = \tfrac{1}{2}(m\alpha^2/\lambda^2). \tag{A11}$$

The states $\tilde{\Phi}$ must be normalized as follows,

$$\langle\tilde{\Phi}|\,(\Gamma_0 - \Gamma_4)\,|\tilde{\Phi}\rangle = 1, \tag{A12}$$

and are not identical with the Schrödinger wavefunctions ψ. The physical normalized solutions of (A5) are then

$$\tilde{\Phi} = (1/n)e^{i\theta T}\,|n\rangle, \tag{A13}$$

where $|n\rangle$ is a basis of the discrete unitary irreducible $O(2, 1)$ representation D_+^j with Casimir operator given in Eq. (A3): $Q^2 = j(j + 1) = \varphi(\varphi + 1)$, $\varphi < 0$. Hence $\varphi = -j - 1$. Therefore, for each j, $n = j + 1, j + 2, \cdots$. Similar equations hold for the continuous spectrum.

The treatment above does not tell us yet what values of j occur; it is yet incomplete. The complete solution is as follows. The operators (A1) together with

$$\begin{aligned}\mathbf{J} &= \mathbf{r} \times \mathbf{p},\\ \mathbf{A} &= \tfrac{1}{2}\mathbf{r}p^2 - \mathbf{p}(\mathbf{r}\cdot\mathbf{p}) - \tfrac{1}{2}\mathbf{r},\\ \mathbf{M} &= \tfrac{1}{2}\mathbf{r}p^2 - \mathbf{p}(\mathbf{r}\cdot\mathbf{p}) + \tfrac{1}{2}\mathbf{r},\\ \mathbf{\Gamma} &= r\mathbf{p}\end{aligned} \tag{A14}$$

satisfy the commutation relations of the Lie algebra of $O(4, 2)$; $\mathbf{J}$ and $\mathbf{A}$ (Runge–Lenz vector) together generate a compact $O(4)$ subgroup that commutes with Γ_0. (Note then that $\mathbf{J}$ and "tilted $\mathbf{A}$" commute with Θ.) The Casimir operators of the Lie algebra of this $O(4, 2)$ are

$$\begin{aligned}&Q_2 = \mathbf{J}^2 + \mathbf{A}^2 - \mathbf{M}^2 - \mathbf{\Gamma}^2 + \Gamma_0^2 - \Gamma_4^2 - T^2 = -3,\\ &Q_3 = 0, \quad Q_4 = 0.\end{aligned} \tag{A15}$$

In an irreducible representation of $SO(4, 2)$, for each $j = 0, 1, 2, 3, \cdots$ we have $n = j + 1, j + 2, \cdots$. Or, for each $n = 1, 2, \cdots$ we have $j = 0, 1, \cdots n - 1$. The energy levels depend only on n^2, which is the basis of the $O(4)$-symmetry.

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$SL(2, C)$ Representations in Explicitly "Energy-Dependent" Basis. I

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Unitary and nonunitary representations of the $SL(2, C)$ group are investigated in such a basis, in which the subgroup diagonalized is that one which in the four-dimensional representation leaves invariant the 4-vector $p_\mu = (\frac{1}{2}(1+v), 0, 0, \frac{1}{2}(1-v))$ for an arbitrary real value of $p_\mu^2 = v$. The split of the representation space into irreducible subspaces changes smoothly when varying the value of v. The formalism is of importance in physical theories which postulate analyticity requirements and Lorentz invariance simultaneously (e.g., Regge and Lorentz pole theory). In this paper we construct explicit basis functions of the representation spaces.

1. INTRODUCTION

The representation theory of the $SL(2, C)$ group is of great importance in physics, and a lot of work has been devoted to construct its representations explicitly. It is, however, surprising that attention has hardly been paid to constructing and investigating them in an explicitly "analytically continuable" form. We mean the following: The representations of the $SL(2, C)$ group are usually given in an $SU(2)$, $SU(1,1)$, or $E(2)$ basis, i.e., the representation space is given as a direct sum (integral) of subspaces invariant with respect to the little groups of the 4-vectors $(1,0,0,0)$, $(1,0,0,1)$, and $(0,0,0,1)$, respectively. Physical theories, which postulate analyticity requirements together with Lorentz invariance, necessitate the construction of $SL(2, C)$ representations over such spaces, which are split into subspaces invariant with respect to the little group of an appropriately chosen 4-vector, e.g., $p_\mu = (\frac{1}{2}(1+v), 0, 0, \frac{1}{2}(1-v))$. Its length $p_\mu^2 = v$ is kept a free parameter. Moreover, we want the representations to be analytic in this variable v in the sense that the split of the representation space into irreducible subspaces changes smoothly when we vary the value of this parameter.

In this paper we will explicitly construct the basis states for such representations. We shall apply a standard procedure.[1] This method consists first of choosing a subgroup which one wants to be diagonal in the basis to be constructed and second of determining the eigenfunctions of the Casimir operator of this subgroup.

For this purpose, we must obviously specify such a subgroup of $SL(2, C)$ which, depending on the value of a suitable parameter, becomes deformed from $SU(2)$ through $E(2)$ to $SU(1,1)$. Then one must determine the representation matrix elements of this group, which is the second point of the previous program above. These problems have already been treated[2,3] but without embedding this group into $SL(2, C)$. [The term "interpolating group (IG)" was introduced for this group[3]; we are going to use it in this paper as well.]

After having constructed the basis with the above specified properties in the $SL(2, C)$ representation space, we naturally examine the matrix elements of finite $SL(2, C)$ transformations and the problem of the transformation coefficients between different basis sets.[4] Here we give them only in integral forms, as the explicit calculations can be found in a separate paper.[5]

In Sec. 2, we shall summarize the results of the