Primer Parcial 2013 C2

p1)

$$H = \frac{L_x^2 + L_y^2}{2I} + \frac{L_z^2}{2I_3}$$

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|1\,1\rangle + |1\,0\rangle + |1\,-1\rangle\right)$$

$$H = \frac{L^2 - L_z^2}{2I} + \frac{L_z^2}{2I_3} = \frac{L^2}{2I} + \frac{L_z^2}{2}\left(\frac{1}{I_3} - \frac{1}{I}\right)$$

autoestados de H son de L^2 y L_z: $[H, L^2] = 0$, $[H, L_z] = 0$

$|l\,m\rangle$

$$H\,|l\,m\rangle = \underbrace{\left\{\frac{\hbar^2 l(l+1)}{2I} + \frac{\hbar^2 m^2}{2}\left(\frac{1}{I_3} - \frac{1}{I}\right)\right\}}_{\mathcal{E}_{lm}}|l\,m\rangle$$

l	m	$\mathcal{E}_{lm}$
1	0	$\frac{\hbar^2}{I}$
1	1	$\hbar^2\left[\frac{1}{I} + \frac{1}{2}\left(\frac{1}{I_3} - \frac{1}{I}\right)\right]$
1	-1	(degenerados!)

Estado después

$\text{Prob}(\mathcal{E}_{10}) = \frac{1}{3}$ $\qquad |1\,0\rangle$

$\text{Prob}(\mathcal{E}_{11} = \mathcal{E}_{1-1}) = \frac{2}{3}$ $\qquad \frac{1}{\sqrt{2}}\left(|1\,1\rangle + |1\,-1\rangle\right)$

L^2 es degenerado

L_z determina los autoestados unívocamente.

$$|E\ m\rangle$$

CCOC : $\{H, L_z\}$

Midiendo L_z da $\frac{1}{3}$ de probabilidad de medir $\{\hbar, 0, -\hbar\}$ equiprobable.

$$H = \frac{L_x^2 + L_y^2}{2I} + \frac{L_z^2}{2I_3} + \lambda L_x^2$$

Conviene escribir:

$$H = H_0 + \lambda L_x. \qquad \text{si } L^2 = l(l+1)\hbar^2$$
$$l = 1.$$

Los estados degenerados en energía son

$$|1\ 1\rangle \quad y \quad |1\ -1\rangle$$

$$L_x|1\ 1\rangle = \frac{1}{2}(L_+ + L_-)|1\ 1\rangle = \frac{\sqrt{2}\hbar}{2}|1\ 0\rangle$$

$$L_x|1\ -1\rangle = \frac{1}{2}(L_+ + L_-)|1\ -1\rangle = \sqrt{2}\hbar\,|1\ 0\rangle$$

$$L_\pm|l\ m\rangle = \hbar\sqrt{l(l+1) - m(m\pm1)}\,|l, m\pm1\rangle$$

$$L_x^2|1\ 1\rangle = \frac{\sqrt{2}}{2}\hbar\cdot\frac{1}{2}(L_+ + L_-)|1\ 0\rangle = \frac{\sqrt{2}\hbar^2}{4}(\sqrt{2}|1\ 1\rangle + \sqrt{2}|1\ -1\rangle)$$

$$L_x^2|1\ -1\rangle = \frac{\sqrt{2}}{2}\hbar\frac{1}{2}(L_+ + L_-)|1\ 0\rangle = \frac{\sqrt{2}\hbar^2}{4}(\sqrt{2}|1\ 1\rangle + \sqrt{2}|1\ -1\rangle)$$

P1) cont.

$$\mathbb{V} = \frac{\lambda \hbar^2}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Los autovectores son: $\frac{1}{\sqrt{2}}(|11\rangle + |1-1\rangle)$

autovalor: $\lambda \hbar^2$

y $\frac{1}{\sqrt{2}}(|11\rangle - |1-1\rangle)$

autovalor: 0

Las energías cambian a:

$$\left(\begin{array}{l} \hbar^2\left[\frac{1}{I} + \frac{1}{2}\left(\frac{1}{I_3} - \frac{1}{I}\right)\right] \\ \\ \hbar^2\left[\frac{1}{I} + \frac{1}{2}\left(\frac{1}{I_3} - \frac{1}{I}\right)\right] + \lambda \hbar^2. \end{array} \right.$$

a primer orden.

P2) $H = \hbar\omega\left(a^+a + \frac{1}{2}\right) + \hbar\omega\frac{\lambda}{2}\left(a^{+2} + a^2\right)$ (1)

$$b^+ = \frac{1}{\sqrt{1-\epsilon^2}}\left(a^+ + \epsilon a\right) \quad (2)$$

$$b = \frac{1}{\sqrt{1-\epsilon^2}}\left(a + \epsilon a^+\right) \quad (3)$$

$(2) - \epsilon(3)$: $\quad b^+ - \epsilon b = \dfrac{a^+(1-\epsilon^2)}{\sqrt{1-\epsilon^2}}$

despejando:

$$\Rightarrow \begin{cases} a^+ = \dfrac{1}{\sqrt{1-\epsilon^2}}\left(b^+ - \epsilon b\right) & (4) \\[2ex] a = \dfrac{1}{\sqrt{1-\epsilon^2}}\left(b - \epsilon b^+\right) & (5) \end{cases}$$

i) Probar que $[b, b^+] = 1$

$$[\ ] = \frac{1}{1-\epsilon^2}\left[a + \epsilon a^+, a^+ + \epsilon a\right]$$

$$= \frac{1}{1-\epsilon^2}\left(\underbrace{[a, a^+]}_{1} + \epsilon^2\underbrace{[a^+, a]}_{-1}\right)$$

$[b, b^+] = 1.$

ii) de (4) y (5):

$$a^+a = \frac{1}{(1-\epsilon^2)}\left(b^+b - \epsilon b^{+2} - \epsilon b^2 + \epsilon^2 b b^+\right)$$

$$a^2 = \frac{1}{(1-\epsilon^2)}\left(b^2 - \epsilon b b^+ - \epsilon b^+ b + \epsilon^2 b^{+2}\right)$$

$$a^{+2} = \frac{1}{1-\epsilon^2}\left(b^{+2} - \epsilon b^+ b - \epsilon b b^+ + \epsilon^2 b^2\right)$$

$$b b^+ = b^+ b + [b, b^+]$$

$$H = \frac{\hbar\omega}{1-\sigma^2}\Big[(1+\sigma^2 - 2\lambda\sigma)\, b^+ b$$
$$+ \left(-\sigma + \frac{\lambda\sigma^2}{2} + \frac{\lambda}{2}\right) b^{+2}$$
$$+ \left(-\sigma + \frac{\lambda\sigma^2}{2} + \frac{\lambda}{2}\right) b^2$$
$$+ (\sigma^2 - \lambda\sigma)\Big] + \frac{\hbar\omega}{2} \quad (6)$$

Los términos cuadráticos se anulan si:

$$\lambda\sigma^2 - 2\sigma + \lambda = 0 \quad (7)$$

ó si: $$\sigma = \frac{2 \pm \sqrt{4 - 4\lambda^2}}{2\lambda} = \frac{1 \pm \sqrt{1-\lambda^2}}{\lambda}$$

Se puede probar que*:

$$\frac{1 + \sigma^2 - 2\lambda\sigma}{1-\sigma^2} = \sqrt{1-\lambda^2} \quad \text{si} \quad \sigma = \frac{1 - \sqrt{1-\lambda^2}}{\lambda}$$

y que: $$\frac{\sigma^2 - \sigma\lambda}{1-\sigma^2} + \frac{1}{2} = \frac{\sqrt{1-\lambda^2}}{2}$$

$$\therefore \quad H = \hbar\omega\delta\left[b^+ b + \frac{1}{2}\right] \quad (8) \quad \delta = \sqrt{1-\lambda^2}$$

* de (7) $\sigma^2 = \frac{2\sigma}{\lambda} - 1 \qquad 1 - \sigma^2 = \frac{2\lambda - 2\sigma}{\lambda}$

el procedimiento es largo. Para la frecuencia se obtiene: $\frac{\sigma(1-\lambda^2)}{\lambda - \sigma} = \delta$, lo mismo para $\frac{\sigma^2 - \lambda\sigma}{1-\sigma^2} + \frac{1}{2} = \frac{\delta}{2}$

$\therefore$ el espectro de energías es: $\mathcal{E}_n = \hbar\omega\delta\left(n + \frac{1}{2}\right)$

$$\mathcal{E}_0 = \frac{\hbar\omega\delta}{2} \simeq \frac{\hbar\omega}{2}\left(1 - \frac{\lambda^2}{2}\right) \qquad \lambda \text{ chico.}$$

$$\mathcal{E}_0 \simeq \frac{\hbar\omega}{2} - \frac{\hbar\omega\lambda^2}{4} \quad (9)$$

Teoría de perturbaciones:

$$V = \lambda \frac{\hbar\omega}{2}\left(a^{+2} + a^2\right)$$

$$E^{(0)} = \frac{\hbar\omega}{2}$$

$$E^{(1)} = \langle 0 | V | 0 \rangle = 0$$

$$E^{(2)} = \sum_{n\neq 0} \frac{|V_{n0}|^2}{E_0^{(0)} - E_n^{(0)}}$$

sólo contribuye $n=2$

$$E_0^{(0)} - E_2^{(0)} = -2\hbar\omega$$

$$V_{20} = \lambda \frac{\hbar\omega}{2}\sqrt{2}$$

$\therefore \quad E^{(2)} = -\frac{\lambda^2 \hbar\omega}{4}$ ok con el exacto. 9.

P3) spin 1/2.

$t<0$ $$H_0 = -\left(\frac{eB}{mc}\right) S_z = \omega S_z$$

$t=0$ se enciende un campo en x:

$$H = \omega S_z + \lambda S_x = \sqrt{\omega^2+\lambda^2}\, \vec{S}\cdot\hat{n}$$

$$\hat{n} = \left(\frac{\omega}{\sqrt{\omega^2+\lambda^2}}\hat{z} + \frac{\lambda}{\sqrt{\omega^2+\lambda^2}}\hat{x}\ 0\right)$$

a) $$|\cancel{\alpha},0\rangle = |+\rangle$$

$$|\alpha,t\rangle = e^{-i\frac{Ht}{\hbar}}|\alpha,0\rangle$$

$$= e^{-i\frac{\gamma\vec{S}\cdot\hat{n}}{\hbar}}|\alpha,0\rangle$$

con $\gamma = \sqrt{\omega^2+\lambda^2}\, t$

usamos operador de rotaciones en $s=1/2$

$$e^{-\frac{i\gamma\vec{S}\cdot\hat{n}}{\hbar}} = \mathbb{I}\cos\frac{\gamma}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\gamma}{2}$$

$$e^{-\frac{i\gamma\vec{S}\cdot\hat{n}}{\hbar}}|+\rangle = \cos\frac{\gamma}{2}|+\rangle - i\sin\frac{\gamma}{2}\, n_x|-\rangle - i\sin\frac{\gamma}{2}\, n_z|+\rangle$$

usamos que: $\sigma_x|+\rangle = |-\rangle$, $\sigma_z|+\rangle = |+\rangle$

$$|\alpha,t\rangle = \left(\cos\frac{\gamma}{2} - i n_z\sin\frac{\gamma}{2}\right)|+\rangle - i n_x\sin\frac{\gamma}{2}|-\rangle$$

check: $\langle\alpha,t|\alpha,t\rangle = 1$ ok!

Alternativamente, diagonalizamos H ; $|S.\hat{n},+\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$; $|S.\hat{n},-\rangle = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$
se obtiene fácil que autovalores de H : $\pm\frac{\hbar}{2}\sqrt{\omega^2+\lambda^2}$
con 1 ecuación obtenemos : $\tan\theta = \frac{\sqrt{\lambda^2+\omega^2}}{\lambda}$

b) $\langle -|\alpha, t\rangle = -i\, n_x \sin\frac{\gamma}{2}$

$$\mathrm{Prob}(|-\rangle) = \frac{\lambda^2}{\omega^2+\lambda^2} \cdot \sin^2\left(\frac{\sqrt{\omega^2+\lambda^2}}{2}\, t\right).$$

si λ es chico $(\lambda \ll \omega)$

$$\mathrm{Prob}(|-\rangle) \simeq \frac{\lambda^2}{\omega^2}\sin^2\frac{\omega t}{2}$$

c) Perturbaciones dependiente del tiempo:

$$c_-^{(1)} = -\frac{i}{\hbar}\int_0^t dt\, e^{\frac{i(\varepsilon_f-\varepsilon_i)t}{\hbar}} \langle -|V|+\rangle$$

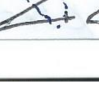
$$\varepsilon_f - \varepsilon_i = -\omega\hbar \qquad \langle -|\lambda S_x|+\rangle = \frac{\lambda\hbar}{2}$$

$$c_-^{(1)} = -i\frac{\lambda}{2}\left[\frac{e^{-i\omega t}-1}{-i\omega}\right] = \frac{\lambda}{\omega}e^{-\frac{i\omega t}{2}}\left(-i\sin\frac{\omega t}{2}\right)$$

$$c_-^{(1)} = -i\frac{\lambda}{\omega}e^{-\frac{i\omega t}{2}}\sin\frac{\omega t}{2}$$

$$\mathrm{Prob} = |c_-^{(1)}|^2 = \frac{\lambda^2}{\omega^2}\sin^2\frac{\omega t}{2} \qquad \text{ok!}$$

d) Sabemos que $[S^2, S_i] = 0 \Rightarrow [S^2, H] = 0 \Rightarrow S^2(t) = \text{cte}$
Como $[\vec{S}.\hat{n}, H] = 0 \Rightarrow \vec{S}.\hat{n} = \text{cte}$
$\langle \vec{S} \rangle$ evoluciona con módulo fijo y con proyección en $\hat{n}$ fijo $\Rightarrow$



$\hat{n}$ $\langle S\rangle$ precesa en $\hat{n}$.