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[10] R. Feynman gave an intersting account of a semiconscious self-monitoring of the dreamer in *Surely You're Joking, Mr. Feynman* (Norton, New York, 1985).

[11] The main obstacle to any experimental investigation of this subject is the difficulty of preparing such complex systems in reproducible states (reproducible as far as the relevant properties are concerned). For a very preliminary exploration of ideas, see C. H. Woo, Found. Phys. **11**, 933 (1981).

Supersymmetry in quantum mechanics

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We give some illustrations and interpretations of supersymmetry in quantum mechanics in simple models. We show that the value of 2 for the g factor of the electron expresses the presence of supersymmetry in the Hamiltonian for an electron in a uniform magnetic field. The problem is considered both in the Schrödinger and Dirac formulations. We also show that the radial Coulomb problem with orbital angular momentum l, nuclear charge Z, and principal quantum number n, is supersymmetrically linked to the similar problem with charge $Z(1-1/n)$ and quantum number $n-1$. Thereby the dependence of Coulomb energies only on the combination Z/n is seen as a manifestation of the supersymmetry in the radial Coulomb problem. Other examples of supersymmetry we consider are the Morse potential, the three-dimensional isotropic oscillator, the states of the helium atom and those of the hydrogen atom in an extremely strong magnetic field.

I. INTRODUCTION

To the beginning student of quantum mechanics, the concept of degeneracy of energy levels may seem elementary, not very deep, and definitely a source of confusion when doing perturbation theory. Of course the fascination focused on this subject is due to the connections between degeneracy, symmetry, and conservation laws. The classic example is rotational invariance leading to conservation of angular momentum $(l\hbar)$ which in turn implies the $2l+1$ degeneracy of energy levels.

There are other degeneracies which can be related to symmetries and conservation laws which are more subtle than the "obvious" ones. The $2p-2s$ degeneracy in hydrogen (similarly for nl, $l=0,1,...n-1$) is sometimes misnamed "accidental." The corresponding conservation law and symmetry was found long ago by Pauli.[1,2] He showed that the degeneracy was due to an $0(4)$ symmetry (invariance under rotations in a four-dimensional space) that could be traced to a special property of the Coulomb problem: the eccentricity and direction of the major axis of a Kepler elliptical orbit form constants of the motion. Another interesting example is the three-dimensional harmonic oscillator which also has degeneracies corresponding to a higher symmetry. In this case it is the symmetry group $SU(3)$ (complex unitary "rotations" of the three coordinates) which accounts for the degeneracy.[2]

Recent studies of supersymmetric quantum mechanics[3] have turned up some more interesting degeneracies. For any Hamiltonian with one degree of freedom, a companion Hamiltonian can be constructed such that the resulting system as a whole is supersymmetric. That is not to say that the combined system always exists in nature. In a sense this is a solution looking for a problem. One virtue of this construction is that these simple results help to demystify supersymmetry and demonstrate the workings of a symmetry transformation which is more general than a Lie group or point group. There is currently considerable ferment in particle physics in supersymmetry[4] but, given the complexities of that subject, it seems useful pedagogically to look at supersymmetry in a variety of simple quantum mechanical Hamiltonians.

Supersymmetry transformations in field theory mix half-integral and integral angular momentum states and hence the corresponding multiplets in general contain both fermions and bosons. This remarkable property suggests that this symmetry may provide a unifying principle in elementary particle physics. Since its discovery[5] in 1974, many further remarkable properties of model field theories have been uncovered. A notable one is that supersymmetric theories seem to be less divergent, that is, they are not as plagued by the infinities which arise in most field theories. Roughly speaking, the putting together of bosonic and fermionic elements leads to a cancellation of the infinities which would otherwise appear separately. Also, most field theories require sensitive adjustment of parameters and here again supersymmetric theories seem to require less of this "fine tuning." Finally, gravity seems to be more naturally incorporated with other interactions within a supersymmetric context in a theory called supergravity. For these reasons, there is great hope and excitement, although at this writing, in spite of the over 2800 papers on this subject,[6] there is no clear indication of how the beautiful mathematics of supersymmetry is actually related to a theory of the known elementary particles! More recently, without getting involved in the complexities of field theories, applications of supersymmetry have been proposed in nuclear physics,[7] and atomic physics.[8]

This paper does not deal with the intricacies of field theories. Rather our aim is to bring out aspects of supersymmetry by viewing the concept through simple quantum mechanical examples. Since the quantum mechanics of a particle moving in a potential is like a field theory in zero dimensions, we illustrate the supersymmetric construction[3] through simple potential problems, point out connections to known results and in some cases, to new ones. For instance we feel it is not as widely known as it should be that supersymmetry resolves a puzzle in quantum mechanics. There is a pattern of degeneracy of Landau levels for a spin $\frac{1}{2}$ particle in a uniform magnetic field for the special case $g = 2$, where g is the gyromagnetic ratio. (Neglecting quantum electrodynamic radiative corrections $g = 2$, of course, for the electron.) The above mentioned 0(4) invariance of the Coulomb problem and $SU(3)$ invariance of the three-dimensional oscillator are covered in textbooks,[2] whereas this $g = 2$ degeneracy is not similarly enshrined to our knowledge. There are a number of reviews of supersymmetry at this level[3,4] and one noted the relation of $g = 2$ to supersymmetry.[9] We carry things further and show that the Dirac equation also leads to supersymmetry in the same sense but only after the negative energy states are filled and the vacuum is redefined. This result is an illustration of supersymmetry when reduced to one degree of freedom and does not reflect a supersymmetry of the field theory. Supersymmetric quantum electrodynamics contains other particles which are not seen. Ordinary quantum electrodynamics are not supersymmetric.

The arrangement of this paper is as follows. In Sec. II we consider the basic steps in constructing a supersymmetric Hamiltonian for any one-dimensional quantum mechanical problem whose ground state energy and wave function are known. The example of a one-dimensional harmonic oscillator in Sec. III is connected to the states of an electron in a magnetic field, first through analysis of the Schrödinger equation and then the Dirac equation. The value $g = 2$ plays a key role in this connection. In Sec. IV we consider the radial Coulomb, the one-dimensional Morse, and the radial part of the isotropic oscillator problems (of interest to atomic, molecular, and nuclear physics) from the point of view of supersymmetry, pointing out alternative and new interpretations to results which have recently appeared in the literature. Finally, Sec. V considers a couple of more nontrivial problems in atomic physics where again the Hamiltonian exhibits supersymmetry.

II. SUPERSYMMETRIC QUANTUM MECHANICS

A one-dimensional quantum mechanical Hamiltonian

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}$$

is said to be supersymmetric[3] if the corresponding potentials $V_\pm(x)$ is related according to

$$V_\pm(x) = (U')^2/8 \mp U''/4\,, \tag{1}$$

where primes denote derivatives with respect to x of a function $U(x)$. In such a situation, the bosonic (H_+) and fermionic (H_-) components have a spectrum of eigenvalues that coincide except that the former has one extra state, namely a lowest state with eigenvalue zero and normalizable wave function $\exp[-\frac{1}{2}U(x)]$.

Let us state this result in another way. Given a potential V one can find the ground state energy and eigenfunction

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} + V(x)\right]\psi_0(x) = E_0\psi_0(x)\,. \tag{2}$$

Since the ground state eigenfunction has no nodes we can write $U = -2\log\psi_0(x)$ and U is real. Equation (2) then gives

$$V - E_0 = \frac{U'^2}{8} - \frac{U''}{4}\,. \tag{3a}$$

Therefore V_+ is V up to a redefinition of the zero of energy and V_- can be calculated from U,

$$V_+ = V - E_0\,, \quad V_- = V - E_0 + (U''/2)\,. \tag{3b}$$

Note also another feature of supersymmetry, that the lowest state of V_+ lies at zero energy.

The demonstration that H is supersymmetric hinges on the existence of the generators of supersymmetry Q, $\bar{Q}$ which together with H satisfy the commutation and anticommutation relations:

$$[Q,H] = [\bar{Q},H] = 0\,, \tag{4a}$$

$$\{\bar{Q},\bar{Q}\}_+ = \{Q,Q\}_+ = 0\,, \tag{4b}$$

$$\{\bar{Q},Q\}_+ = 2H\,. \tag{4c}$$

Equation (4a) states the invariance of the Hamiltonian under this symmetry. Equation (4b) expresses the fact that Q and $\bar{Q}$ are "fermionlike" and their anticommutator properties are relevant. Note that Eq. (4b) simply states that the square of the generators is zero. (We write these as anticommutators in deference to the four-space supersymmetry algebra where the Q's have indices.[4]) Finally Eq. (4c) closes the algebra through an anticommutator of Q with $\bar{Q}$. Since this algebra involves commutators and anticommutators, it does not form a Lie Algebra but rather a Graded Lie Algebra. From the algebra one can build up infinitesimal transformations $(1 + \theta Q + \bar{\theta}\bar{Q} + Ht)$. The unusual property here is that θ and $\bar{\theta}$ are anticommuting numbers and hence it is difficult to visualize this transformation. One thing we can conclude is that a transformation in the Q direction followed by one in the $\bar{Q}$ direction can generate time translations through Eq. (4c).

The generators have been constructed[3] that operate on the x variable and on the 2×2 matrix space of $H_\pm$:

$$Q \equiv [p - i(U'/2)]\sigma^+\,, \quad \bar{Q} \equiv [p + i(U'/2)]\sigma^-\,, \tag{5}$$

where $\sigma^\pm$ are the 2×2 matrices

$$\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

Because of the relations $\{\sigma^-,\sigma^+\}_+ = 1$ and $[\sigma^+,\sigma^-] = \sigma_z$, it is easily verified that Eqs. (4) are satisfied and that

$$H = \frac{1}{2}(Q\bar{Q} + \bar{Q}Q) = \frac{1}{2}\left(p^2 + \frac{U'^2}{4}\right)I + \frac{U''}{4}\sigma_z\,, \tag{7}$$

in agreement with Eq. (1).

The two constituents, $H_+ = \frac{1}{2}\bar{Q}Q$ and $H_- = \frac{1}{2}Q\bar{Q}$, of the supersymmetric Hamiltonian H have, respectively, the potentials V_+ and V_-. That the spectra of these two potentials are simply related can be seen as follows. If ψ_n is an eigenfunction of H_+ with eigenvalue E_n, that is,

$$\tfrac{1}{2}\bar{Q}Q\psi_n = E_n\psi_n\,, \tag{8}$$

then $Q\psi_n$ is an eigenfunction of $H_- = \frac{1}{2} Q\bar{Q}$ with the same eigenvalue because

$$\tfrac{1}{2} Q\bar{Q}(Q\psi_n) = Q(\tfrac{1}{2}\bar{Q}Q\psi_n) = E_n(Q\psi_n) . \quad (9)$$

Thus the eigenvalues of H_+ and H_- coincide, except for one, namely the lowest eigenstate ψ_0 of H_+ where it follows from Eq. (5) and $\psi_0 = \exp(-\frac{1}{2}U)$ that $Q\psi_0 = 0$. The lowest eigenvalue at zero is, therefore, present only in the H_+ spectrum but all other eigenvalues are duplicated in H_+ and H_-. Such a spectrum constitutes a supersymmetric spectrum in quantum mechanics and we can refer to the two potentials, V_+ and V_-, as supersymmetric partners. In the context of supersymmetry in field theories, the bosons correspond to H_+ and their counterpart fermions to H_-.

III. ASSOCIATION OF SUPERSYMMETRY WITH $g = 2$ FOR THE ELECTRON

It is universally accepted that the simplest quantum mechanical problem is that of the one-dimensional harmonic oscillator (perhaps, equally so in classical mechanics!). It also affords the simplest illustration of supersymmetry and has been extensively discussed,[3] although the specific association we wish to develop in this section with the problem of $g = 2$ electrons in a magnetic field has not been appreciated as much.

The Hamiltonian $\frac{1}{2}(p^2 + x^2)$ has eigenvalues $n + \frac{1}{2}$ and $U = x^2$. (We have chosen units with the mass m, oscillator frequency ω, and $\hbar$ set equal to unity.) Therefore, from Eq. (7) the supersymmetric Hamiltonian is

$$H = \tfrac{1}{2}(p^2 + x^2) + \tfrac{1}{2}\sigma_z . \quad (10)$$

The two potentials are

$$V_\pm = \tfrac{1}{2}x^2 \mp \tfrac{1}{2} \quad (11)$$

and the supersymmetric spectrum is shown in Fig. 1. We will now identify the physical system described by this spectrum.

A. Schrödinger–Pauli equation

Consider an electron in a uniform magnetic field $\mathbf{B} = (0,0,B)$. Let the gauge be so chosen that the vector potential is $\mathbf{A} = (0,Bx,0)$. The nonrelativistic Schrödinger–Hamiltonian in which the supplementary assumption is made that the spin magnetic moment couples to B

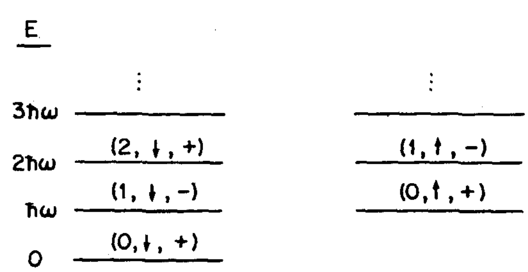

Fig. 1. Spectrum of the supersymmetric Schrödinger–Hamiltonian for an electron in a magnetic field. Levels have been labelled by (n, σ_z, π), where n is the Landau quantum number of orbital motion, σ_z the spin projection, and π the parity of the state.

with a g factor g, is given by[10]

$$H = \frac{1}{2m}\left(p_y + \frac{eBx}{c}\right)^2 + \frac{1}{2m}(p_x^2 + p_z^2) + g\left(\frac{e\hbar}{2mc}\right) s_z B , \quad (12)$$

where $s_z = \frac{1}{2}\sigma_z$, and m is the mass of the electron. The spectrum of this Hamiltonian is well known[10] and, in fact, has been of considerable interest in the last decade in atomic physics and astrophysics, both for cyclotron and synchrotron emission in strong magnetic fields and, with an additional Coulomb attraction included, for the structure of atoms in such fields.[11]

For our purposes here, what is of interest is that the y and z motions in Eq. (12) correspond to free motion and the only nontrivial part of H is the one-dimensional motion in x. Here it reduces to a one-dimensional oscillator centered at $x = -(cp_y/eB)$ with cyclotron frequency $\omega = eB/mc$, along with the additional constant term in $s_z B$. In units of the cyclotron energy $\hbar\omega$, the energy levels are given by[10]

$$E/\hbar\omega = n + \tfrac{1}{2} \pm g/4 . \quad (13)$$

For the special case $g = 2$, these levels display the characteristic degeneracy of the supersymmetric pattern shown in Fig. 1.

B. Dirac equation

It is well known that the Dirac electron in an electromagnetic field can be treated in terms of a two component formalism from which the four-component spinor solutions can then be constructed.[12] The two component equation for an electron in a uniform magnetic field is identical to the Schrödinger equation with g set equal to 2 and E replaced by $[E^2 - (mc^2)^2]/(2mc^2)$.[13] In spite of this latter difference, supersymmetry can still be associated with $g = 2$.

The result can also be established directly for the Dirac equation,

$$H_D\Psi = i\hbar\frac{\partial}{\partial t}\Psi , \quad (14)$$

with

$$H_D = c\left(p_y + \frac{eBx}{c}\right)\alpha_y + cp_x\alpha_x + cp_z\alpha_z + \beta mc^2 . \quad (15)$$

This Hamiltonian has positive energy states of the form shown in Fig. 2(a). But there are of course negative energy states and hence the spectrum in Fig. 2(a) does not carry the hallmark of supersymmetry shown in Fig. 1. However, with the negative states filled and the holes reinterpreted as antiparticles, the resulting Hamiltonian is supersymmetric as we will now show.

Let $\Psi = \exp[(i/\hbar)(p_x x + p_y y + p_z z)]\psi(x)$, and further choose $p_y = p_z = 0$. There remains the one-dimensional equation

$$H_D\psi = (eBx\alpha_y + cp_x\alpha_x + \beta mc^2)\psi = E\psi . \quad (16)$$

Introduce dimensionless harmonic oscillator ladder operators $a^\dagger$ and a

$$a^\dagger + a = x(\hbar c/2eB)^{-1/2} ,$$
$$i(a^\dagger - a) = p_x(eB\hbar/2c)^{-1/2} , \quad (17)$$

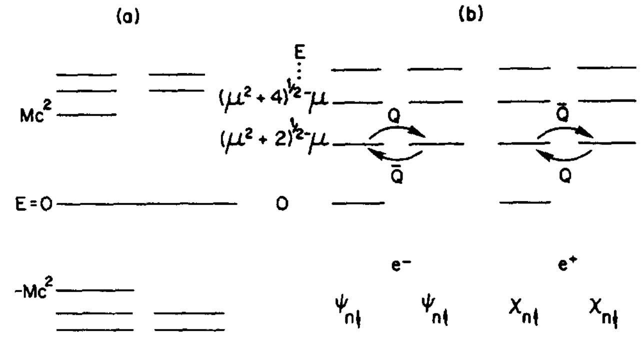

Fig. 2. (a) Spectrum of the Dirac Hamiltonian for an electron/positron in a magnetic field. (b) Spectrum of H, the supersymmetric Hamiltonian in Eq. (24), showing states, energy eigenvalues and supersymmetry transformations connecting the states. All other operations of Q and $\bar{Q}$ annihilate the states.

and rewrite Eq. (16) as

$$H_D=(i/\sqrt{2})a^\dagger(\alpha_x-i\alpha_y)-(i/\sqrt{2})a(\alpha_x+i\alpha_y)+\mu\beta; \tag{18}$$

all energies are measured in units of $[(mc^2)(\hbar eB/mc)]^{1/2}$ and μ is the square root of the dimensionless ratio of the rest mass energy to the cyclotron energy, $\mu=mc(c/\hbar eB)^{1/2}$. Using the representation

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

the eigenstates of H_D are

$$\psi_{n\uparrow}=\begin{pmatrix}C_+|n-1\rangle\\ 0\\ 0\\ iC_-|n\rangle\end{pmatrix},\quad \psi_{n\downarrow}=\begin{pmatrix}0\\ C_+|n\rangle\\ -iC_-|n-1\rangle\\ 0\end{pmatrix},$$

$$\chi_{n\uparrow}=\begin{pmatrix}C_-|n-1\rangle\\ 0\\ 0\\ -iC_+|n\rangle\end{pmatrix},\quad \chi_{n\downarrow}=\begin{pmatrix}0\\ C_-|n\rangle\\ iC_+|n-1\rangle\\ 0\end{pmatrix}, \tag{19}$$

where

$$C_\pm\equiv[(\epsilon_n\pm\mu)/2\epsilon_n]^{1/2}$$

and

$$\epsilon_n\equiv(\mu^2+2n)^{1/2},\quad n=0,1,2,\cdots. \tag{20}$$

The oscillator states are normalized as $\langle n|n'\rangle=\delta_{nn'}$ and $|0\rangle$ is the lowest. That Eq. (19) are the eigenstates of H_D with eigenvalues in Eq. (20) can be verified through use of $a|n\rangle=\sqrt{n}\,|n-1\rangle$:

$$H_D\psi_n=\epsilon_n\psi_n,$$
$$H_D\chi_n=-\epsilon_n\chi_n. \tag{21}$$

The $n=0$ states are singlets, all others are doublets in spin projection.

Using the eigenstates, Eq. (21), one can construct a spectral representation for the Hamiltonian:

$$H_D=\sum_{n=0}^{\infty}\epsilon_n\{\psi_{n\uparrow}\psi_{n\uparrow}^\dagger+\psi_{n\downarrow}\psi_{n\downarrow}^\dagger-\chi_{n\downarrow}\chi_{n\downarrow}^\dagger-\chi_{n\uparrow}\chi_{n\uparrow}^\dagger\}. \tag{22}$$

One can check that this agrees with the definition, Eq. (18), and satisfies the eigenvalue equations Eq. (21). The negative energy spectrum coming from the third and fourth terms extends to minus infinity which means there is no ground state. Dirac's resolution of the problem takes one out of the single particle framework. Recognizing that this Hamiltonian describes fermions, one supposes that the negative states are filled to obtain the physical ground state. Then the removal of an electron from a negative energy state takes positive energy. The single particle Dirac equation then describes an electron (ψ states) or lack of an electron (χ states), both of which correspond to positive energy with respect to the physical ground state. This reinterpretation results in the change in the signs of the χ terms in Eq. (22).[14] The new Hamiltonian $\hat{H}_D$, based on this new vacuum state is

$$\hat{H}_D=\sum_n\epsilon_n\begin{pmatrix}|n-1\rangle\langle n-1| & & & \\ & |n\rangle\langle n| & & \\ & & |n-1\rangle\langle n-1| & \\ & & & |n\rangle\langle n|\end{pmatrix}. \tag{23}$$

We used the identities $a=\Sigma_n n^{1/2}|n-1\rangle\langle n|$, and $a^\dagger=\Sigma_n n^{1/2}|n\rangle\langle n-1|$. By subtracting off the ground state energy one obtains a supersymmetric Hamiltonian:

$$H=\hat{H}_D-mc^2, \tag{24}$$

which has the spectrum shown in Fig. 2(b). For $n=0$, $\epsilon_n=\mu=mc^2$ so that H has a zero eigenvalue. The eigenstates from Eq. (19) are $\psi_{0\downarrow}$ and $\chi_{0\uparrow}$. For other values of n, there are the four states given in Eq. (19). Thus Fig. 2(b) duplicates the spectrum shown in Fig. 1, with the additional degeneracy now of states of the positron with the opposite spin projection. Note also that the eigenvalue ϵ_n-mc^2 for $\mu^2\gg 2n$ reduces precisely to $n(\hbar\omega)$ with $\omega=eB/mc$, as in Eq. (13), with $g=2$.

The supersymmetry generators are readily found:

$$Q=\sqrt{2}\sum_n(\epsilon_n-\mu)^{1/2}\times\begin{pmatrix}0 & |n-1\rangle\langle n| & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -|n\rangle\langle n-1| & 0\end{pmatrix},$$

$$\bar{Q}=Q^\dagger. \tag{25}$$

The algebra satisfied by these operators together with H is identical to the algebra given by Eqs. (4).

In summary, states of an electron (or positron) in a magnetic field with opposite spin projections are identified as supersymmetric partners. The degeneracy between these states which makes supersymmetry a good symmetry in the problem rests on g taking the value 2. Note that in this quantum mechanical problem, the bimodal attributes of spin projection (conjugates under time reversal) and parity play the role that the boson/fermion aspect does in supersymmetry in field theories. The operators Q and $\overline{Q}$ switch between these attributes.

C. A more general context for Eq. (10)

The Hamiltonian that we have discussed as describing the states of an electron (with $g=2$) in a magnetic field has a more general interpretation as well. It describes the states of any two level system that is coupled to an oscillator field. Once again, the case $g=2$ corresponds to the degenerate case, when the energy difference between the two levels is identical to the oscillator level spacing ($g\neq 2$ would correspond to a mismatch between these two energies). Thus, we can read Eq. (10) as the simplest Hamiltonian that combines a spin $\frac{1}{2}$ fermion (the two-level system) and a boson field (the oscillator). The so-called "Jaynes–Cummings" model,[15] which is extensively used in quantum optics[16] and in describing a "two-level atom" (such as Rydberg states of atoms with $|m| = l = n-1$, these symbols having their usual meaning in atomic physics) coupled to the radiation field,[17] adds to H in Eq. (10) the coupling between the two fields and can be described in our notation as

$$H_{JC} = H + k(Q + \overline{Q}) , \tag{26}$$

where H is as in Eq. (10), k a coupling constant, and Q and $\overline{Q}$ as in Eq. (5), i.e.,

$$Q = (p - ix)\sigma^{+} \equiv a\sigma^{+} ,$$
$$\overline{Q} = (p - ix)\sigma^{-} \equiv a^{\dagger}\sigma^{-} , \tag{27}$$

a and $a^{\dagger}$ being the annihilation and creation operators of the oscillator field. The coupling term in Eq. (26), therefore, raises or lowers the two-level system with an attendant absorption or emission of the oscillator quantum. The interest in H_{JC} is that it is an exactly soluble model. Given our discussion of the supersymmetry of H, the eigenvalues can, in fact, be written down trivially. A pair of degenerate levels of the supersymmetric H are coupled by the $(Q+\overline{Q})$ operator; the eigenstates of the full H_{JC} are, therefore, equal admixtures of the two states, with coefficients $\pm 2^{-1/2}$, and with eigenvalues

$$E^{\pm} = n + \tfrac{1}{2} \pm k(n+1)^{1/2} . \tag{28}$$

The degeneracy is lifted as a result of the coupling term in k breaking the supersymmetry.

IV. SUPERSYMMETRY IN THE THREE-DIMENSIONAL COULOMB AND OSCILLATOR HAMILTONIANS

For potentials with spherical symmetry in three dimensions, the Schrödinger equation reduces essentially to a single variable equation in the radial coordinate r, the dependence of the wave function on the angular variables being in the standard form of a spherical harmonic $Y_{lm}(\theta,\phi)$. The one-dimensional construction of supersymmetry can, therefore, be applied to the radial equation. This leads to some interesting results which also give a new slant to the $0(4)$ and $SU(3)$ symmetries mentioned earlier for the $-1/r$ and r^2 potentials, respectively.[2]

A. The hydrogen atom

A supersymmetric construction has been given[8] for the radial Schrödinger equation of the hydrogen atom,

$$\left(-\frac{1}{2}\frac{d^2}{dy^2} - E_n - \frac{1}{y} + \frac{l(l+1)}{2y^2}\right)\chi_{nl}(y) = 0 ,$$
$$\chi_{nl}(0) = 0 , \tag{29}$$

with $E_n = -1/2n^2$, $y = (mZe^2/\hbar^2)r$. With $U(y)$ defined as

$$U(y) = 2y/(l+1) - 2(l+1)\ln y , \tag{30}$$

Eq. (29) at fixed l defines the series of Bohr levels with $n \geqslant (l+1)$ in the potential

$$V_{+} = \tfrac{1}{2}(l+1)^{-2} - (1/y) + l(l+1)/2y^2 , \tag{30a}$$

and with energies $[(l+1)^{-2} - n^{-2}]/2$, the lowest eigenvalue defining the zero of the energy scale. Correspondingly, from Eq. (1) and Eq. (30), we have the partner

$$V_{-} = \tfrac{1}{2}(l+1)^{-2} - (1/y) + (l+1)(l+2)/2y^2 , \tag{30b}$$

which is easily recognized as also defining a series of Bohr levels but starting with $n = l+2$. We are speaking throughout of the same partial wave l, the angular part $Y_{lm}(\theta\phi)$ being common to both $V_{\pm}$ and playing no role in the discussion. But for the radial equation, the supersymmetric partner to Eq. (29) coincides formally with the radial equation for the next higher l value, i.e., with $l \to l+1$. As an example, consider $l=0$. V_{+} describes the ns states of the hydrogen atom with $n \geqslant 1$. The eigenvalues of V_{-} on the other hand, coincide formally with those of the np states, $n \geqslant 2$. Since there is the well-known hydrogenic degeneracy between ns and np ($n \geqslant 2$) states, we have a supersymmetric spectrum as described in Sec. II.

The authors[8] who advanced the construction of $V_{\pm}$ in the previous paragraph have interpreted the results as describing a supersymmetric connection between atoms in a column of the Periodic Table. Thus the $l=0$ case considered above is seen as describing the s states of H through V_{+} and of Li ($1s^2ns$ $n \geqslant 2$) through V_{-}. This interpretation is, however, problematical[18] and we see the connection somewhat differently. The key point of departure is that the radial equation (29) is not a true one-dimensional problem, being only on the half-line $(0,\infty)$. To apply the considerations of Sec. II, we should therefore transform first to a problem on the full-line $(-\infty,\infty)$, which is accomplished by a well-known[19] change of variables from y to $x = \ln y$. As we will see below, this converts the radial $1/r$ problem into that of a Morse potential and the supersymmetric construction gives two Morse potentials $V_{\pm}(x)$. Although algebraically one passes trivially from the Morse to the Coulomb problem, the interpretation of the supersymmetric spectrum is quite different. The double degeneracy of the excited states is not between states of the same n and different l (for example, s and p) but rather between states of the same l but different *n and, simultaneously, different* Z. Therefore, instead of providing connections between states of different atoms (with different numbers of electrons), we see the connection rather as between isoelectronic ions (one electron always). This connection is a straightforward expression of the simultaneous scaling of one electron atomic energies with Z and n.

Changing the variable y in Eq. (29) to $x = \ln y$ and simultaneously transforming the wave function $\chi(y)$ to $\psi(x) = \exp(-x/2)\chi$ gives

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} - E_n\, e^{2x} - e^x + \frac{1}{2}\left(l + \frac{1}{2}\right)^2 \right] \psi(x) = 0 . \tag{31}$$

With $E_n = -1/2n^2$, this describes a one-dimensional Morse potential $(e^{2x}/2n^2) - e^x$ whose eigenvalues are $-\frac{1}{2}(l+\frac{1}{2})^2$. Translating the zero of the energy scale to the lowest eigenvalue, with $l = n-1$, we define the potential

$$V_+(x) = (e^{2x}/2n^2) - e^x + \tfrac{1}{2}\,(\tfrac{1}{2} - n)^2 . \tag{32}$$

The spectrum of $V_+(x)$ for any fixed n has eigenvalues $\frac{1}{2}(n+l)(n-l-1)$, that is, a finite ($l \leqslant n-1$) number of eigenvalues starting at zero (for $l = n-1$). To carry out the supersymmetric construction with V_+ given as above, we have from Eq. (3) that the corresponding function $U(x)$ is

$$U(x) = 2(e^x/n) + (1-2n)x . \tag{33}$$

Alternatively, the basic definition $U(x) = -2 \log \psi_0(x)$, where $\psi_0(x)$ is the ground state wave function of Eq. (31), leads to the same result.

With $U(x)$ in hand, Eq. (3) is used again to obtain the supersymmetric partner

$$V_-(x) = e^{2x}/2n^2 - (1 - 1/n)e^x + \tfrac{1}{2}\,(\tfrac{1}{2} - n)^2 , \tag{34}$$

which has the same set of eigenvalues $\frac{1}{2}(n+l)(n-l-1)$ except for the missing zero eigenvalue (Fig. 3). Note that $V_-(x)$ is also a Morse potential that goes to the same asymptotic limits at $x = \pm\infty$ as does $V_+(x)$ but departs in between. See Fig. 4. Remarkably, the two different potentials have the same eigenvalues except for the lowest in V_+. From a different point of view, it has been noted before that there exist Morse potentials with different strengths which have such coincidences in their eigenvalue spectrum.[20] Also, during the course of our writing, we have come across another paper giving similar supersymmetric constructions for the $1/r$ and Morse potentials.[21]

Having carried out the construction with the true one-dimensional variable x, we can now transform back to y and the wave function $\chi(y)$ to get as a partner to Eq. (29),

$$\left[-\frac{1}{2}\frac{d^2}{dy^2} + \frac{1}{2n^2} - \left(1 - \frac{1}{n}\right)\frac{1}{y} + \frac{l(l+1)}{2y^2} \right] \chi(y) = 0 . \tag{35}$$

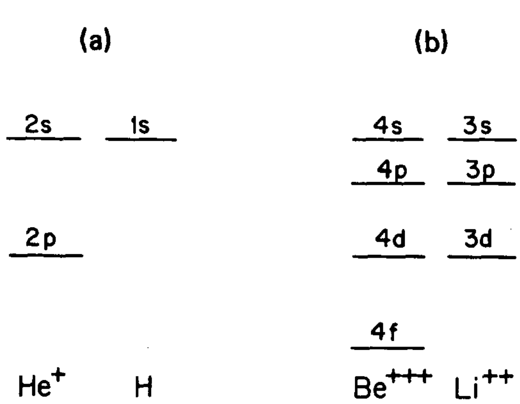

Fig. 3. Supersymmetric spectrum of states n,l and $(n-1),l$. States in atomic notation and eigenvalues $(n+l)(n-l-1)$ shown for (a) $n=2$, (b) $n=4$.

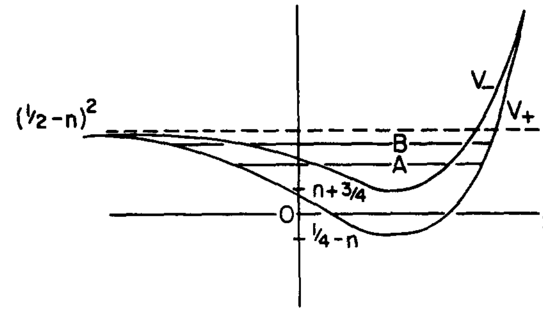

Fig. 4. Supersymmetric pair of Morse potentials $V_\pm(x)$: schematic. Both potentials support a finite number of bound states $A,B,\ldots$ that coincide in energy. In addition V_+ has a zero eigenvalue.

Unlike the relationship between Eq. (30a) and Eq. (30b), the link between Eq. (29) and Eq. (35) is that they both describe Coulomb states of the same l but different values of the principal quantum number. To see this, we observe that the only difference between the two equations lies in the coefficient of the $1/y$ term and that y is proportional to Z. Therefore, upon dividing Eq. (35) through by $(1-1/n)^2$ and redefining $(1-1/n)\,y$ as a new variable y, the two equations are identical except that Eq. (29) applies to the value n, and Eq. (35) to the value $n-1$. By absorbing the $(1-1/n)$ factor that multiplies y into the nuclear charge Z, this correspondence can be stated in terms of dimensional energies: Eq. (29) describes states with l and n and nuclear charge Z and having energy $-(Z^2/n^2)\,(me^4/\hbar^2)$, whereas Eq. (35) describes states with l and $n-1$ and charge $Z(1-1/n)$ but with the same eigenenergy. Figure 5 illustrates a few simple cases for specific choices of Z and n.

Thus supersymmetry links states of isoelectronic ions under the simultaneous change $n \rightarrow n-1$, $Z \rightarrow Z(1-1/n)$. This reflects, of course, the fact that the atomic energy expression involves Z and n only in the combination Z/n and therefore remains invariant under this simultaneous interchange. Further, as regards the radial wave function in Eq. (29), it is well known that y and n enter only in the combination y/n. This is clear from the very structure of this equation, that the only combinations involved are Z/n and y/n. Thus, the supersymmetry link amounts to a "dilatation" transformation $D_{n/(n-1)}$, such a transformation being one that scales coordinates:

$$D_a f(x) = f(ax) . \tag{36}$$

The relevance of the operator $D_{n/(n-1)}$ for group symme-

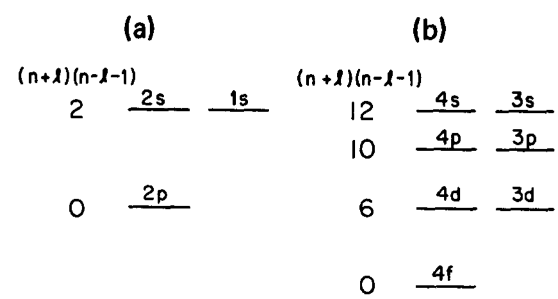

Fig. 5. Redrawing of Fig. 3 to show simultaneous scaling $n \rightarrow n-1$, $Z \rightarrow Z(1-1/n)$ that relates supersymmetric partners (a) $n=4$, $Z=4$, (b) $n=Z=2$.

tries of the hydrogenic radial functions has been noted before[22] but the above considerations through supersymmetry cast a new light on the problem.

B. The isotropic oscillator

The difference between working with the variables x and y for three-dimensional radial problems is also exemplified by applying these considerations to the isotropic oscillator:

$$\left(-\frac{1}{2}\frac{d^2}{dy^2}+\frac{1}{2}y^2+\frac{l(l+1)}{2y^2}-E_n\right)\chi_n(y)=0\,, \quad (37)$$

with $y=(m\omega/\hbar)^{1/2}r$ and $E_n=n+3/2$, $n=l, l+2,\ldots$. The procedure followed in Eqs. (30), (30a), (30b) would now give

$$U(y)=y^2-2(l+1)\ln y\,, \quad (38a)$$

and correspondingly

$$V_+(y)=\tfrac{1}{2}y^2+[l(l+1)/2y^2]-(l+\tfrac{3}{2})\,, \quad (38b)$$

$$V_-(y)=\tfrac{1}{2}y^2+[(l+1)(l+2)/2y^2]-(l+\tfrac{1}{2})\,. \quad (38c)$$

This would again establish as partners states with l and $l+1$ but, unlike in the Coulomb problem, the oscillator does not have such degeneracies (Fig. 6); rather, the degeneracies are staggered by two units in l; or, for the same l, the energy eigenvalues are staggered by two units. This connection emerges more naturally, albeit as a more trivial statement, when one works with a variable x on the full line.

Defining x slightly differently from our previous usage through $x=2\ln y$, and setting $\chi=\exp(x/4)\psi(x)$, Eq. (37) is transformed into

$$\left(-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{8}e^{2x}-\frac{1}{4}E_ne^x+\frac{(l+\frac{1}{2})^2}{8}\right)\psi(x)=0\,. \quad (39)$$

Once again, we have a Morse potential with

$$V_+(x)=\tfrac{1}{8}e^{2x}-\tfrac{1}{4}(n+\tfrac{3}{2})e^x+\tfrac{1}{8}(n+\tfrac{1}{2})^2\,, \quad (40)$$

and eigenvalue $(n-l)(n+l+1)/8$. For fixed l and $n\geqslant l$, the lowest eigenvalue is again at zero. Together with $U(x)=e^x-(n+\frac{1}{2})x$, this defines a supersymmetric partner

$$V_-(x)=\tfrac{1}{8}e^{2x}-\tfrac{1}{4}(n-\tfrac{1}{2})e^x+\tfrac{1}{8}(n+\tfrac{1}{2})^2\,. \quad (41)$$

We have here another pair of Morse potentials somewhat similar to the pair Eq. (32) and Eq. (34) and the sketch in Fig. 4. The coefficients and, in particular, their dependence on n are different but $V_\pm$ again have the same asymptotic limits, differ in between, and have all their eigenvalues except one coincide with each other.

Transforming Eq. (41) back to the variable y gives the supersymmetric partner to the isotropic oscillator in Eq. (37) as

$$\left(-\frac{1}{2}\frac{d^2}{dy^2}+\frac{1}{2}y^2+\frac{l(l+1)}{2y^2}+2-E_n\right)\chi=0\,, \quad (42)$$

with E_n again equal to $n+\frac{3}{2}$. This time the l value and the oscillator parameter are unchanged (in the Coulomb problem, the Coulomb strength Z was scaled to a new value), and the difference between Eq. (42) and Eq. (37) is a trivial shift by 2 units ($2\hbar\omega$ in real energy units) of the two potentials. The corresponding shift in eigenvalues for states of the same l is precisely that exhibited by the spectrum of the isotropic oscillator (Fig. 6). Note that this conclusion is very similar to the results established in Sec. III for the one-dimensional oscillator.

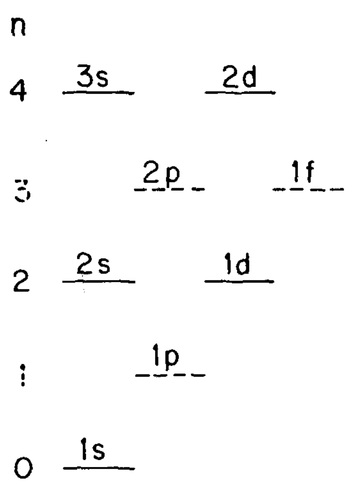

Fig. 6. Spectrum of the isotropic oscillator. States indicated by nuclear notation $\frac{1}{2}(n-l)+1,l$.

V. OTHER SUPERSYMMETRIC HAMILTONIANS IN ATOMIC PHYSICS

In the previous section we examined the supersymmetric spectra associated with the basic potentials of atomic and molecular physics, namely, the Coulomb and Morse potentials, respectively. Other more complicated atomic spectra also seem to exhibit supersymmetry in their spectra and we consider two examples here.

A. Singlet/triplet supersymmetry

In Sec. III we examined the one electron problem in a magnetic field as exhibiting supersymmetry, the two partners in the supersymmetric ladder of spectral states differing in the alternative projections σ_z of the electron spin. Interestingly, an analogous situation but with respect to the alternative spin values, $S=0$ (singlet) and $S=1$ (triplet), of two electrons appears in the spectrum of a two-electron atomlike helium. Consider singly excited states of total orbital angular momentum $L=0$, that is, the sequence of configurations $1s2s$, $1s3s$, ... $1sns$, ... Each of these configurations has $S=0$ or 1. Together with the ground state $1s^2$ this set forms the $L=0$ family of states of the helium atom below the ionization threshold.

The requirements of the Pauli principle, that the total wave function be antisymmetric, has two consequences. The ground state has only $S=0$ and the excited states have spatial wave functions that are, respectively, symmetric and antisymmetric for $S=0$ and 1. This leads to the well-known result that the energy of these states takes the form[23]

$$E=\langle T+V_{en}\rangle+\langle V_{ee}^{d}\rangle\pm\langle V_{ee}^{\rm exch}\rangle \quad \text{for} \quad S={}^0_1\,, \quad (43)$$

where T, V_{en}, V_{ee}^{d}, and $V_{ee}^{\rm exch}$ are, respectively, the kinetic energy, electron–nuclear potential energy, and the "direct" and "exchange" parts of the electron–electron interaction. The first two, which are the one-electron terms, and V_{ee}^{d}, which is the repulsion between two-electron density distri-

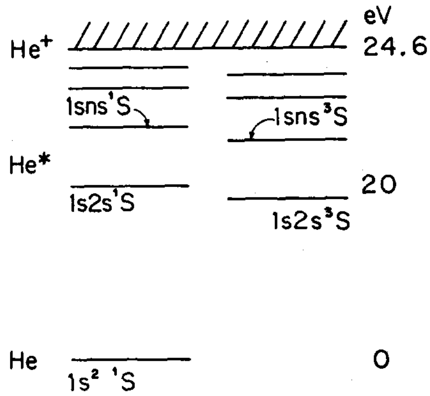

Fig. 7. $L = 0$ states of the He atom below the first ionization threshold.

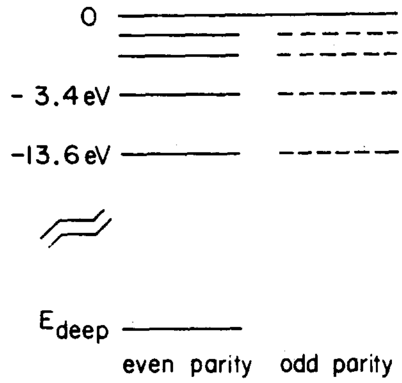

Fig. 8. Spectrum of the "one-dimensional Coulomb" potential, $-(|z| + \rho_c)^{-1}$ for $\rho_c \to 0$ (not to scale). Break on the energy scale represents that E_{deep} lies much lower and $E_{\text{deep}} \to -\infty$ as $\rho_c \to 0$.

butions, are common to $S = 0$ and 1, $\pm$ signs in the spatial wave function due to the requirements of symmetry/antisymmetry always squaring to $+1$. It is only in the exchange contribution, which is purely quantum mechanical and depends explicitly on the wave functions (not just densities), that the $\pm$ signs matter.[23] The net result is the canonical spectrum (Fig. 7), with each of the $1sns\ {}^3S$ states lying slightly lower that the $1sns\ {}^1S$ states. Neglect of the exchange term, the so-called Hartree approximation in atomic physics, makes the two-electron spectrum in Fig. 7 into a supersymmetric pattern, with a nondegenerate ground state, $1s^2\ {}^1S$, and pairs of degenerate, $1sns\ {}^1S$ and $1sns\ {}^3S$, excited states. Exchange can then be viewed as breaking this supersymmetry in the two-electron interaction.

B. Hydrogen atom in an infinitely strong magnetic field

The discovery of pulsars, which are believed to be neutron stars with very strong ($\simeq 10^{12}$ G) magnetic fields, led to considerable study of atomic structure in such a situation.[6] The strong field, say in the z direction, can be seen as restraining motion in the transverse x and y directions to distances of the order of the cyclotron radius $\rho_C = (c\hbar/eB)^{1/2}$, so that the Coulomb field in the hydrogen atom becomes effectively one dimensional (in z) in character: $-1/r \to -(\rho_C^2 + z^2)^{-1/2}$. In the limit of infinite field strength when $\rho_C \to 0$, we have then a "one-dimensional Coulomb potential," $-1/|z|$. In condensed matter physics, excitons, which are hydrogenlike systems, show similar magnetic field effects even with laboratory field strengths.

The spectrum of the $-1/|z|$ potential is known, and was first reported in this journal.[24] It is shown in Fig. 8 and consists of one odd parity and one even parity (under $z = -z$) state at each of the Bohr energies $-13.6\ \text{eV}/n^2$, $n = 1,2,\ldots$. In addition, the ground state, which is of even parity, has a logarithmically infinite binding energy. [The origin of this is easily made plausible by noting that whereas $\int_0 r^2 dr(1/r)$ is finite, $\int_0 dz(1/|z|)$ is logarithmically singular.] It is clear from Fig. 8 that here again we have a supersymmetric pattern. For the problem of a hydrogen atom in a very strong (but not infinite) magnetic field, the odd and even parity excited states coincide in first order but are slightly nondegenerate in higher-order terms in ρ_C, and the "deep" ground state is not at minus infinity but lies at $-13.6\ \text{eV}\ \ln^2(a_0/\rho_C)^2$, where a_0 is the Bohr radius.[25] Exact supersymmetry obtains in the one-dimensional limit, $\rho_c = 0$, but is slightly broken when ρ_c is small and finite.

A recent paper[26] noted the supersymmetry in the $-1/|z|$ problem, and the coincidence of the excited state energies in Fig. 8 with the usual Bohr energy values of the three-dimensional hydrogen atom, as an aspect of the so-called "shifted $1/N$ expansions" wherein the dimension N of the system is artifically varied to speed up convergence of the perturbation expansions. In this problem a connection is made between the $-1/r$ potential with $N = 3$ and the $-1/|z|$ potential with $N = 1$. Our discussion above makes contact with a concrete physical situation, namely a strong magnetic field, which has this effect of reducing the effective dimension by 2. Elsewhere,[27] one of us discussed this problem as an example of the compactification of dimensions, a subject again of considerable current interest in field theories.

In the context of the current discussion, note that any symmetric one dimensional potential well which supports only one bound state [such as, for instance, a delta function well, $-a\delta(x)$] can also be viewed as having a supersymmetric spectrum. All the continuum states are doubly degenerate, of odd and even parity, and together with the nodeless even-parity ground state constitute a supersymmetric family just as in the other examples we have discussed (except that the excited states are now all in the continuum).

To end this section, we note that in the two examples discussed in this section, we could not readily proceed to construct explicitly $V_\pm$ and the $Q,\bar{Q}$ operators. This is because the exact ground state wave function ψ_0 and therefore, U, are inaccessible unlike in the exactly solvable examples considered in Secs. III and IV. Nevertheless, from the nature of the spectrum and, in particular, the hallmark of a supersymmetric pattern that there is one nondegenerate ground state and the excited states are duplicated, it is easy to recognize a quantum mechanical Hamiltonian as possessing supersymmetry.

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On traveling round without feeling it and uncurving curves

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We discuss an elementary example of how, in a strong gravitational field, the basic kinematical concepts of acceleration and circular motion seem to have paradoxical properties. This allows an insight into the physical significance of space-time curvature without the use of difficult mathematical formalism.

Our intuition about the motion of macroscopic bodies is based on Newtonian physics. This is why some properties of fast motions and strong gravitational fields appear paradoxical to us. The so-called twin paradox is well known. Here we present and discuss another striking paradox of this type. We follow the tradition of many textbooks and articles on relativity (e.g., Ref. 1) that use free-falling elevators or accelerating spacecraft as an illustrative device.

A spacecraft can stay at a fixed distance from a spherical celestial body by using its engines to balance the gravitational attraction. There is, however, another possibility: by moving around the body along a circular orbit a centrifugal force is introduced, and less help is therefore needed from the engine to overcome the pull of gravity. The orbital velocity can even be such that the gravitational and centrifugal forces are equal. On such a *free* orbit the engine must obviously be switched off.

For orbital velocities smaller than the free one the engines must point *down* in order to reduce the attraction of gravity; in the opposite case, the engines will point *up* in order to reduce the effect of the centrifugal force. In both cases the thrust of the engine is correlated with the orbital velocity: the bigger the difference between the orbital speed and the free velocity, the stronger the engine power has to be to prevent the spacecraft from leaving the orbit. All that is rather obvious and everybody would certainly agree that Fig. 1 makes sense and describes a general situation.

Figure 2 shows spacecraft with exactly the same engines working with exactly the same thrust, but moving with different orbital speeds at a fixed distance from the central body. Is this possible? Could it be that orbiting spacecraft which use *the same* engine thrust circle with *different* orbital speeds? In other words: is it possible that acceleration on a given circular orbit does not depend on angular velocity. However paradoxical it might sound, the answer is "yes."

The situation described in Fig. 2 is possible on a circular orbit at which the velocity of free motion equals the velocity

