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[4]For a more detailed discussion of the necessary and sufficient conditions for symmetrical ideal solutions, see Ref. 5, Chap. 4.

[5]A. Ben-Naim, *Water and Aqueous Solutions* (Plenum, New York, 1974).

[6]A. Ben-Naim, J. Phys. Chem. **82**, 792 (1978).

[7]R. W. Gurney, *Ionic Processes in Solutions* (McGraw-Hill, New York, 1953).

[8]C. Tanford, *The Hydrophobic Effect* (Wiley-Interscience, New York, 1973).

[9]A. Ben-Naim and Y. Marcus, J. Chem. Phys. **80**, 4438 (1984).

[10]A. Ben-Naim and Y. Marcus, J. Chem. Phys. **81**, 2016 (1984).

[11]In experiment I*', let $N_A^\alpha = N_A^\beta$ be the initial concentrations of A in the two phases α and β. We can choose the solvation free energies of A in these two phases in such a way that the partition coefficient at equilibrium is $K_A = (\bar{N}_A^\beta/\bar{N}_A^\alpha)_{eq} = \exp\{W(A|\alpha) - W(A|\beta)/kT\}$, which fulfills the condition $\Delta G^{I*'}/kT(N_A^\alpha + N_A^\beta) = \ln 2 + \ln\{\sqrt{K_A}/(1+K_A)\} = -\ln 2$. A choice of $K_A = 12.276$ or $K_A = (12.276)^{-1}$ is a solution of this equation. For the "demixing" experiment I*, we start with a homogeneous distribution of A and B in the system. We define K_A and K_B as above and require that the Gibbs energy change be $\Delta G^{I*} = -kT(N_A + N_B)\ln 2$, where N_A and N_B are the total number of A and B molecules in the system. A choice of $K_A = 12.276$ and $K_B = (12.276)^{-1}$ will lead to an almost complete demixing of A and B (about 93% of A's in one compartment and 93% of B's in the other).

[12]Isaiah 11:3.

Coherent states of a harmonic oscillator

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The coherent states of a harmonic oscillator are introduced following Schrödinger's definition and the equivalence with other definitions is established. The basic properties of these states are discussed in some detail.

I. INTRODUCTION

The coherent states of a harmonic oscillator (hereafter referred to as CS) have been used in the quantum mechanical description of coherent light sources and in communication theory at optical frequencies. Although this set of states is recognized as a basic mathematical tool for dealing with those physical situations in which the harmonic oscillator is a model, it has not been given adequate attention in the usual textbooks on quantum mechanics. Discussions of these states are found in technical papers[1] and in specialized books on quantum optics[2] and communication theory.[3] In this article we discuss the basic properties of the CS that can be derived with the help of elementary quantum mechanics.

The history of the CS goes back to the early days of quantum mechanics, when around 1926, Schrödinger[4] reported the existence of a certain class of states that displayed, in some sense, the classical behavior of the oscillator. To be more precise, the average energy of the oscillator in such a state is equal to the corresponding classical energy (minus the quantum-mechanical zero-point energy $\hbar\omega/2$) and the averages of x and p have the same oscillatory forms as in the classical case with definite phase. The CS have been called in the literature the minimum uncertainty coherent states, the Schrödinger coherent states, or the Glauber coherent states (when applied to the radiation oscillators).

On the basis of Schrödinger's work, we find that these states have the following properties:

(1) They are a subset of the three-parameter family of minimum uncertainty states[5] (at some fixed time, say $t = 0$):

$$\psi(x) = (\pi\hbar\lambda)^{-1/4} \exp[ixp_0/\hbar - (x - x_0)^2/(2\hbar\lambda)]. \quad (1)$$

The subset is fixed by the choice $\lambda = (m\omega)^{-1}$, where m is the mass and ω is the angular frequency of the oscillator.[6] Here, p_0 and x_0 are the averages of p and x, respectively, in the state $\psi(x)$. They correspond to the classical values of p and x at $t = 0$.

(2) The states follow the classical motion in the sense that $\psi(x)$ in (1), with $\lambda = (m\omega)^{-1}$, evolves in time, under the harmonic potential $V(x) = m\omega^2x^2/2$, into a similar state $\psi(x,t)$ that is obtainable from (1) by replacing p_0 and x_0 by their classical values $p_{cl}(t)$ and $x_{cl}(t)$, respectively.[5,6]

(3) The fluctuations Δx and Δp in such a state are independent of time and $\Delta x \Delta p = \hbar/2$ for all time.[5]

Schrödinger's main motivation appears to be directed towards the possibility of discovering states with similar properties for potentials other than the harmonic potential. In a letter to Max Planck, Schrödinger states the following[7]:

..."I was to produce a wave-packet, ..., which was practically confined to a small special region, and which as a matter of fact revolves in precisely the harmonic ellipses described by classical mechanics for an arbitrary long time without dispersing! I believe that it is only a question of computational skill to accomplish the same thing for the electron in the hydrogen atom. The transition from microscopic characteristic oscillations to the macroscopic "orbits" of classical mechanics will then be clearly visible,... ."

Wave packets describing the Kepler orbits in a hydrogenlike atom are yet to be discovered. This type of investigation is being carried out by Nieto *et al.*[6,8] From their work it appears that the harmonic oscillator is the only case for which we can talk about coherent states.

The CS in a particular form have also been used by Block and Nordsieck[9] and subsequently by Schwinger[10] in connection with certain aspects of the electromagnetic radiation. Glauber[11(a),(b)] (first introduced the coherent state), Klauder and Sudarshan,[2] and many others used these states extensively for dealing with the coherent properties and photon statistics of optical fields.

In the traditional form of the quantum electrodynamics, due originally to Dirac, the radiation field (the electric and magnetic fields) in a cavity is expressed as a linear superposition of normal modes.[12] This is where the harmonic oscillator enters the show. The field Hamiltonian is then expressible as the sum of oscillator-type Hamiltonians. The state of the radiation field is describable by the eigenstates of the radiation oscillators that are the well-known number states. However, these number states are found to be unsuitable for calculations when the description of the field involves the phase and the amplitude variables. A radiation mode such as a plane propagating wave is best described by a CS that is a particular linear combination of the number states expressing the cooperative behavior of the photons.

The coherent state of an oscillator of mass m and angular frequency ω has been introduced in the literature in the following equivalent ways:

(a) The CS are generated from the ground state $|0\rangle$ of the oscillator by the displacement operator $D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a)$ acting on it, where a and $a^\dagger$ are the annihilation and creation operators, respectively, of the standard harmonic oscillator theory,[13] and α is a complex number, the star denoting complex conjugate (Sec. III).

(b) CS are the eigenstates of the annihilation operator a (Sec. III).

(c) CS are those minimum uncertainty states ($\Delta x\Delta p=\hbar/2$) that have the additional property $\Delta p=m\omega\Delta x$. The rest of the minimum uncertainty states are called squeezed states (Sec. V).

(d) CS are those minimum uncertainty states for which $(2m)^{-1}[(\Delta p)^2+m^2\omega^2(\Delta x)^2]$ takes the minimum value $\hbar\omega/2$ (Sec. V).

We shall establish the interrelationship among these definitions and that of Schrödinger's. The two definitions (a) and (b) are equivalent for the harmonic oscillator due to the particular type of commutation relations of the elements I, a, $a^\dagger$, and $a^\dagger a$ that form the Heisenberg–Weyl algebra.

Although we are dealing here with a mechanical oscillator, the theory can be readily adapted to the case of a single mode of the radiation field by setting $m=1$ and interpreting x as the normal coordinate and p as its time derivative. We shall give a few examples of the use of CS connected with radiation theory.

In Sec. II we have collected the relevant formulas in the harmonic oscillator theory that are assumed to be well known[13] and that will be constantly used in this article. In Sec. III we introduce the CS as a quasiclassical state following Schrödinger's ideas and establish the definitions (a) and (b). Various useful properties are then derived following Klauder and Sudarshan[2] in Sec. IV. In Sec. V we calculate some important fluctuations and establish the equivalence of (c) and (d).

II. REVIEW OF THE PROPERTIES OF a AND $a^\dagger$

The quantum theory of an oscillator[13] is conveniently treated in terms of the two operators a and $a^\dagger$, defined below, instead of the usual x and p. These operators naturally appear in the theory. Consider an oscillator with the Hamiltonian

$$H(t)=(2m)^{-1}[\,p^2(t)+m^2\omega^2x^2(t)\,], \tag{2}$$

where $x(t)$ and $p(t)$ are operator-valued functions of time t in the Heisenberg picture. They satisfy the commutation relations[14]

$$[x(t),p(t)]=i\hbar. \tag{3}$$

The Heisenberg equations of motion for $x(t)$ and $p(t)$ are

$$\dot{x}(t)=(i\hbar)^{-1}[x(t),H(t)]=p(t)/m, \tag{4}$$

$$\dot{p}(t)=-m\omega^2x(t). \tag{5}$$

These equations have exactly the same form as the corresponding classical equations. To solve this coupled system of equations, we introduce the two operators $a(t)$ and $a^\dagger(t)$ defined by

$$a(t)=[m\omega x(t)+ip(t)]/\sqrt{(2m\hbar\omega)}, \tag{6}$$

$$a^\dagger(t)=[m\omega x(t)-ip(t)]/\sqrt{(2m\hbar\omega)}. \tag{7}$$

These correspond to the normal coordinate and its complex conjugate, which are introduced to decouple the classical system of equations similar in form to (4) and (5).

Equations (4) and (5) then reduce to

$$\dot{a}(t)=[a(t),H]/(i\hbar)=-i\omega a(t), \tag{8a}$$

$$\dot{a}^\dagger(t)=i\omega a^\dagger(t). \tag{8b}$$

Hence,

$$a(t)=a(0)\exp(-i\omega t), \tag{9a}$$

$$a^\dagger(t)=a^\dagger(0)\exp(i\omega t), \tag{9b}$$

where $a(0)=a$, $a^\dagger(0)=a^\dagger$ are the operators $a(t)$ and $a^\dagger(t)$, respectively, at $t=0$. [Any $A(0)$ will be written as A.]

From the commutation relation

$$[x(t),p(t)]=[x,p]=i\hbar,$$

it follows, by the use of (6) and (7) that

$$[a(t),a^\dagger(t)]=[a,a^\dagger]=1. \tag{10}$$

The Hamiltonian (2) becomes

$$H=\hbar\omega[a^\dagger(t)a(t)+1]$$

$$=\hbar\omega[a^\dagger a+1]. \tag{11}$$

The operator $N=a^\dagger a$, called the number operator, commutes with H. Hence they have simultaneous eigenstates. It is well known[13] that eigenvalues of N are 0,1,2,... . The eigenstates corresponding to the eigenvalue n is denoted by $|n\rangle$. The following properties of $|n\rangle$ are also well known[14]:

$$a|0\rangle=0, \tag{12}$$

$$a|n\rangle=\sqrt{n}\,|n-1\rangle, \tag{13}$$

$$a^\dagger|n\rangle=\sqrt{n+1}\,|n+1\rangle \tag{14}$$

$$|n\rangle=(a^{\dagger n}/\sqrt{n!})|0\rangle, \tag{15}$$

$$\langle n|n'\rangle=\delta(n,n'), \tag{16}$$

$$\sum_{n=0}^{\infty}|n\rangle\langle n|=I, \tag{17}$$

where I is the identity operator. The set $\{|n\rangle\}$ forms a basis for the Hilbert space of the harmonic oscillator and is more convenient than the usual $|x\rangle$ or $|p\rangle$ basis to deal with the properties of the harmonic oscillator. With the help of the completeness relation (17) we can write any state $|\psi\rangle$ and any operator (acting on the Hilbert space of the oscillator) in terms of $|n\rangle$. For example

$$|\psi\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n|\psi\rangle, \tag{18}$$

$$A = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |n\rangle \langle n|A|m\rangle \langle m|. \tag{19}$$

In particular

$$a = \sum_{n=0}^{\infty} \sqrt{n}|n-1\rangle \langle n|, \tag{20}$$

$$a^\dagger = \sum_{n=0}^{\infty} \sqrt{(n+1)}|n+1\rangle \langle n|. \tag{21}$$

These formulas, which are assumed to be well known, will be constantly used throughout this article.

III. COHERENT STATES, DEFINITIONS

Following the original ideas of Schrödinger we intend to find a state $|\alpha(t)\rangle$, called the coherent state of the oscillator (a state that will satisfy the time-dependent Schrödinger equation of the oscillator) that satisfies the following two conditions[15,16]

(a) $\langle\alpha(t)|x|\alpha(t)\rangle = x_{\rm cl}(t)$

and

(b) $\langle\alpha(t)|H|\alpha(t)\rangle = E_{\rm cl}$, the classical energy.

The classical coordinate $x_{\rm cl}(t)$ of the oscillator has the form

$$x_{\rm cl}(t) = \lambda\,[\alpha \exp(-i\omega t) + \alpha^* \exp(i\omega t)], \tag{22}$$

where $\alpha = |\alpha|\exp(i\theta)$ and λ is a suitably chosen number (to be conveniently fixed later on). $E_{\rm cl} = 2m\lambda^2|\alpha|^2\omega^2$. The energy is being measured from the ground-state energy of the oscillator, that is the zero point energy $\hbar\omega/2$ (which may be considered to be used in confining the particle in the wave packet) is substracted from the Hamiltonian. H is then simply equal to $h\omega a^\dagger a$.

Using (6) and (7), for $t=0$, we have

$$x = \left[\sqrt{\hbar/(2m\omega)}\right](a + a^\dagger) \equiv x_0(a + a^\dagger). \tag{23}$$

Since

$$|\alpha(t)\rangle = \exp[-i\omega t(a^\dagger a)]|\alpha\rangle, \tag{24}$$

condition (a) reduces to

$$x_{\rm cl}(t) = x_0\langle\alpha|[\exp(i\omega t a^\dagger a)](a + a^\dagger)[\exp(i\omega t a^\dagger a)]|\alpha\rangle. \tag{25}$$

The right-hand side of (24) is simplified by making use of the operator identity[17]

$$[\exp(\xi a^\dagger a)]\, f(a,a^\dagger)\,[\exp(-\xi a^\dagger a)] = f[a\exp(-\xi), a^\dagger \exp(\xi)], \tag{26}$$

whenever f is expandable in a power series. With $\xi = i\omega t$, we have, from (25) and (26),

$$x_0\langle\alpha|a|\alpha\rangle \exp(-i\omega t) + x_0\langle\alpha|a^\dagger|\alpha\rangle \exp(i\omega t) = \lambda\alpha\exp(-i\omega t) + \lambda\alpha^* \exp(i\omega t).$$

With the choice of $\lambda = x_0$, condition (a) leads to

$$\langle\alpha|a|\alpha\rangle = \alpha. \tag{27}$$

We shall see presently that (27) does not uniquely define $|\alpha\rangle$. However, (27) with condition (b) uniquely fixes $|\alpha\rangle$, as we shall see below.

We now introduce two operators a' and $a'^\dagger$ defined by

$$a' = a + \alpha, \tag{28a}$$

$$a'^\dagger = a^\dagger + \alpha^*. \tag{28b}$$

Evidently $[a,a^\dagger] = [a',a'^\dagger]$. A displacement operator $D(\alpha)$ that is unitary and generates the transformation (28) can now be defined by the equations:

$$a' = D^\dagger(\alpha) a D(\alpha), \tag{29a}$$

$$a'^\dagger = D^\dagger(\alpha) a^\dagger D(\alpha), \tag{29b}$$

$$D(\alpha)D^\dagger(\alpha) = D^\dagger(\alpha)D(\alpha) = I. \tag{30}$$

Let $|\alpha'\rangle$ be the state defined by

$$|\alpha'\rangle = D^\dagger(\alpha)|\alpha\rangle. \tag{31}$$

From Eqs. (29) and (31) we then obtain

$$\langle\alpha'|a'|\alpha'\rangle = \langle\alpha|D(\alpha)a'D^\dagger(\alpha)|\alpha\rangle = \langle\alpha|a|\alpha\rangle = \alpha. \tag{32}$$

Hence, it follows that

$$\langle\alpha'|a|\alpha'\rangle = \langle\alpha'|a' - \alpha|\alpha'\rangle = 0. \tag{33}$$

Since Eq. (33) can be satisfied by the choice of $|\alpha'\rangle$ as any of the eigenstates $|n\rangle$, there is no unique $|\alpha\rangle$ that satisfies Eq. (27). However, condition (b) uniquely fixes $|\alpha\rangle$ in the following way.

As mentioned before, we are measuring energies from the ground states so that the oscillator Hamiltonian $H = \hbar\omega a^\dagger a$. We also define $H' = \hbar\omega a'^\dagger a'$, where a' and $a'^\dagger$ are given by Eq. (28). Now using Eqs. (24), (31), and (33) we have

$$\begin{aligned}\langle H\rangle &= \langle\alpha(t)|H|\alpha(t)\rangle \\ &= \langle\alpha(t)|D(\alpha)H'D^\dagger(\alpha)|\alpha(t)\rangle \\ &= \langle\alpha'(t)|H'|\alpha'(t)\rangle \\ &= \hbar\omega|\alpha|^2 + \langle\alpha'|H|\alpha'\rangle \\ &= E_{\rm cl} + \langle\alpha'|H|\alpha'\rangle.\end{aligned} \tag{34}$$

Equation (34) shows that (b) is satisfied if and only if

$$\langle\alpha'|H|\alpha'\rangle = \hbar\omega\langle\alpha'|a^\dagger a|\alpha'\rangle = 0. \tag{35}$$

There is only one $|\alpha'\rangle \equiv |0\rangle$ that satisfies Eq. (35). Hence the state satsifying (a) and (b) is unique and using Eq. (31) is given by

$$|\alpha(t)\rangle = [\exp(-i\omega t a^\dagger a)]D(\alpha)|0\rangle, \tag{36a}$$

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{36b}$$

We note that the ground state $|0\rangle$ of the oscillator is a

member of the set $\{|\alpha\rangle\}$ corresponding to $\alpha = 0$, since $D(0) = I$. We also remind the reader that the notation[2] $|\alpha\rangle$ should be written as $|\alpha_1,\alpha_2\rangle$, where $\alpha = \alpha_1 + i\alpha_2$, α_1, α_2 real so that a CS such as $|1\rangle \equiv |1,0\rangle$ may not be confused with the number state $|1\rangle$. Whenever a chance for confusion will appear, we shall use the extended notation.

Now we set to find an explicit form for $D(\alpha)$. To this effect we assume that there exists a Hermitian operator $h(\alpha)$ such that

$$D(\alpha) = \exp[ih(\alpha)],$$

the right-hand side being unitary. Then Eqs. (29) reduce to

$$\exp[-ih(\alpha)]a\exp[ih(\alpha)] = a + \alpha,$$

$$\exp[ih(\alpha)]a^\dagger \exp[ih(\alpha)] = a^\dagger + \alpha^*.$$

Using the operator identity[18]

$$\exp(A)B\exp(-A) = B + [A,B] + (\tfrac{1}{2})[A,[A,B]] + \cdots, \quad (37)$$

we get

$$\exp[ih(\alpha)]a\exp[ih(\alpha)] = a - i[h(\alpha),a] + \cdots, \quad (38)$$

and similar expression for the equation involving $a^\dagger$. Hence, we must have

$$[h(\alpha),a] = i\alpha$$

$$[h(\alpha),a^\dagger] = -i\alpha^*.$$

It is easily verified that these two equations are satisfied if we take

$$h(\alpha) = \lambda a^\dagger + \mu a, \quad (39)$$

where λ and μ are constants, with

$$\lambda = -i\alpha,$$

$$\mu = i\alpha^*.$$

Then the explicit form of $D(\alpha)$ may be expressed as

$$D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a], \quad (40)$$

which is found to satisfy Eq. (29).

The inverse of $D(\alpha)$ is easily found to be

$$D^{-1}(\alpha) = \exp[-\alpha a^\dagger + \alpha^* a] = D(-\alpha). \quad (41)$$

We also have from Eqs. (36b), (28), and (29),

$$a|\alpha\rangle = aD(\alpha)|0\rangle = D(\alpha)(a+\alpha)|0\rangle = \alpha|\alpha\rangle, \quad (42)$$

which expresses $|\alpha\rangle$ as the eigenstate of a with eigenvalue α, where α is any complex number. Equation (42) has been used by Glauber[11] as the defining property of CS. The expression

$$|\alpha\rangle = \exp[\alpha a^\dagger - \alpha^* a]|0\rangle. \quad (43)$$

$$= \exp[-|\alpha|^2/2]\exp(\alpha a^\dagger)|0\rangle, \quad (44)$$

can also be obtained by solving Eq. (42) as an eigenvalue equation. To derive Eq. (44) from (43) we have made use of the operator identity[18]

$$\exp(A+B) = \exp A \exp B \exp(-[A,B]/2),$$

where $[A,[A,B]] = 0$.

From Eq. (44), we obtain by using

$$\exp(\alpha a^\dagger) = \sum_{n=0}^{\infty} \left(\frac{\alpha^n}{n!}\right)(a^\dagger)^n,$$

and Eq. (15),

$$|\alpha\rangle = \exp\left(\frac{-|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\left(\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle, \quad (45)$$

which is the expression for $|\alpha\rangle$ in the number basis. The x and p representations of $|\alpha\rangle$ can also be derived from Eq. (45).

The number state $|n\rangle$ represents the state with exactly n photons. Hence, the probability that the state $|\alpha\rangle$ has n photons is obtained from Eq. (45):

$$|\langle n|\alpha\rangle|^2 = (|\alpha|^{2n}/n!)\exp(-|\alpha|^2), \quad (46)$$

which is a Poisson distribution with average number of photons $\langle n\rangle = |\alpha|^2$, average square number $\langle n^2\rangle = |\alpha|^4 + |\alpha|^2$, and fluctuation in n about its mean $\langle n\rangle$ equal to $|\alpha|^2$. The idea becomes more relevant when we talk about a radiation mode where a sine wave is known to deliver photons according to the Poisson distribution. In this sense $|\alpha\rangle$ provides a quantum description of the classical field.

IV. PROPERTIES OF COHERENT STATES

Before going into the physical properties of CS, we investigate certain mathematical aspects that make these states useful in situations where the harmonic oscillator is a model. The important mathematical properties of these states are the following:

(1) Two different CS are not orthogonal. This follows from Eq. (45):

$$\langle\alpha|\beta\rangle = \exp\frac{-|\alpha|^2}{2}\exp\frac{-|\beta|^2}{2}\sum_n\sum_m \frac{(\alpha^{*n}\beta^m)}{\sqrt{(n!m!)}}\langle n|m\rangle$$

$$= \exp\frac{-|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^*\beta. \quad (47)$$

The overlap $\langle\alpha|\beta\rangle$ is never zero. It is also a continuous function of the complex variables α and β over the whole of the complex plane.

(2) The set $\{|\alpha\rangle\}$ is linearly dependent. However, any finite number of distinct $|\alpha\rangle$'s are linearly independent.

We have noticed that α is any complex number so that there are infinitely many $|\alpha\rangle$'s. The first part of the theorem effectively says that any one of the CS can be expressed linearly in terms of all the others.

Many different types of relations exist exhibiting the linear dependence of CS. We state one simple case here. Writing $\alpha = r\exp(i\theta)$ and $d^2\alpha \equiv r\,dr\,d\theta$, we obtain from Eq. (45),

$$\int \alpha^m|\alpha\rangle d^2\alpha = \sum_{n=0}^{\infty}\left(\frac{|n\rangle}{\sqrt{n!}}\right)\int_0^\infty r^{n+m+1}\times\exp\left(\frac{-r^2}{2}\right)dr \times\int_0^{2\pi}\exp[i(m+n)\theta]\,d\theta, \quad (48)$$

where m is any nonzero integer. Since the r integral in Eq.

(48) is bounded for every value of n, we have

$$\int \alpha^m |\alpha\rangle d^2\alpha = 0, \tag{49}$$

for every natural number m.

We will show under (3), how any $|\alpha\rangle$ can be expressed in terms of the others. The second part of the theorem is simply demonstrated by taking any linear combination of a finite number n of states $|\alpha_i\rangle$ and setting it to zero:

$$\sum_{k=1}^{n} c_k |\alpha_k\rangle = 0.$$

By taking inner product with $\langle\alpha|$, where α is any complex number, we have

$$\sum_{k=1}^{n} c_k \langle\alpha|\alpha_k\rangle = 0. \tag{50}$$

Since $\langle\alpha|\alpha_k\rangle$ are linearly independent functions of α, expression (50) can vanish identically only when all the c_k's are zero.

(3) The set $\{|\alpha\rangle\}$ is complete, that is, we have a completeness relation of the form

$$\pi^{-1} \int |\alpha\rangle \langle\alpha| d^2\alpha = I, \tag{51}$$

where the integration is over the entire complex plane.

To verify this statement we show that for every $|\psi\rangle$ and $|\phi\rangle$, we have

$$\pi^{-1} \int \langle\psi|\alpha\rangle \langle\alpha|\phi\rangle d^2\alpha = \langle\psi|\phi\rangle. \tag{52}$$

Introducing the number basis and using

$$\langle n|\alpha\rangle = (\alpha^n/\sqrt{n!})\exp(-|\alpha|^2/2), \tag{53}$$

[which is obtained from Eq. (45)] the left-hand side of Eq. (52) can be expressed as

$$\pi^{-1} \sum_n \sum_m \langle n|m\rangle \left[\int_0^\infty \left(\frac{r^{n+m+1}}{\sqrt{(n!m!)}} \right) \exp(-r^2) dr \times \int_0^{2\pi} \exp[i(n-m)\theta] d\theta \right] \langle m|\phi\rangle$$

$$= \sum_n \langle\psi|n\rangle \langle n|\phi\rangle$$

$$= \langle\psi|\phi\rangle,$$

where we have used the fact that the θ integral is equal to $2\pi\delta_{nm}$ and

$$\int_0^\infty r^s e^{-r} dr = s!.$$

The CS provide an alternative (continuous) basis (nonorthogonal) for the Hilbert space of the oscillator the other being the well-known number states. In the following subsections we use Eq. (51) for obtaining the CS representations of vectors and operators.

A. Expansion of a state in the CS basis

Any vector $|\psi\rangle$ (in the Hilbert space of the oscillator) can be expressed in the CS basis, by using Eq. (51):

$$|\psi\rangle = \pi^{-1} \int |\alpha\rangle \langle\alpha|\psi\rangle d^2\alpha. \tag{54}$$

In particular, any of the CS, say $|\beta\rangle$, can be expressed as

$$|\beta\rangle = \pi^{-1} \int |\alpha\rangle \langle\alpha|\beta\rangle d^2\alpha$$

$$= \pi^{-1} \int |\alpha\rangle \exp\left(-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^*\beta \right) d^2\alpha. \tag{55}$$

Equation (55) reveals the linear dependence of the infinite set $\{|\alpha\rangle\}$. The CS are said to be *overcomplete*; a sloppy way of saying the same thing is that there are more states than are necessary for expressing any state in terms of CS. This is usually shown in the following way:

Multiply Eq. (47) by $\exp(-im\theta)$ and integrate over θ and 0 to 2π. Here $\alpha = r\exp(i\theta)$ and m is a positive integer including zero. We then have

$$\int_0^{2\pi} |\alpha\rangle \exp(-im\theta) d\theta = \exp\left(\frac{-r^2}{2} \right) \sum_{n=0}^{\infty} |n\rangle \left(\frac{r^n}{\sqrt{n!}} \right) \times \int_0^{2\pi} \exp[i(n-m)\theta] d\theta$$

$$= r^m \exp\left(\frac{-r^2}{2} \right) \left(\frac{2\pi}{\sqrt{m!}} \right) |m\rangle.$$

Hence

$$|m\rangle = (2\pi)^{-1} r^{-m} \exp\left(\frac{r^2}{2} \right) (\sqrt{m!}) \times \int_0^{2\pi} |\alpha\rangle \exp(-im\theta) d\theta.$$

This shows that the subset $\{|\alpha\rangle_r\}$, obtained by setting $r =$ any constant, supplies adequate number of states for expressing any number state $|m\rangle$ and hence any state $|\psi\rangle$ in the oscillator Hilbert space in terms of the subset.

Using Eq. (54), we can write any number state $|n\rangle$ as

$$|n\rangle = \int \left(\frac{\alpha^*}{\sqrt{n!}} \right) \exp\left(-\frac{|\alpha|^2}{2} \right) |\alpha\rangle d^2\alpha. \tag{56}$$

The expansion coefficients in Eq. (54) can also be expressed as

$$\langle\alpha|\psi\rangle = \sum_n \langle\alpha|n\rangle \langle n|\psi\rangle$$

$$= \exp\left(-\frac{|\alpha|^2}{2} \right) \sum_n \langle n|\psi\rangle \left(\frac{\alpha^{*n}}{\sqrt{n!}} \right). \tag{57}$$

Since $\langle\psi|\psi\rangle = \Sigma_n |\langle n|\psi\rangle|^2 = 1$, the series on the right-hand side of Eq. (57),

$$f_\psi(\alpha^*) \equiv \sum_n \langle n|\psi\rangle \left(\frac{\alpha^{*n}}{\sqrt{n!}} \right), \tag{58}$$

is absolutely convergent in any finite region of the complex plane, showing that $f_\psi(\alpha^*)$ is an *entire* function (analytic in every finite region of the complex plane). We conclude that for every $|\psi\rangle$ in the oscillator Hilbert space, there exists an entire function $f_\psi(\alpha^*)$ such that the CS representation of $|\psi\rangle$ is given by

$$\langle\alpha|\psi\rangle = f_\psi(\alpha^*)\exp(-|\alpha|^2/2). \tag{59}$$

This one-to-one correspondence between $\langle\alpha|\psi\rangle$ and $f_\psi(\alpha^*)$ establishes a relation between the space of the CS representations $\langle\alpha|\psi\rangle$ and the Segal–Bargmann (S–B) space of entire functions $f_\psi(\alpha^*)$. We note that $f_n(\alpha^*)$ cor-

responding to $|\psi\rangle = |n\rangle$ is $(\alpha^*)^n/\sqrt{n!}$. The functions $f_n(\alpha^*)$, $n = 0,1,2,...$, form an orthonormal basis of the Segal–Bargmann space. For any two functions $f(\alpha^*) = \Sigma a_n \alpha^{*n}$, $g(\alpha^*) = \Sigma b_n \alpha^{*n}$, in this space, we have

$$\langle f|g\rangle = \sum_n (n!) a_n^* b_n, \tag{60}$$

$$\langle f|f\rangle = \sum_n (n!)|a_n|^2. \tag{61}$$

The last equation gives the completeness relation. Also, it shows that not every entire function is an element of this space but only those $f(\alpha^*)$, for which $\langle f|f\rangle < \infty$.

Operators such as a, $a^\dagger$ take very simple forms (representations) in the S–B space. It is evident from Eqs. (13) and (14) and the form of $f_n(\alpha^*)$ that the representation of $a^\dagger$ is α^* and of a is $\partial/\partial\alpha^*$. Many complicated equations can be immensely simplified in this representation. (See Ref. 1, p. 17.)

B. Coherent state representations of operators

An operator A acting on the oscillator Hilbert space is expressible in the C–S basis by using Eq. (51) in the form:

$$A = \pi^{-2} \iint |\alpha\rangle \langle\alpha|A|\beta\rangle \langle\beta| d^2\alpha\, d^2\beta. \tag{62}$$

The above expression is not, in general, unique due to the overcompleteness of the C–S basis.[11]

The matrix element $\langle\alpha|A|\beta\rangle$ in Eq. (62) can be expressed in the number basis as

$$\langle\alpha|A|\beta\rangle = \exp\left(-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2}\right) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{\langle n|A|m\rangle}{\sqrt{(n!m!)}}\right)(\alpha^*)^n\beta^m. \tag{63}$$

We assume that $\langle n|A|m\rangle$'s are well-behaved functions so that an entire function $A(\alpha^*,\beta)$ can be defined by

$$A(\alpha^*,\beta) \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\langle n|A|m\rangle\sqrt{(n!m!)}\right]\left(\frac{(\alpha^*)^n(\beta)^m}{(n!m!)}\right). \tag{64}$$

Hence,

$$\langle\alpha|A|\beta\rangle = A(\alpha^*,\beta)\exp(-|\alpha|^2/2 - |\beta|^2/2). \tag{65}$$

We call $A(\alpha^*,\beta)$ the CS representation of A. Treating the right-hand side of (64) as a double Taylor series, we may consider $A(\alpha^*,\beta)$ as the generating function of the matrix element $\langle n|A|m\rangle$. The diagonal element $\langle\alpha|A|\alpha\rangle$ can be obtained from Eq. (64) by setting $\beta = \alpha$, so that we can write

$$\langle n|A|m\rangle\sqrt{(n!m!)} = \left(\frac{\partial}{\partial\alpha^*}\right)\left(\frac{\partial}{\partial\alpha}\right)[\langle\alpha|A|\alpha\rangle\exp(|\alpha|^2)]_{\alpha=0}. \tag{66}$$

This shows that $A(\alpha^*,\beta)$ can be derived from the diagonal term $A(\alpha^*,\alpha)$ This is possible due to the overcompleteness of the C–S basis, and does not take place in any other orthogonal basis.

The representation of the identity operator I is obtained from Eq. (64) by setting $A = I$,

$$I(\alpha^*,\beta) = \exp(\alpha^*\beta). \tag{67}$$

The CS representation of the state $|\phi\rangle = A|\psi\rangle$ can be obtained by using Eqs. (59) and (65):

$$\begin{aligned} f_\phi(\alpha^*) &= \langle\alpha|\phi\rangle\exp(|\alpha|^2/2) \\ &= \pi^{-1}\int \langle\alpha|A|\beta\rangle\langle\beta|\psi_0\rangle\exp\left(\frac{|\alpha|^2}{2}\right)d^2\beta \\ &= \pi^{-1}\int A(\alpha^*,\beta) f_\psi(\beta^*)\exp(-|\beta|^2 d^2\beta) \\ &= \int A(\alpha^*,\beta) f_\psi(\beta^*) d\mu, \end{aligned} \tag{68}$$

where

$$d\mu = \pi^{-1}\exp(-|\beta|^2) d^2\beta. \tag{69}$$

The trace of a Hermitian operator A is given by using Eq. (53),

$$\begin{aligned} \mathrm{Tr}\, A &= \sum_n \pi^{-1}\int \langle n|\alpha\rangle\langle\alpha|n\rangle\langle n|A|n\rangle d^2\alpha \\ &= \pi^{-1}\int A(\alpha^*,\alpha)\exp(-|\alpha|^2) d^2\alpha \\ &= \int A(\alpha^*,\alpha) d\mu. \end{aligned} \tag{70}$$

The CS representation of $D(\lambda)$ is found useful in various situations especially in the theory of radiation detection. We notice first that

$$\begin{aligned} D(\lambda)|\beta\rangle &= D(\lambda)D(\beta)|0\rangle \\ &= D(\lambda+\beta)\exp[(\beta^*\lambda - \lambda\beta^*)/2]|0\rangle \\ &= |\lambda+\beta\rangle\exp[(\beta^*\lambda - \lambda\beta^*)/2]. \end{aligned} \tag{71}$$

Hence,

$$\begin{aligned} D(\alpha^*,\beta;\lambda) &\equiv \langle\alpha|D(\lambda)|\beta\rangle\exp[(|\alpha|^2 + |\beta|^2)/2] \\ &= \exp(-|\lambda|^2/2)\exp(\alpha^*\lambda + \alpha^*\beta - \beta\lambda^*). \end{aligned} \tag{72}$$

The diagonal element is

$$\begin{aligned} \langle\alpha|D(\lambda)|\alpha\rangle &= D(\alpha^*,\alpha;\lambda)\exp(-|\alpha|^2) \\ &= \exp(-|\lambda|^2/2)\exp(\alpha^*\lambda - \alpha\lambda^*). \end{aligned} \tag{73}$$

The number representation of $D(\lambda)$ can be obtained from Eqs. (64) and (72) which lead to

$$\exp(-\lambda|^2/2)\exp(\alpha^*\lambda + \alpha^*\beta - \beta\lambda^*) = \sum_n\sum_m\left(\frac{\alpha^{*n}\beta^m}{\sqrt{(n!m!)}}\right)\langle n|D(\lambda)|m\rangle.$$

For detailed calculation we refer the readers to the paper by Mollow and Glauber.[19,20] We quote the results:

$$\langle n|D(\lambda)|m\rangle\exp(-|\lambda|^2/2)(n!m!)^{1/2} \times(-\lambda^*)^{m-n} L_n^{(m-n)}(|\lambda|^2),$$

for $n \leqslant m$, and

$$\langle n|D(\lambda)|n\rangle = \exp(-|\lambda|^2/2) L_n(|\lambda|^2),$$

where $L_n^{(m-n)}$ is the associated Laguerre polynomial and L_n is the Laguerre polynomial.[21]

C. The P representation of Glauber[11]

The CS representation of a density operator $\rho(a^\dagger,a)$ can be written according to Eq. (62) as

$$\rho(a^\dagger,a)=\pi^{-2}\int R(\alpha,\beta^*)|\alpha\rangle\langle\beta|\times\exp\left(-\frac{(|\alpha|^2+|\beta|^2}{2}\right)d^2\alpha\, d^2\beta, \tag{74}$$

where

$$R(\alpha,\beta^*)\equiv\langle\alpha|\rho(a^\dagger,a)|\beta\rangle.$$

This is known as the R representation of Glauber.

There is, for a variety of density operators, a much simpler representation known as the P representation that was first introduced by Titulaer and Glauber[22]:

$$\rho(a^\dagger,a)=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha, \tag{75}$$

where $P(\alpha)$ is defined over the entire complex plane and is called the weight function. The average value of an operator $A(a^\dagger,a)$ in the state described by ρ is then given by

$$\langle A\rangle=\mathrm{Tr}(\rho A)=\int P(\alpha)\langle\alpha|A|\alpha\rangle d^2\alpha. \tag{76}$$

Although $P(\alpha)$ in Eq. (76) appears like a probability density function over the complex plane of the classical probability theory, it may be highly singular and even negative. However, this blemish disappears in many important cases where $P(\alpha)$ is positive everywhere and the calculations of the averages can be carried out in a manner analogous to classical calculations.

Let us consider some properties of $P(\alpha)$. Since ρ is a density operator it satisfies the following conditions:

$$\rho^\dagger=\rho \tag{77a}$$

$$\mathrm{Tr}(\rho)=1, \tag{77b}$$

$$\rho^2=\rho, \tag{77c}$$

$$0\leqslant\langle\phi|\rho|\phi\rangle=|\langle\phi|\psi\rangle|^2\leqslant 1, \tag{77d}$$

where $|\phi\rangle$ is any state.

The hermiticity of ρ implies that $P(\alpha)$ must be real valued. By setting $A=I$ in Eq. (76) we get

$$\mathrm{Tr}(\rho)=\int P(\alpha)d^2\alpha=1, \tag{78}$$

which shows that $P(\alpha)$ is properly normalized. We have also

$$0\leqslant\int P(\alpha)\exp[-|\beta-\alpha|^2]d^2\alpha\leqslant 1, \tag{79}$$

which is obtained by choosing $|\phi\rangle$ and $|\psi\rangle$ in Eq. (77d) as any two coherent states.

As an example we consider the density operator ρ for a system in the coherent state $|\alpha\rangle$. ρ is then

$$\rho=|\alpha\rangle\langle\alpha|.$$

We set

$$P(\beta)=\delta^{(2)}(\beta-\alpha), \tag{80}$$

where the two-dimensional delta function is defined by

$$\delta^{(2)}(\beta)\equiv\delta(\mathrm{Re}\,\beta)\delta(\mathrm{Im}\,\beta). \tag{81}$$

It can easily be verified that Eqs. (78) and (79) are satisfied. Equation (80) gives the $P(\beta)$ for the pure coherent state.

Consider now the average of the operator $(a^\dagger)^m a^n$. We have

$$\langle(a^\dagger)^m a^n\rangle=\mathrm{Tr}(\rho(a^\dagger)^m a^n=\int P(\alpha)\langle\alpha|a^{\dagger m}a^n|\alpha\rangle d^2\alpha=\int P(\alpha)\alpha^{*m}\alpha^n\, d^2\alpha. \tag{82}$$

We notice that the average of the operator $(a^\dagger)^m a^n$ in the state ρ is simply obtained by taking the average of $\alpha^{*m}\alpha^n$ with the weight function $P(\alpha)$. The average value of a *normally ordered* function of a and $a^\dagger$, $f_n(a,a^\dagger)$ [in which all creation operators are transferred to the left of all annihilation operators with the help of the commutation relation (10)] is given by

$$\langle f_n(a,a^\dagger)\rangle=\mathrm{Tr}[\rho f_n(a,a^\dagger)]=\int P(\alpha)f_n(\alpha,\alpha^*)d^2\alpha, \tag{83}$$

where $f_n(\alpha,\alpha^*)$ is obtained from the normal ordered operator function $f_n(a^\dagger,a)$ by replacing $a^\dagger$ by α^* and a by α. It is essential to have the function in the normal ordered form so that terms of the type $\langle\alpha|a^{\dagger m}a^n|a\rangle$ can simply be replaced by $\alpha^{*m}\alpha^n$ as in Eq. (82).

The significance of the properties is further illustrated by considering the case of an oscillator in thermal equilibrium with a heat reservoir at temperature T, in other words, an oscillator with thermal noise. The density operator for such a system is[3(b)]

$$\rho(a^\dagger,a)=[1-\exp(-\theta)]\exp(-\theta a^\dagger,a), \tag{84}$$

where $\theta=\hbar\omega/k_B T$, ω is the angular frequency of the oscillator, and k_B is the Boltzmann constant. We can verify this by expressing ρ in Eq. (84) in the number representation:

$$\rho(a^\dagger,a)=[1-\exp(-\theta)]\sum_{n=0}^{\infty}\exp(-\theta a^\dagger a)|n\rangle\langle n|=[1-\exp(-\theta)]\sum_{n=0}^{\infty}\exp(-\theta n)|n\rangle\langle n|.$$

Hence

$$\mathrm{Tr}\,\rho=[1-\exp(-\theta)]\sum_{n=0}^{\infty}[\exp(-\theta)]^n=1,$$

and the average number of photons in the state ρ is

$$N=\langle n\rangle=\mathrm{Tr}(a^\dagger a\rho)=[\exp(\theta)-1]^{-1}.$$

This is the Planck distribution as it should be.

A CS representation of this ρ is

$$\rho=(\pi N)^{-1}\int\exp\left(-\frac{|\alpha|^2}{N}\right)|\alpha\rangle\langle\alpha|d^2\alpha, \tag{85}$$

which can easily be verified by calculating $\langle n|\rho|m\rangle$ from Eq. (85) which turns out to be

$$\langle n|\rho|m\rangle=[1-\exp(-\theta)]\exp(-\theta n)\delta_{m,n}. \tag{86}$$

The weight function $P(\alpha)$ for this case is

$$P(\alpha)=(\pi N)^{-1}\exp(-|\alpha|^2/N), \tag{87}$$

which is Gaussian and positive everywhere.

In the classical limit $\hbar\omega/k_B T \ll 1$, the distribution $P(\alpha)$ in Eq. (87) becomes the classical distribution

$$P(\alpha) = (\hbar\omega/\pi k_B T)\exp(-\hbar\omega|\alpha|^2/k_B T). \tag{88}$$

The general case of determining $P(\alpha)$ from ρ has been discussed by Sudarshan.[23]

D. Differential representation of operators

The differential representations of $a^\dagger$ and a can be obtained from the two relations

$$\langle\alpha|a^\dagger|\psi\rangle = \alpha^*\langle\alpha|\psi\rangle \tag{89}$$

and

$$\langle\alpha|a|\alpha'\rangle = \alpha'\langle\alpha|\alpha'\rangle. \tag{90}$$

From Eq. (89) it follows that $a^\dagger$ is represented by multiplication by α^*. Substituting from Eq. (47) for $\langle\alpha|\alpha'\rangle$ in Eq. (90) we have

$$\begin{aligned}\langle\alpha|a|\alpha'\rangle &= \alpha'\exp\left(-\frac{\alpha\alpha^*}{2}-\frac{\alpha'\alpha'^*}{2}+\alpha^*\alpha'\right)\\ &= \left(\frac{\alpha}{2}+\frac{\partial}{\partial\alpha^*}\right)\langle\alpha|\alpha'\rangle.\end{aligned} \tag{91}$$

Hence

$$\begin{aligned}\langle\alpha|a|\psi\rangle &= \pi^{-1}\int\langle\alpha|a|\alpha'\rangle\,\langle\alpha'|\psi\rangle d^2\alpha'\\ &= \left(\frac{\alpha}{2}+\frac{\partial}{\partial\alpha^*}\right)(\pi^{-1})\int\langle\alpha|\alpha'\rangle\,\langle\alpha'|\psi\rangle d^2\alpha'\\ &= \left(\frac{\alpha}{2}+\frac{\partial}{\partial\alpha^*}\right)\langle\alpha|\psi\rangle.\end{aligned} \tag{92}$$

Hence the representation of a is $\alpha/2+\partial/\partial\alpha^*$ in the space of the functions $\langle\alpha|\psi\rangle$. In general, the operator $F(a^\dagger,a)$ has the representation

$$\langle\alpha|F(a^\dagger,a)|\psi\rangle = F\left(\alpha^*,\frac{\alpha}{2}+\frac{\partial}{\partial\alpha^*}\right)\langle\alpha|\psi\rangle. \tag{93}$$

The derivatives inside F act on terms containing α^* in $\langle\alpha|\psi\rangle$ as well as those in F itself. For example,

$$\langle\alpha|aa^\dagger|\beta\rangle = \left(\frac{\alpha}{2}+\frac{\partial}{\partial\alpha^*}\right)\alpha^*\langle\alpha|\beta\rangle.$$

From Eq. (47) we have

$$\left(\frac{\partial}{\partial\alpha^*}\right)\langle\alpha|\beta\rangle = \left(-\frac{\alpha}{2}+\beta\right)\langle\alpha|\beta\rangle. \tag{94}$$

Hence,

$$\langle\alpha|aa^\dagger|\beta\rangle = (\alpha^*\beta+1)\langle\alpha|\beta\rangle.$$

When F is in the normal ordered form, all $a^\dagger$ are on the right of all a in each term of F. In this case there is no term in F where $\partial/\partial\alpha^*$ can act giving nonzero values. Thus the normal ordered form of $aa^\dagger$ is $a^\dagger a+1$. Hence

$$\begin{aligned}\langle\alpha|aa^\dagger|\beta\rangle &= \langle\alpha|a^\dagger a+1|\beta\rangle\\ &= \alpha^*\left(\frac{\partial}{2}+\frac{\partial}{\partial\alpha^*}\right)\langle\alpha|\beta\rangle+\langle\alpha|\beta\rangle\\ &= (\alpha^*\beta+1)\langle\alpha|\beta\rangle.\end{aligned}$$

Using Eq. (94) we can write

$$\langle\alpha|F(a^\dagger,a)|\beta\rangle = \langle\alpha|\beta\rangle F\left(\alpha^*,\beta+\frac{\partial}{\partial\alpha^*}\right).1. \tag{95}$$

The diagonal elements of F are given by

$$\langle\alpha|F(a^\dagger,a)|\alpha\rangle = F\left(\alpha^*,\alpha+\frac{\partial}{\partial\alpha^*}\right).1, \tag{96}$$

which is a very useful formula. The left-hand side of Eq. (96) is a function of α and α^* that we shall write as $F_n(\alpha,\alpha^*)$ which is also equal to $\langle\alpha|:F(a^\dagger,a):|\alpha\rangle$, where $:F(a^\dagger,a):$ mean the normal ordered form of $F(a^\dagger,a)$. When the operator function F is in the normal ordered form, the diagonal element is simply obtained by substituting α for a and α^* for $a^\dagger$.

As an example of the use of the above formalism let us consider the time development of a quantum mechanical system with Hamiltonian $H(a,a^\dagger,t)$. The state vector satisfies the Schrödinger equation

$$i\hbar\left(\frac{\partial}{\partial t}\right)|\psi(t)\rangle = H|\psi(t)\rangle. \tag{97}$$

The unitary evolution operator $U(t)$ is defined by[13]

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle. \tag{98}$$

It is well known that $U(t)$ satisfies the operator equation

$$i\hbar\left(\frac{\partial}{\partial t}\right)U(t) = HU(t), \tag{99}$$

with $U(0)=1$. The problem is to determine $U(t)$ by solving Eq. (99).

Operator equations are usually complicated to solve. However, we can obtain an equation for the normal ordered associated function $U_n(t)$ as follows:

$$\begin{aligned}i\hbar\left(\frac{\partial}{\partial t}\right)U_n(\alpha^*\alpha,t) &= i\hbar\left(\frac{\partial}{\partial t}\right)\langle\alpha|U(a^\dagger,a,t)|\alpha\rangle\\ &= \langle\alpha|H(a^\dagger,a,t)U(a^\dagger,a,t)|\alpha\rangle\\ &= H\left(\alpha^*,a+\frac{\partial}{\partial\alpha^*},t\right)\\ &\quad\times U\left(\alpha^*,a+\frac{\partial}{\partial\alpha^*},t\right)\\ &= H\left(\alpha^*,\alpha+\frac{\partial}{\partial\alpha^*},t\right)U_n(\alpha^*,\alpha,t),\end{aligned} \tag{100}$$

which is an equation involving c numbers only.

Let us consider the forced harmonic oscillator (interaction of a classical current with a single mode of the em field) for which

$$H = \hbar\omega a^\dagger a+\hbar[f(t)a+f^*(t)a^\dagger].$$

Equation (100) then becomes

$$\begin{aligned}i\hbar\left(\frac{\partial}{\partial t}\right)U_n(\alpha^*,\alpha,t) = \Big[&\hbar\omega\alpha^*\left(\alpha+\frac{\partial}{\partial\alpha^*}\right)\\ &+\hbar f(t)\left(\alpha+\frac{\partial}{\partial\alpha^*}\right)\\ &+\hbar f^*(t)\alpha^*\Big]U_n(\alpha^*,\alpha,t),\end{aligned}$$

which is a c-number equation that can be solved in various ways. Similarly the Liouville equation for the density matrix can be reduced to a c-number equation that is comparatively easier to handle than the original operator equation.[24]

V. FLUCTUATIONS OF OPERATORS

The fluctuation $(\Delta A)_\psi$ of a Hermitian operator in any state $|\psi\rangle$ is defined by

$$(\Delta A)_\psi \equiv \sqrt{\langle\psi|(A-\langle A\rangle_\psi)^2|\psi\rangle}\,,$$

where

$$\langle A\rangle_\psi \equiv \langle\psi|A|\psi\rangle.$$

The fluctuations $(\Delta x)_n$ and $(\Delta p)_n$ in the number state $|n\rangle$ is easily evaluated:

$$(\Delta x)_n = \sqrt{(m\hbar\omega/2)}\,\sqrt{(2n+1)}\,, \tag{101}$$

$$(\Delta p)_n = \sqrt{\hbar/(2m\omega)}\,\sqrt{(2n+1)}\,, \tag{102}$$

where we have used $\langle x\rangle_n = \langle p\rangle_n = 0$. The uncertainty product is then

$$\langle\Delta x\rangle_n\langle\Delta p\rangle_n = \hbar(n+\tfrac{1}{2}). \tag{103}$$

It follows that among the number states the ground state $|0\rangle$ is the only minimum uncertainty state with $(\Delta x)(\Delta p) = \hbar/2$.

To deal with the fluctuations in x and p in the CS, we define for convenience,

$$a_1 \equiv \sqrt{[m\omega/(2\hbar)]}\,x,$$

$$a_2 \equiv \sqrt{[1/(2m\hbar\omega)]}\,p, \tag{104}$$

so that from the definition of a in Eq. (6), we have

$$a = a_1 + ia_2.$$

For a radiation oscillator, a_1 and a_2 are identified as the amplitudes of the two quadrature phases (the negative and positive frequency components) of the field.

From the commutation relation between x and p it follows that

$$[a_1, a_2] = i/2. \tag{105}$$

The uncertainty relation corresponding to Eq. (105) is

$$(\Delta a_1)^2(\Delta a_2)^2 \geqslant 1/16, \tag{106}$$

where the averages are calculated for any arbitrary state. For the CS, $|\alpha\rangle$, we have

$$\langle a_1\rangle = \langle\alpha|a_1|\alpha\rangle = \mathrm{Re}\,\alpha \tag{107}$$

$$\langle a_1^2\rangle = (\mathrm{Re}\,\alpha)^2 + \tfrac{1}{4}. \tag{108}$$

We can derive similar expressions for a_2. Hence in any coherent state $|\alpha\rangle$, we have

$$(\Delta a_1)^2 = \langle a_1^2\rangle - \langle a_1\rangle^2 = \tfrac{1}{4}, \tag{109}$$

$$(\Delta a_2)^2 = \tfrac{1}{4}. \tag{110}$$

Hence all coherent states are minimum uncertainty states with equal fluctuations in the two quadratures. [Definition (c) in Sec. I.] We also have

$$(\Delta a_1)^2 + (\Delta a_2)^2 = \tfrac{1}{2}. \tag{111}$$

[Definition (d) in Sec. I.]

The equivalence of the definitions of the CS given in Sec. I is now established for the harmonic oscillator. The equivalence is due to the particular commutation relations of the operators $I, a, a^\dagger, aa^\dagger$, that is, the particular structure of the Lie algebra of those operators. If in the case of a certain system the operators satisfy different type of commutation relations, the definitions (a) and (b) in Sec. I, may not be at all equivalent.[25]

The average value of an operator in a coherent state can be easily calculated first by normal ordering the operator and then substituting α for a and α^* for $a^\dagger$ in the normal ordered form. The fluctuations can then be evaluated.

VI. CONCLUSION

We have discussed here the properties of coherent states of the harmonic oscillator. These are the states for which the energy of the oscillator is a constant of motion with the same value as that of a classical particle moving with the center of the packet under the harmonic potential (the zero-point energy is subtracted). These states are generated by applying a time-varying force to the oscillator initially in the ground state.

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