Operadores Tensoriales Vectoriales

Un operador tensorial de primer rango son tres operadores que se pueden expresar en

- componentes esféricas $\{T_{+1}, T_0, T_{-1}\}$, que se transforman según $D^1_{mm'}(\alpha, \beta, \gamma)$
- componentes cartesianas, $\{T_x, T_y, T_z\}$, que se transforman según $R_{ij}(\alpha, \beta, \gamma)$

Las relaciones de transformación entre éstas es:

$$\begin{cases} T_{+1} = \frac{1}{\sqrt{2}}\,(T_x + iT_y) \\ T_{-1} = \frac{-1}{\sqrt{2}}\,(T_x - iT_y) \\ T_0 = T_z \end{cases} \qquad \begin{cases} T_x = \frac{-1}{\sqrt{2}}\,(T_{+1} - T_{-1}) \\ T_y = \frac{i}{\sqrt{2}}\,(T_{+1} + T_{-1}) \\ T_z = T_0 \end{cases}$$

Operadores Tensoriales

$$U(R)\; T_q^k\; U^\dagger(R) = \sum_{q'} \; T_{q'}^k \; D_{q'q}^k(R)$$

La definición es equivalente a:

$$\begin{cases} [J_z, T_q^k] \;=\; \hbar\, q\, T_q^k \\ [J_+, T_q^k] \;=\; \hbar \sqrt{k(k+1)-q(q+1)}\;\; T_{q+1}^k \\ [J_-, T_q^k] \;=\; \hbar \sqrt{k(k+1)-q(q-1)}\;\; T_{q-1}^k \end{cases}$$

Para una rotación infinitesimal, $\boldsymbol{D^k_{q'q}(\varepsilon) = \langle\, kq' \,|\, U(\varepsilon) \,|\, kq \,\rangle = \langle\, kq' \,|\, (1 - \frac{i}{\hbar}\, \varepsilon \cdot J) \,|\, kq \,\rangle}$:

$$U(\boldsymbol{\varepsilon})\; T^k_q\; U^\dagger(\boldsymbol{\varepsilon}) = \sum_{q'} T^k_{q'}\; D^k_{q'q}(\boldsymbol{\varepsilon})$$

$$(1 - \tfrac{i}{\hbar}\, \boldsymbol{\varepsilon} \cdot \boldsymbol{J})\; T^k_q\; (1 + \tfrac{i}{\hbar}\, \boldsymbol{\varepsilon} \cdot \boldsymbol{J}) = \sum_{q'} T^k_{q'}\; \langle\, kq' \,|\, (1 - \tfrac{i}{\hbar}\, \boldsymbol{\varepsilon} \cdot \boldsymbol{J}) \,|\, kq \,\rangle$$

$$-\tfrac{i}{\hbar}\, \boldsymbol{\varepsilon} \cdot [\, \boldsymbol{J}, T^k_q \,] = -\tfrac{i}{\hbar}\, \boldsymbol{\varepsilon} \cdot \sum_{q'} T^k_{q'}\; \langle\, kq' \,|\, \boldsymbol{J} \,|\, kq \,\rangle$$

$$[\, \boldsymbol{J}, T^k_q \,] = \sum_{q'} T^k_{q'}\; \langle\, kq' \,|\, \boldsymbol{J} \,|\, kq \,\rangle$$

$$\begin{cases} [\, J_z, T^k_q \,] = \displaystyle\sum_{q'} T^k_{q'} \quad \hbar\, q \quad \langle\, kq' \,|\, k\, q \,\rangle \quad = \hbar\, q\; T^k_q \\ [\, J_+, T^k_q \,] = \displaystyle\sum_{q'} T^k_{q'}\; \hbar \sqrt{k(k+1) - q(q+1)}\, \langle\, kq' \,|\, k\, q+1 \rangle = \hbar \sqrt{k(k+1) - q(q+1)}\; T^k_{q+1} \\ [\, J_-, T^k_q \,] = \displaystyle\sum_{q'} T^k_{q'}\; \hbar \sqrt{k(k+1) - q(q-1)}\, \langle\, kq' \,|\, k\, q-1 \rangle = \hbar \sqrt{k(k+1) - q(q-1)}\; T^k_{q-1} \end{cases}$$

Descomposición en tensores irreducibles

$$\boldsymbol{T^k_q = \sum_{q_1 q_2} X^{k_1}_{q_1}\ Y^{k_2}_{q_2}\ \langle k_1\, k_2, q_1\, q_2 | \, kq\rangle}$$

$$\hbar^{-1}\left[\boldsymbol{J_+}, \boldsymbol{T^k_q}\right] = \sum_{q_1 q_2} \hbar^{-1}\left\{ \left[J_+, X^{k_1}_{q_1}\right] Y^{k_2}_{q_2} + X^{k_1}_{q_1}\left[J_+, Y^{k_2}_{q_2}\right]\right\} \langle k_1\, k_2, q_1\, q_2 | \, kq\rangle =$$

$$= \sum_{q_1 q_2} X^{k_1}_{q_1+1}\ Y^{k_2}_{q_2}\ \sqrt{k_1(k_1+1) - q_1(q_1+1)}\ \langle k_1\, k_2, q_1\, q_2 | \, kq\rangle +$$

$$+ \sum_{q_1 q_2} X^{k_1}_{q_1}\ Y^{k_2}_{q_2+1}\ \sqrt{k_2(k_2+1) - q_2(q_2+1)}\ \langle k_1\, k_2, q_1\, q_2 | \, kq\rangle =$$

$$= \sum_{q_1 q_2} X^{k_1}_{q_1}\ Y^{k_2}_{q_2}\left\{ \underbrace{\sqrt{k_1(k_1+1) - q_1(q_1-1)}\ \langle k_1\, k_2, q_1\text{-}1\, q_2 | \, kq\rangle + \sqrt{k_2(k_2+1) - q_2(q_2-1)}\ \langle k_1\, k_2, q_1\, q_2\text{-}1 | \, kq\rangle}_{\sqrt{k(k+1) - q(q+1)}\ \langle k_1 k_2, q_1 q_2 | \, k\, q+1\rangle}\right\}$$

$$= \sqrt{k(k+1) - q(q+1)}\ \sum_{q_1 q_2} X^{k_1}_{q_1}\ Y^{k_2}_{q_2}\ \langle k_1 k_2, q_1 q_2 | \, k\, q+1\rangle$$

$$= \boldsymbol{\sqrt{k(k+1) - q(q+1)}\ \ T^k_{q+1}}$$

Teorema de Wigner-Eckart

$$\langle \alpha' j' m' | T^k_q | \alpha j m \rangle = \langle j' m' | k j, q m \rangle \langle \alpha' j' \| T \| \alpha j \rangle$$

$$\frac{1}{\hbar} \langle \alpha' j' m' | [J_-, T^k_q] | \alpha j m \rangle =$$

$$\sqrt{k(k+1)-q(q-1)} \langle \alpha' j' m' | T^k_{q-1} | \alpha j m \rangle =$$

$$\sqrt{j'(j'+1)-m'(m'+1)} \langle \alpha' j' m'+1 | T^k_q | \alpha j m \rangle - \sqrt{j(j+1)-m(m-1)} \langle \alpha' j' m' | T^k_q | \alpha j m-1 \rangle$$

Comparar con la relación de recurrencia de los coeficientes de CG:

$$\sqrt{j_1(j_1+1)-m_1(m_1-1)} \langle jm | j_1 j_2, m_1-1\, m_2 \rangle =$$

$$\sqrt{j(j+1)-m(m+1)} \langle j m+1 | j_1 j_2, m_1 m_2 \rangle - \sqrt{j_2(j_2+1)-m_2(m_2-1)} \langle jm | j_1 j_2, m_1\, m_2-1 \rangle$$

$$k \to j_1 \qquad q \to m_1 \qquad j \to j_2 \qquad m \to m_2 \qquad j' \to j \qquad m' \to m$$