Guía 6: Oscilador armónico.

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \qquad \begin{cases} a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \\ a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right) \end{cases}$$

Solución Algebraica

$\Rightarrow [a, a^+] = 1$

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$H = \hbar\omega\,(\underbrace{a^+a}_{N} + \tfrac{1}{2}) \qquad \begin{matrix}[N, a] = -a \\ [N, a^+] = a^+\end{matrix}$$

$$E_n = \hbar\omega(n + \tfrac{1}{2}) \qquad n = 0, 1, \ldots$$

$$|n\rangle = \frac{1}{\sqrt{n!}}\,a^{+n}|0\rangle$$

P1: Valores medios, incertezas,

$\langle n|x|n\rangle = \langle n|p|n\rangle = 0$ (No más clásico)

Probar que en estado $|n\rangle$:

$$\mathrm{Var}(x)\,\mathrm{Var}(p) = (n + \tfrac{1}{2})^2\hbar^2$$

$n = 0$ da la mínima incerteza

$\Rightarrow$ es una gaussiana centrada en el origen.

Cálculo de $\varphi_0(x) = \langle x|0\rangle$:

$$\langle x|a|0\rangle = 0$$

$$\underbrace{\langle x|\cancel{x}|0\rangle}_{x\varphi_0(x)} + \frac{i}{m\omega}\underbrace{\langle x|p|0\rangle}_{-i\hbar\frac{\partial}{\partial x}\varphi_0(x)} = 0$$

$$\Rightarrow \quad \varphi_0(x) = N e^{-\frac{x^2}{4\sigma^2}}$$

$$\sigma^2 = \frac{\hbar}{2m\omega}$$

Guía 6: Oscilador armónico

$[a, a^\dagger] = 1$

P4 Estados coherentes.

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$\alpha \in \mathbb{C}$

a: no hermítico

(a) $\langle\alpha|a|\alpha\rangle = \alpha \Rightarrow \langle\alpha|a^\dagger|\alpha\rangle = \alpha^*$

$\langle\alpha|a^2|\alpha\rangle = \alpha^2 \Rightarrow \langle\alpha|a^{\dagger 2}|\alpha\rangle = \alpha^{*2}$

(b) $N = a^\dagger a$

$$\langle\alpha|N|\alpha\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$$

$a|\alpha\rangle = \alpha|\alpha\rangle$

$\langle\alpha|a^\dagger = \alpha^*\langle\alpha|$

$$Var(N)_{|\alpha\rangle} = \langle\alpha|N^2|\alpha\rangle - (\langle\alpha|N|\alpha\rangle)^2$$

$\langle\alpha|a^\dagger a a^\dagger a|\alpha\rangle$

$|\alpha|^2\langle\alpha|a^\dagger a + \underbrace{[a, a^\dagger]}_{+1}|\alpha\rangle$

$$Var(N)_{|\alpha\rangle} = |\alpha|^2$$

(c) $H = \hbar\omega(N + \frac{1}{2}) \Rightarrow \langle\alpha|H|\alpha\rangle = \hbar\omega(|\alpha|^2 + \frac{1}{2})$

$$Var(H)_{|\alpha\rangle} = \langle\alpha|H^2|\alpha\rangle - (\langle\alpha|H|\alpha\rangle)^2$$

$$= \hbar^2\omega^2\langle\alpha|N^2 + N + \frac{1}{4}|\alpha\rangle - \hbar^2\omega^2(|\alpha|^2 + \frac{1}{2})^2$$

$$= \hbar^2\omega^2(\cancel{|\alpha|^4} + \cancel{|\alpha|^2} + |\alpha|^2 + \cancel{\tfrac{1}{4}}) - \hbar^2\omega^2(\cancel{|\alpha|^4} + \cancel{|\alpha|^2} + \cancel{\tfrac{1}{4}})$$

$$Var(H) = \hbar^2\omega^2|\alpha|^2$$

(d) $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger)$ $\qquad p = \sqrt{\frac{\hbar m\omega}{2}}\,\frac{(a-a^\dagger)}{i}$

$\langle\alpha|x|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,(\alpha+\alpha^*)$ $\qquad \langle\alpha|p|\alpha\rangle = \sqrt{\frac{\hbar m\omega}{2}}\,\frac{1}{i}(\alpha-\alpha^*)$

$\langle x\rangle_{|\alpha\rangle} = 2\sqrt{\frac{\hbar}{2m\omega}}\,\mathrm{Re}\,\alpha$ $\qquad \langle p\rangle_{|\alpha\rangle} = 2\sqrt{\frac{\hbar m\omega}{2}}\,\mathrm{Im}(\alpha)$

- $\mathrm{Var}(x) = \frac{\hbar}{2m\omega}\left[\langle\alpha|a^2+aa^\dagger+a^\dagger a+a^{\dagger 2}|\alpha\rangle - (\alpha+\alpha^*)^2\right]$

$= \frac{\hbar}{2m\omega}\left[\cancel{\alpha^2}+\cancel{2|\alpha|^2}+1+\cancel{\alpha^{*2}}-\cancel{\alpha^2}-\cancel{2|\alpha|^2}-\cancel{\alpha^{*2}}\right]$

$\mathrm{Var}(x) = \frac{\hbar}{2m\omega} = \sigma^2$ No depende de α.

(e) $\mathrm{Var}(p) = \frac{\hbar m\omega}{2}\left[\langle\alpha|a^2-aa^\dagger-a^\dagger a+a^{\dagger 2}|\alpha\rangle - (\alpha-\alpha^*)^2\right]$

$= \frac{\hbar m\omega}{2}\left[\cancel{\alpha^2}-\cancel{2|\alpha|^2}-1-\cancel{\alpha^{*2}}-\cancel{\alpha^2}+\cancel{2|\alpha|^2}-\cancel{\alpha^{*2}}\right]$

$\mathrm{Var}(p) = \frac{\hbar m\omega}{2}$ No depende de α

(f) $\mathrm{Var}(x)\,\mathrm{Var}(p) = \frac{\hbar^2}{4} = \frac{|\langle\alpha|[x,p]|\alpha\rangle|^2}{4}$

Mínima incerteza.

$\Rightarrow$ El estado coherente es una gaussiana con los valores medios $\langle x\rangle$ y $\langle p\rangle$:

$$\langle x|\alpha\rangle = \frac{1}{(2\pi\sigma^2)}\, e^{i\frac{\langle p\rangle x}{\hbar}}\, e^{-\frac{(x-\langle x\rangle)^2}{4\sigma^2}}$$

$$\langle x|\alpha\rangle = \langle x|e^{i\frac{\langle p\rangle x}{\hbar} - i\frac{\langle x\rangle p}{\hbar}}|0\rangle\, e^{\frac{i\langle x\rangle\langle p\rangle}{2\hbar}}$$

Baker-Campbell-Hausdorf: $e^{A+B} = e^A e^B e^{-[A,B]/2}$

Guía 6: Oscilador armónico

Cont $\boxed{P4}$

(9) Calcular $\langle \alpha | x(t) | \alpha \rangle$ $\langle \alpha | p(t) | \alpha \rangle$

Vamos a hallar $a(t)$:

$$\dot{a}(t) = \frac{[a, H]}{i\hbar} = \frac{\hbar\omega [a, a^+a]}{i\hbar}$$

$$\dot{a}(t) = \frac{\omega}{i}\left(\underbrace{[a, a^+]}_{1} a + a^+[a, a]\right)$$

$$\dot{a}(t) = -i\omega \Rightarrow \boxed{\begin{aligned} a(t) &= a(0)e^{-i\omega t} \\ a^+(t) &= a^+(0)e^{i\omega t} \end{aligned}}$$

$$x(t) = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{\frac{m\omega}{2\hbar}}\left(x(0) + i\frac{p(0)}{m\omega}\right)e^{-i\omega t} + \sqrt{\frac{m\omega}{2\hbar}}\left(x(0) - i\frac{p(0)}{m\omega}\right)e^{i\omega t}\right]$$

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t.$$

$$p(t) = \sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{\frac{m\omega}{2\hbar}}\cdot\frac{1}{i}\left(x(0) + i\frac{p(0)}{m\omega}\right)e^{-i\omega t} - \sqrt{\frac{m\omega}{2\hbar}}\,\frac{1}{i}\left(x(0) - i\frac{p(0)}{m\omega}\right)e^{i\omega t}\right]$$

$$p(t) = -m\omega\, x(0)\sin\omega t + p(0)\cos\omega t$$

$$\langle x(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle a(t)\rangle + \langle a^+(t)\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\underbrace{\alpha e^{-i\omega t}}_{\alpha(t)} + \underbrace{\alpha^* e^{+i\omega t}}_{\alpha^*(t)}\right).$$

$$\boxed{\langle x(t)\rangle = 2\sqrt{\frac{\hbar}{2m\omega}}\, Re(\alpha(t))}$$

$$\boxed{\langle p(t)\rangle = 2\sqrt{\frac{\hbar m\omega}{2}}\, Im(\alpha(t))}$$

Si $\alpha(0) = |\alpha| e^{-i\varphi_0}$

$$\alpha(t) = |\alpha| e^{-i(\omega t + \varphi_0)}$$

$$\alpha^*(t) = |\alpha| e^{i(\omega t + \varphi_0)}$$

h) e i) están después del (k).

l) Usamos el punto (h):

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Si se mide H:

$$Prob(\hbar\omega(n+\tfrac{1}{2}) \,|\, |\alpha\rangle) = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$ Distrib. de Poisson

$$\langle H\rangle = \hbar\omega|\alpha|^2$$

(m) $\frac{Sdv(N)}{N} \sim \frac{1}{|\alpha|}$ $\frac{Sdv(H)}{H} \sim \frac{1}{|\alpha|}$

Si $|\alpha|$ crece $\Rightarrow$ la incerteza relativa tiende a cero.

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Cont. $\boxed{PA}$

(K) Probar que: $|\alpha(t)\rangle = U(t,0)|\alpha\rangle$
es un estado coherente, hallar $\alpha(t)$.

$$a|\alpha(t)\rangle = a\,U(t,0)|\alpha\rangle$$

$$1 = U_{(t)}U^{\dagger}_{(t)}$$

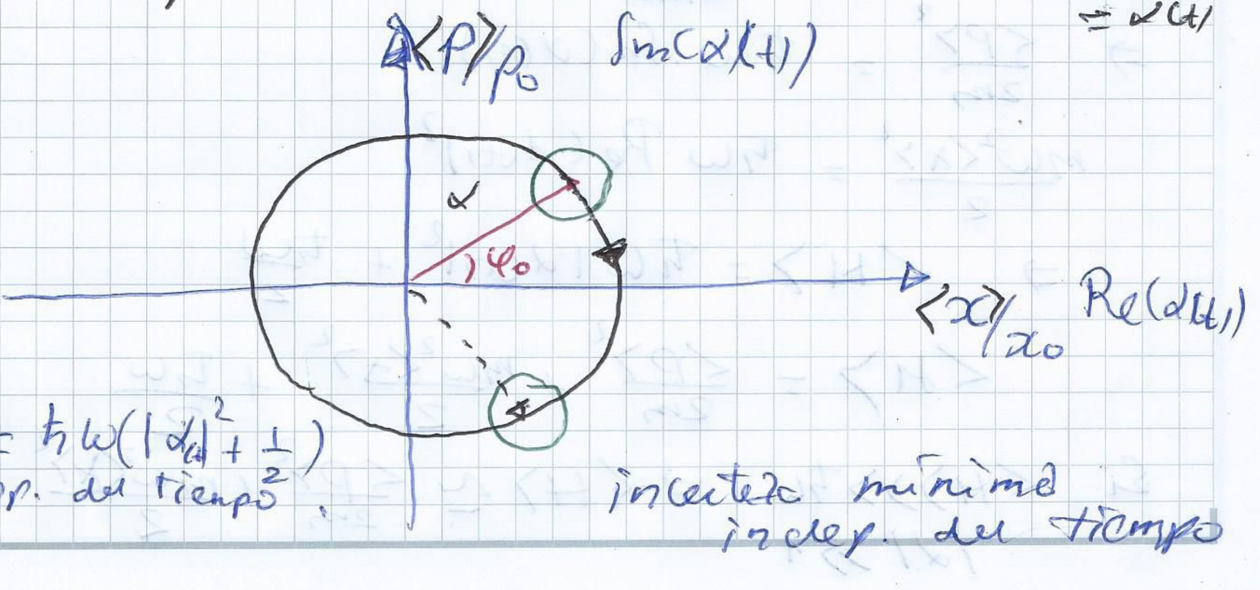
$$= U(t,0)\underbrace{U^{\dagger}(t,0)\,a\,U(t,0)}_{a_H(t) = e^{-i\omega t}a}|\alpha\rangle$$

$$= U(t,0)\,e^{-i\omega t}\alpha|\alpha\rangle$$

$$a|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle$$

con $\boxed{\alpha(t) = e^{-i\omega t}\alpha}$

Todos los cálculos previos con $|\alpha\rangle$ valen para $|\alpha(t)\rangle$ cambiando $\alpha \to \alpha e^{-i\omega t} = \alpha(t)$



$\langle H\rangle = \hbar\omega(|\alpha_{(t)}|^2 + \frac{1}{2})$
no dep. del tiempo.

incerteza mínima
indep. del tiempo

Si $|\alpha|$ es muy grande $(|\alpha| \gg 1)$ el comportamiento es muy similar al clásico, sus valores medios oscilan y la incerteza relativa es muy chica.
[Esto no pasa en los autoestados $|n\rangle$ cuyo valor medio es $\langle x\rangle = \langle p\rangle = 0$]

(k) Sea $|\alpha\rangle = \sum_n C_n |n\rangle \Rightarrow a|\alpha\rangle = \alpha|\alpha\rangle$

$$\sum_n C_n a|n\rangle = \sum_n \alpha C_n |n\rangle$$

$$C_n = \alpha \frac{C_{n-1}}{\sqrt{n}} \Longleftarrow \sum_n C_n \sqrt{n}\,|n-1\rangle = \sum_n \alpha C_n |n\rangle$$

$$C_0 = 1 \quad \Rightarrow C_n = \frac{\alpha^n}{\sqrt{n!}} \quad \therefore \boxed{|\alpha\rangle = N \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

(l) $\langle\alpha|\alpha\rangle = 1 \Rightarrow N^2 = \sum_n \frac{|\alpha|^n}{n!} \Rightarrow \boxed{N = e^{-\frac{|\alpha|^2}{2}}}$

$$\left.\begin{array}{l} |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{|\alpha|^n}{\sqrt{n!}} |n\rangle \\ |\beta\rangle = e^{-\frac{|\beta|^2}{2}} \sum_m \frac{|\beta|^m}{\sqrt{m!}} |m\rangle \end{array}\right\} \langle\beta|\alpha\rangle = e^{-\frac{|\alpha|^2 + |\beta|^2}{2}} \sum_n \frac{(\beta^* \alpha)^n}{n!}$$

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}\left[(|\alpha|^2 + |\beta|^2) - 2\alpha\beta^*\right]}$$

(n)

$$\langle H\rangle = \hbar\omega\left(|\alpha(t)|^2 + \frac{1}{2}\right)$$

$$\langle p\rangle = 2\sqrt{\frac{\hbar m\omega}{2}}\, \operatorname{Im} \alpha(t)$$

$$\langle x\rangle = 2\sqrt{\frac{\hbar}{2m\omega}}\, \operatorname{Re} \alpha(t).$$

$$\Rightarrow \frac{\langle p\rangle^2}{2m} = \hbar\omega\, \operatorname{Im}(\alpha(t))^2$$

$$\frac{m\omega^2\langle x\rangle^2}{2} = \hbar\omega\, \operatorname{Re}(\alpha(t))^2$$

$$\Rightarrow \langle H\rangle = \hbar\omega|\alpha(t)|^2 + \frac{\hbar\omega}{2}$$

$$\langle H\rangle = \frac{\langle p\rangle^2}{2m} + \frac{m\omega^2\langle x\rangle^2}{2} + \frac{\hbar\omega}{2}$$

Si $\langle H\rangle \gg \hbar\omega \Rightarrow \langle H\rangle \simeq \frac{\langle p\rangle^2}{2m} + \frac{m\omega^2\langle x\rangle^2}{2}$ relación clásica

$|\alpha| \gg 1$

Guía 6: Oscilador armónico

[P5] Operador desplazamiento en el espacio de fases.

Motivación: del punto (f) [P4] podemos concluir que a menos de una fase:

$$|\alpha\rangle = e^{i\frac{\langle p\rangle}{\hbar}x - i\frac{\langle x\rangle}{\hbar}p}|0\rangle$$

$$i\frac{\langle p\rangle}{\hbar}X = i\frac{2}{\hbar}\sqrt{\frac{\hbar m\omega}{2}}\cdot\sqrt{\frac{\hbar}{2m\omega}}(a+a^{+})\,\mathrm{Im}(\alpha)$$

$$-i\frac{\langle x\rangle p}{\hbar} = -i\,2\sqrt{\frac{\hbar}{2m\omega}}\cdot\sqrt{\frac{\hbar m\omega}{2}}\,\frac{(a-a^{+})}{i}\,\mathrm{Re}(\alpha)$$

$$= i(a+a^{+})\,\mathrm{Im}(\alpha) - (a-a^{+})\,\mathrm{Re}(\alpha)$$

$$= a^{+}\alpha - a\alpha^{*}$$

∴ $$|\alpha\rangle = \underbrace{e^{(\alpha a^{+} - \alpha^{*}a)}}_{\mathcal{D}(\alpha)}|0\rangle$$

$\mathcal{D}(\alpha)$ es el operador de desplazamiento en el espacio de fases.

Desplaza en $\underbrace{\langle x\rangle}_{2\sqrt{\frac{\hbar}{2m\omega}}\mathrm{Re}\,\alpha}$ y en $\underbrace{\langle p\rangle}_{2\sqrt{\frac{\hbar m\omega}{2}}\mathrm{Im}\,\alpha}$.

(a) $$\mathcal{D}(\alpha) = e^{\alpha a^{+} - \alpha^{*}a} = e^{A}$$

$$\mathcal{D}^{+}(\alpha) = e^{\alpha^{*}a - \alpha a^{+}} = e^{A^{+}} = e^{-A}$$

∴ $$\mathcal{D}(\alpha)\mathcal{D}^{+}(\alpha) = \mathcal{D}^{+}(\alpha)\mathcal{D}(\alpha) = \mathbb{1}$$

$$\mathcal{D}^{-1}(\alpha) = \mathcal{D}(-\alpha) \qquad \mathcal{D}(\alpha) \text{ es unitario.}$$

(b) $$\mathcal{D}(\alpha+\beta) = e^{(\alpha+\beta)a^\dagger - (\alpha+\beta)^* a}$$
$$= e^{(\alpha a^\dagger - \alpha^* a) + (\beta a^\dagger - \beta^* a)}$$
$$= e^{A+B}$$

$$[A,B] = [\alpha a^\dagger - \alpha^* a, \beta a^\dagger - \beta^* a]$$
$$= \alpha\beta^* - \alpha^*\beta = 2i\,\mathrm{Im}(\alpha\beta^*)\ \mathbb{1}$$

$$[A,[A,B]] = [B,[A,B]] = 0$$

aplica BCH: $$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

$$\boxed{\mathcal{D}(\alpha+\beta) = \mathcal{D}(\alpha)\mathcal{D}(\beta)\, e^{-i\,\mathrm{Im}(\alpha\beta^*)}}$$

(c) $$\mathcal{D}^\dagger(\alpha)\, a\, \mathcal{D}(\alpha) = e^{A} a e^{-A} \qquad -A = \alpha a^\dagger - \alpha^* a$$

pero: $$[A,a] = -[\alpha a^\dagger - \alpha^* a, a] = -\alpha \underbrace{[a^\dagger, a]}_{-1} = +\alpha$$

$$\Rightarrow [A,[A,a]] = [a,[A,a]] = 0$$

$$\mathcal{D}^\dagger(\alpha)\, a\, \mathcal{D}(\alpha) = a + \underbrace{[A,a]}_{\alpha} + \frac{1}{2!}\underbrace{[A,[A,a]]}_{0} + \dots$$

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$$\therefore \boxed{\begin{aligned} \mathcal{D}^\dagger(\alpha)\, a\, \mathcal{D}(\alpha) &= a + \alpha \\ \Rightarrow \mathcal{D}^\dagger(\alpha)\, a^\dagger\, \mathcal{D}(\alpha) &= a^\dagger + \alpha^* \end{aligned}}$$ (dagamos).

$$\mathcal{D}^\dagger(\alpha)\, X\, \mathcal{D}(\alpha) = \sqrt{\frac{\hbar}{2m\omega}}\,(a + \alpha + a^\dagger + \alpha^*) = X + 2\sqrt{\frac{\hbar}{2m\omega}}\,\mathrm{Re}(\alpha)$$
$$\mathcal{D}^\dagger(\alpha)\, P\, \mathcal{D}(\alpha) = \sqrt{\frac{\hbar m\omega}{2}}\,\frac{1}{i}\,(a + \alpha - a^\dagger - \alpha^*) = P + 2\sqrt{\frac{\hbar m\omega}{2}}\,\mathrm{Im}(\alpha)$$

Guía 6: Oscilador armónico

Cont. $\boxed{P5}$

(d) Probar que $|\alpha\rangle = D(\alpha)|0\rangle$ es autoestado de a: (aniquilación) (bajada)

$$a|\alpha\rangle = \underset{\hat{1}}{a} D(\alpha)|0\rangle$$
$$= D(\alpha)\underbrace{D^{+}(\alpha)\,a\,D(\alpha)}_{a+\alpha}|0\rangle$$
$$= D(\alpha)(a+\alpha)|0\rangle \qquad a|0\rangle = 0$$
$$= \alpha D(\alpha)|0\rangle = \alpha|\alpha\rangle$$

$\therefore$ $|\alpha\rangle = D(\alpha)|0\rangle$ es un estado coherente.

(e) $$D(\alpha) = e^{\alpha a^{+} - \alpha^{*} a} = e^{\alpha a^{+}} e^{-\alpha^{*} a} e^{-\frac{|\alpha|^2}{2}}$$

$$[\alpha a^{+}, -\alpha^{*} a] = -|\alpha|^2 \underbrace{[a, a^{+}]}_{1} = -|\alpha|^2$$

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^{+}} \underbrace{e^{-\alpha^{*} a}|0\rangle}_{|0\rangle}$$

$\therefore$ $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^{+}}|0\rangle$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{(\alpha a^{+})^n}{n!}|0\rangle$$

pero: $a^{+n}|0\rangle = \sqrt{n!}\,|n\rangle$

$$\boxed{|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle}$$

(f) • Si $\operatorname{Im}(\alpha)=0 \Rightarrow$ no se desplaza en momento

$$D^{\dagger}(\alpha)\, x\, D(\alpha) = x + \underbrace{\sqrt{\frac{\hbar}{2m\omega}}\, 2\operatorname{Re}\alpha}_{\Delta x}$$

$\therefore$ $D(\alpha)$ es el operador de desplazamiento.

$$D(\alpha) = e^{-i\frac{p\Delta x}{\hbar}}$$

• Si $\operatorname{Re}(\alpha)=0 \Rightarrow$ no se desplaza en posición

$$D^{\dagger}(\alpha)\, p\, D(\alpha) = p + \underbrace{\sqrt{\frac{\hbar m\omega}{2}}\, 2\operatorname{Im}\alpha}_{\Delta p}.$$

$$D(\alpha) = e^{i\frac{x\Delta p}{\hbar}}$$

• En general desplaza en posición y en momento.

(g) $$\langle\alpha|x|\alpha\rangle = \langle 0|D^{\dagger}(\alpha)\, x\, D(\alpha)|0\rangle$$
$$= \langle 0|x + 2\sqrt{\frac{\hbar}{2m\omega}}\operatorname{Re}\alpha|0\rangle =$$

$$\therefore \langle\alpha|x|\alpha\rangle = 2\sqrt{\frac{\hbar}{2m\omega}}\operatorname{Re}\alpha.$$

$$\langle\alpha|p|\alpha\rangle = \langle 0|D^{\dagger}(\alpha)\, p\, D(\alpha)|0\rangle$$
$$= \langle 0|p + 2\sqrt{\frac{\hbar m\omega}{2}}\operatorname{Im}(\alpha)|0\rangle$$

$$\therefore \langle\alpha|p|\alpha\rangle = 2\sqrt{\frac{\hbar m\omega}{2}}\operatorname{Im}\alpha$$

donde se usó $\langle 0|x|0\rangle = 0 = \langle 0|p|0\rangle$

$$\langle\alpha|x^2|\alpha\rangle = \langle 0|\underbrace{D^{\dagger}(\alpha)\, x\, D(\alpha)}_{x+\langle x\rangle}\,\underbrace{D^{\dagger}(\alpha)\, x\, D(\alpha)}_{x+\langle x\rangle}|0\rangle$$
$$= \langle 0|x^2 + 2x\langle x\rangle + \langle x\rangle^2|0\rangle$$
$$= \langle 0|x^2|0\rangle + \langle x\rangle^2$$

$$\operatorname{Var}(x) = \langle\alpha|x^2|\alpha\rangle - (\langle\alpha|x|\alpha\rangle)^2 = \underbrace{\langle 0|x^2|0\rangle}_{\frac{\hbar}{2m\omega}}$$

$$\operatorname{Var}(p) = \underbrace{\langle 0|p^2|0\rangle}_{\frac{\hbar m\omega}{2}}$$