Física Teórica 2 - Guía 6: Oscilador Armónico

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Problema 7

a) En este ítem conviene usar la ecuación demostrada en el problema 4

$$\langle\beta|\alpha\rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 - 2\alpha\beta^*\right)\right]. \tag{1}$$

Luego,

$$\begin{aligned}\langle\psi|\psi\rangle &= |N|^2\left(\langle\alpha|\alpha\rangle + \langle\alpha|-\alpha\rangle + \langle-\alpha|\alpha\rangle + \langle-\alpha|-\alpha\rangle\right)\\ &= |N|^2\left(1 + e^{-2|\alpha|^2} + e^{-2|\alpha|^2} + 1\right) = 2|N|^2\left(1 + e^{-2|\alpha|^2}\right).\end{aligned} \tag{2}$$

Esto implica que

$$|N|^2 = \frac{1}{2\left(1 + e^{-2|\alpha|^2}\right)}. \tag{3}$$

b) En este ítem, conviene usar la expresión demostrada en el problema 5

$$|\alpha\rangle = D(\alpha)|0\rangle, \tag{4}$$

donde

$$D(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right). \tag{5}$$

Sabiendo que

$$x = \frac{1}{\sqrt{2}}\sigma\left(a + a^\dagger\right),\ p = -\frac{\imath}{\sqrt{2}}\frac{\hbar}{\sigma}\left(a - a^\dagger\right),\ \sigma = \sqrt{\frac{\hbar}{m\omega}} \tag{6}$$

se llega a

$$D(\alpha) = \exp\left(\frac{\imath p_\alpha \hat{x}}{\hbar} - \frac{\imath x_\alpha \hat{p}}{\hbar}\right), \tag{7}$$

con

$$x_\alpha = \langle\alpha|x|\alpha\rangle = \sqrt{2}\sigma\Re(\alpha) \tag{8}$$

y

$$p_\alpha = \langle\alpha|p|\alpha\rangle = \sqrt{2}\sigma\Im(\alpha). \tag{9}$$

Por último, usando la propiedad

$$\exp(A+B) = \exp(A)\exp(B)\exp\left(-\frac{1}{2}[A,B]\right) \tag{10}$$

si A y B conmutan con $[A,B]$ vista en clases anteriores se tiene que

$$D(\alpha) = \exp\left(\frac{\imath p_\alpha \hat{x}}{\hbar}\right)\exp\left(-\frac{\imath x_\alpha \hat{p}}{\hbar}\right)\exp\left(-\imath\frac{x_\alpha p_\alpha}{2\hbar}\right). \tag{11}$$

Por lo tanto, sabiendo que $\exp\left(-\frac{\imath x_0 \hat{p}}{\hbar}\right)$ es el operador desplazamiento satisfaciendo la propiedad $\langle x|\exp\left(-\frac{\imath x_0\hat{p}}{\hbar}\right)|\psi\rangle = \langle x - x_0|\psi\rangle$ se obtiene

$$\begin{aligned}\langle x|\alpha\rangle &= \langle x|D(\alpha)|0\rangle = \langle x|\exp\left(\tfrac{\imath p_\alpha\hat{x}}{\hbar}\right)\exp\left(-\tfrac{\imath x_\alpha\hat{p}}{\hbar}\right)\exp\left(-\imath\tfrac{x_\alpha p_\alpha}{2\hbar}\right)|0\rangle = \exp\left(-\imath\tfrac{x_\alpha p_\alpha}{2\hbar}\right)\exp\left(\tfrac{\imath p_\alpha x}{\hbar}\right)\langle x|\exp\left(-\tfrac{\imath x_\alpha\hat{p}}{\hbar}\right)|0\rangle\\ &= \exp\left(-\imath\tfrac{x_\alpha p_\alpha}{2\hbar}\right)\exp\left(\tfrac{\imath p_\alpha x}{\hbar}\right)\langle x - x_\alpha|0\rangle = \exp\left(-\imath\tfrac{x_\alpha p_\alpha}{2\hbar}\right)\exp\left(\tfrac{\imath p_\alpha x}{\hbar}\right)\tfrac{1}{\sqrt{\sigma\sqrt{\pi}}}\exp\left(-\tfrac{(x-x_\alpha)^2}{2\sigma^2}\right)\\ &= \tfrac{1}{\sqrt{\sigma\sqrt{\pi}}}\exp\left(-\tfrac{\imath}{\hbar}\left(\tfrac{x_\alpha p_\alpha}{2} - p_\alpha x\right)\right)\exp\left(-\tfrac{(x-x_\alpha)^2}{2\sigma^2}\right).\end{aligned} \tag{12}$$

Analogamente, cambiando α por $-\alpha$ y notando que los x_α y p_α cambian de signo queda

$$\langle x|-\alpha\rangle = \frac{1}{\sqrt{\sigma\sqrt{\pi}}}\exp\left(-\frac{\imath}{\hbar}\left(\frac{x_\alpha p_\alpha}{2}+p_\alpha x\right)\right)\exp\left(-\frac{(x+x_\alpha)^2}{2\sigma^2}\right). \tag{13}$$

Luego,

$$\begin{aligned}\langle x|\psi\rangle &= N\left(\langle x|\alpha\rangle + \langle x|-\alpha\rangle\right)\\ &= \frac{N}{\sqrt{\sigma\sqrt{\pi}}}\exp\left(-\frac{\imath x_\alpha p_\alpha}{2\hbar}\right)\left(\exp\left(\frac{\imath p_\alpha x}{\hbar}\right)\exp\left(-\frac{(x-x_\alpha)^2}{2\sigma^2}\right)+\exp\left(-\frac{\imath p_\alpha x}{\hbar}\right)\exp\left(-\frac{(x+x_\alpha)^2}{2\sigma^2}\right)\right)\end{aligned} \tag{14}$$

y por lo tanto, usando que $|A+B|^2 = |A|^2+|B|^2+2\Re\left(AB^*\right)$ queda

$$\begin{aligned}&|\psi(x)|^2\\ &= \frac{|N|^2}{\sigma\sqrt{\pi}}\left(\exp\left(-\frac{(x-x_\alpha)^2}{\sigma^2}\right)+\exp\left(-\frac{(x+x_\alpha)^2}{\sigma^2}\right)+2\Re\left\{\exp\left(\frac{2\imath p_\alpha x}{\hbar}\right)\exp\left(-\frac{(x-x_\alpha)^2}{2\sigma^2}\right)\exp\left(-\frac{(x+x_\alpha)^2}{2\sigma^2}\right)\right\}\right)\end{aligned}, \tag{15}$$

o bien,

$$|\psi(x)|^2 = \frac{|N|^2}{\sigma\sqrt{\pi}}\left(\exp\left(-\frac{(x-x_\alpha)^2}{\sigma^2}\right)+\exp\left(-\frac{(x+x_\alpha)^2}{\sigma^2}\right)+2\cos\left(\frac{2p_\alpha x}{\hbar}\right)\exp\left(-\frac{x^2+x_\alpha^2}{2\sigma^2}\right)\right). \tag{16}$$

En cuanto a la dependencia temporal, basta con notar que la evolución temporal de un estado coherente $|\alpha\rangle$ es otro estado coherente $|\alpha(t)\rangle$ donde

$$\alpha(t) = \alpha\exp\left(-\imath\omega t\right). \tag{17}$$

Por lo tanto,

$$x_\alpha(t) = \sqrt{2}\sigma\Re\left(\alpha(t)\right) = \sqrt{2}\sigma\Re\left(\alpha\exp\left(-\imath\omega t\right)\right) \tag{18}$$

y

$$p_\alpha(t) = \sqrt{2}\sigma\Im\left(\alpha(t)\right) = \sqrt{2}\sigma\Im\left(\alpha\exp\left(-\imath\omega t\right)\right). \tag{19}$$

La cuenta para calcular $|\psi(p)|^2$ es análoga, sabiendo que $\exp\left(-\frac{\imath p_0\hat{x}}{\hbar}\right)$ es el operador desplazamiento en impulsos satisfaciendo la propiedad $\langle p|\exp\left(\frac{\imath p_0\hat{x}}{\hbar}\right)|\psi\rangle = \langle p-p_0|\psi\rangle$.

c) En cuanto a los valores medios, se tiene que

$$\langle x\rangle(t) = |N|^2\left(\langle\alpha(t)|x|\alpha(t)\rangle+\langle\alpha(t)|x|-\alpha(t)\rangle+\langle-\alpha(t)|x|\alpha(t)\rangle+\langle-\alpha(t)|x|-\alpha(t)\rangle\right). \tag{20}$$

Por lo visto anteriormente,

$$\langle\alpha(t)|x|\alpha(t)\rangle = x_{\alpha(t)} \tag{21}$$

y por ende, como $x_{-\alpha} = -x_\alpha$,

$$\langle x\rangle(t) = |N|^2\left(\langle\alpha(t)|x|-\alpha(t)\rangle+\langle-\alpha(t)|x|\alpha(t)\rangle\right) = 2|N|^2\Re\left\{\langle-\alpha(t)|x|\alpha(t)\rangle\right\}. \tag{22}$$

Para hacer esta cuenta, usamos que $a|\alpha\rangle = \alpha|\alpha\rangle \Leftrightarrow \langle\alpha|a^\dagger = \langle\alpha|\alpha^*$ obteniendo

$$\langle-\alpha|x|\alpha\rangle = \frac{1}{\sqrt{2}}\sigma\left(\langle-\alpha|a|\alpha\rangle+\langle-\alpha|a^\dagger|\alpha\rangle\right) = \frac{1}{\sqrt{2}}\sigma\left(\langle-\alpha|\alpha|\alpha\rangle+\langle-\alpha|-\alpha^*|\alpha\rangle\right) = \frac{1}{\sqrt{2}}\sigma\left(\alpha-\alpha^*\right)\langle-\alpha|\alpha\rangle = \frac{1}{\sqrt{2}}\sigma\left(\alpha-\alpha^*\right)e^{-2|\alpha|^2}. \tag{23}$$

Esto implica que

$$\Re\left\{\langle-\alpha|x|\alpha\rangle\right\} = 0 \tag{24}$$

y por lo tanto

$$\langle x\rangle(t) = 0. \tag{25}$$

Analogamente,

$$\langle p\rangle(t) = 2|N|^2\Re\left\{\langle-\alpha(t)|p|\alpha(t)\rangle\right\} \tag{26}$$

y

$$\langle-\alpha|p|\alpha\rangle = -\frac{\imath}{\sqrt{2}}\frac{\hbar}{\sigma}\left(\langle-\alpha|a|\alpha\rangle-\langle-\alpha|a^\dagger|\alpha\rangle\right) = \frac{\imath}{\sqrt{2}}\frac{\hbar}{\sigma}\left(\alpha+\alpha^*\right)e^{-2|\alpha|^2}. \tag{27}$$

Luego, al igual que antes,

$$\Re\left\{\langle-\alpha|p|\alpha\rangle\right\} = 0 \tag{28}$$

y

$$\langle p\rangle(t) = 0. \tag{29}$$