Julia Bertero Di Tella
gmail: juliabertero@gmail.com

Física Teórica 2 - ~~guía 6~~.

Ejercicio Integrador guía 6.

(a) verifico que $S(r)$ es unitorio, es decir, que se cumple que $S(r)\,S^{+}(r) = \mathbb{1}$:

$$S(r)\,S^{+}(r) = e^{\frac{r}{2}(a^2-a^{+2})}\, e^{\frac{r}{2}(a^{+2}-a^2)} \underset{\uparrow}{=} 1$$

si llamo $A = \frac{r}{2}(a^2-a^{+2})$ y $B=1$

$[A,B]=0$

y se cumple $e^{A}Be^{-A} = e^{0} = 1$

$$S^{-1}(r) = \left(e^{\frac{r}{2}(a^2-a^{+2})}\right)^{+} \underset{\uparrow}{=} e^{\frac{r}{2}(a^{+2}-a^2)} = e^{-\frac{r}{2}(a^2-a^{+2})} = S(-r)$$

como $S(r)S^{+}(r)=1$

$S^{+}(r) = S^{-1}(r)$

(b) uso la propiedad $e^{A}Be^{-B} = e^{c}B$ donde $[A,B]=cB$

y llamo $A = \frac{r}{2}(a^{+2}-a^2)$ y $B=x$

$$[A,B] = \frac{r}{2}[a^{+2}-a^2, x] = \frac{r}{2}\frac{m\omega}{2\hbar}\left[\left(x+\frac{ip}{m\omega}\right)^2-\left(x-\frac{ip}{m\omega}\right)^2, x\right]$$

$$= \frac{m\omega r}{4\hbar}\left[x^2+\frac{i(xp+px)}{m\omega}-\frac{p^2}{m^2\omega^2}-\left(x^2+(xp-px)\frac{i}{m\omega}-\frac{p^2}{m^2\omega^2}\right), x\right]$$

$$= \frac{m\omega r}{4\hbar}\left[\frac{2i}{m\omega}(xp+px), x\right] = \frac{ri}{2\hbar}[xp+px, x]$$

$$\underset{\uparrow}{=} \frac{ri}{2\hbar}\cdot(-2i\hbar x) = -ri^2x = -rx \Rightarrow c=-r$$

$$[xp, x] = x\underbrace{[p,x]}_{-i\hbar} + \underbrace{[x,x]}_{=0}p = -i\hbar x$$

$$[px, x] = p[x,x] + [p,x]x = -i\hbar x$$

$$\Rightarrow e^{A}Be^{-A} = S^{+}(r)\,x\,S(r) = e^{-r}x$$

(c) $Q_r \equiv e^r x \qquad P_r \equiv e^{-r} p \qquad a_r = \sqrt{\frac{m\omega}{2\hbar}} \left(x e^r + p \frac{i e^{-r}}{m\omega} \right)$

$[Q_r, P_r] = [e^r x, e^{-r} p] = e^r e^{-r} [x, p] = i\hbar$

$\Rightarrow$ Cumplen conmutación canónica.

$[a_r, a_r^+] \underset{\uparrow}{=} \frac{m\omega}{2\hbar} \left[\left(x e^r + p \frac{i e^{-r}}{m\omega} \right), \left(x e^r - p \frac{i e^{-r}}{m\omega} \right) \right]$

uso $\quad a_r^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(x e^r - p \frac{i e^{-r}}{m\omega} \right)$

Cálculo auxiliar:

- $a_r a_r^+ = \frac{m\omega}{2\hbar} \left(x e^r + p \frac{i e^{-r}}{m\omega} \right) \left(x e^r - p \frac{i e^{-r}}{m\omega} \right) =$

$= \frac{m\omega}{2\hbar} \cdot \left(x^2 e^{2r} \underbrace{- \frac{i}{m\omega} xp + \frac{i}{m\omega} px}_{-\frac{i}{m\omega}(xp - px) = -\frac{i}{m\omega}[x,p] = \frac{-i \cdot i\hbar}{m\omega} = \frac{\hbar}{m\omega}} + \frac{p^2 e^{-2r}}{m^2\omega^2} \right)$

$= \frac{m\omega}{2\hbar} \left(x^2 e^{2r} + \frac{\hbar}{m\omega} + \frac{p^2 e^{-2r}}{m^2\omega^2} \right)$

- $a_r^+ a_r = \frac{m\omega}{2\hbar} \left(x e^r - p \frac{i e^{-r}}{m\omega} \right) \left(x e^r + p \frac{i e^{-r}}{m\omega} \right)$

$= \frac{m\omega}{2\hbar} \left(x^2 e^{2r} + \frac{i}{m\omega} \underbrace{(xp - px)}_{= [x,p] = i\hbar} + \frac{p^2}{m^2\omega^2} \right)$

$= \frac{m\omega}{2\hbar} \left(x^2 e^{2r} - \frac{\hbar}{m\omega} + \frac{p^2}{m^2\omega^2} \right)$

$[a_r, a_r^+] = a_r a_r^+ - a_r^+ a_r = 2 \cdot \frac{m\omega}{2\hbar} \cdot \frac{\hbar}{m\omega} = 1$

a_r y a_r^+ cumplen con las condiciones de conmutación de los operadores de creación y destrucción, y Q_r y P_r son los análogos ~~a~~ los operadores de ~~creación y destrucción~~ posición y momento.

(d)

$$a_r |\alpha, r\rangle = a_r S(r)|\alpha\rangle = S(r) S^+(r) a_r S(r)|\alpha\rangle$$
$$\mathbb{1} = S(r) S^+(r)$$

$$= S(r) \cdot \sqrt{\frac{m\omega}{2\hbar}}\, S^+(r)\left(x e^{r} + \frac{p i e^{-r}}{m\omega}\right) S(r)\,|\alpha\rangle$$

$$= S(r) \cdot \sqrt{\frac{m\omega}{2\hbar}} \left[e^{r} \underbrace{S^+(r) x S(r)}_{e^{-r} x} + \frac{i e^{-r}}{m\omega} \overbrace{S^+(r) p S(r)}^{e^{r} p} \right] |\alpha\rangle$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\, S(r)\left[e^{r} e^{-r} x + \frac{i\, e^{-r}}{m\omega} e^{r} p \right]|\alpha\rangle$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\, S(r)\left[x + \frac{i p}{m\omega}\right]|\alpha\rangle$$

$$= S(r) \cdot a\, |\alpha\rangle \underset{a|\alpha\rangle = \alpha|\alpha\rangle}{=} S(r)\, \alpha |\alpha\rangle$$

$$= S(r)\,\alpha|\alpha\rangle = \alpha S(r)\,|\alpha\rangle = \alpha\,|\alpha, r\rangle$$

$\Rightarrow$ $|\alpha, r\rangle$ es ~~autovalor~~ autoestado de a_r con autovalor α.

Esto implica que al hacer un cambio de variables tal que $x \rightarrow Q_r = e^{r} x$ y $p \rightarrow P_r = e^{-r} p$ los estados coherentes siguen la transformación $|\alpha\rangle \longrightarrow |\alpha, r\rangle = S(r)|\alpha\rangle$ y mantienen ~~el misma energía~~ ~~ante un cambio la energía del estado~~ mismo autovalor.

(e)

$$\langle \alpha, r | x | \alpha, r\rangle \underset{\uparrow}{=} \langle \alpha | S^+(r) \, x \, S(r) | \alpha\rangle = \langle \alpha | e^{-r} x | \alpha \rangle = e^{-r} \langle \alpha | x | \alpha\rangle$$

$$\langle \alpha, r | = \langle \alpha | S^+(r)$$

$$\langle x \rangle = e^{-r} \sqrt{\frac{\hbar}{2m\omega}} (\alpha + \alpha^*)$$

$$\langle \alpha, r | p | \alpha, r\rangle = \langle \alpha | S^+(r) \, p \, S(r) | \alpha\rangle = e^{r} \langle \alpha | p | \alpha \rangle =$$

$$= \frac{e^{r}}{i} \sqrt{\frac{\hbar m \omega}{2}} (\alpha - \alpha^*)$$

$$\langle \alpha, r | x^2 | \alpha, r\rangle = \langle \alpha | S^+(r) \, x^2 S(r) | \alpha\rangle = e^{-2r} \langle \alpha | x^2 | \alpha \rangle =$$

$$= e^{-2r} \frac{\hbar}{2m\omega} (\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1)$$

$$S^+(r) x^2 S(r) \underset{\uparrow}{=} e^{-2r} x^2$$

$$\left[\frac{r}{2}(a^{+2} - a^2), x^2\right] = x \underbrace{\left[\frac{r}{2}(a^{+2} - a^2), x\right]}_{-rx} + \underbrace{\left[\frac{r}{2}a^{+2} - a^2, x\right]}_{-rx} x = -2r x^2$$

$$\langle \alpha, r | p^2 | \alpha, r \rangle = e^{2r} \langle \alpha | p^2 | \alpha \rangle = \frac{-e^{2r}}{2} \hbar m \omega (\alpha^2 + \alpha^{*2} - 2|\alpha|^2 - 1)$$

$$\Rightarrow \mathrm{Var}(x) = \frac{\hbar}{2m\omega} e^{-2r} (\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1) - \frac{\hbar}{2m\omega} e^{-2r} (\alpha^2 + \alpha^{*2} + 2|\alpha|^2)$$

$$= \frac{\hbar}{2m\omega} e^{-2r}$$

$$\mathrm{Var}(p) = -e^{2r} \frac{\hbar m \omega}{2} (\alpha^2 + \alpha^{*2} - 2|\alpha|^2 - 1)$$

$$- \left(-e^{2r} \frac{\hbar m \omega}{2}\right) (\alpha^2 + \alpha^{*2} - 2|\alpha|^2)$$

$$= e^{2r} \frac{\hbar m \omega}{2}$$

$$\Rightarrow \mathrm{Var}(x)\,\mathrm{Var}(p) = \frac{\hbar}{2m\omega} e^{-2r} e^{2r} \frac{\hbar m \omega}{2} = \frac{\hbar^2}{4}$$

$\Rightarrow$ son los estados de mínima incerteza lo cual implica que su función de onda es de la forma de un paquete gaussiano. Los estados $|\alpha, r\rangle$ son una traslación en el espacio de fases del estado fundamental

Donde se estaban los siguientes resultados:

$$\langle\alpha|x|\alpha\rangle = \langle\alpha|\sqrt{\frac{\hbar}{2m\omega}}(a+a^{+})|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|a+a^{+}|\alpha\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\langle\alpha|a|\alpha\rangle + \langle\alpha|a^{+}|\alpha\rangle\right) =$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,(\alpha^{*}+\alpha)$$

Calculo auxiliar:

- $\langle\alpha|a^{+}|\alpha\rangle = \alpha^{*}\langle\alpha|\alpha\rangle = \alpha^{*}$
  - $a|\alpha\rangle = \alpha|\alpha\rangle$
  - $\langle\alpha|a^{+} = \langle\alpha|\alpha^{*}$
- $\langle\alpha|a^{2}|\alpha\rangle = \alpha\langle\alpha|a|\alpha\rangle = \alpha^{2}$
- $\langle\alpha|a^{+2}|\alpha\rangle = \alpha^{*2}$
- $\langle\alpha|a^{+}a|\alpha\rangle = \langle\alpha|\alpha^{*}\alpha|\alpha\rangle = |\alpha|^{2}$
- $\langle\alpha|a|\alpha\rangle = \alpha\langle\alpha|\alpha\rangle = \alpha$

$$\langle\alpha|p|\alpha\rangle = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\langle\alpha|a-a^{+}|\alpha\rangle = \sqrt{\frac{\hbar m\omega}{2}}\,\frac{\alpha-\alpha^{*}}{i}$$

$$\langle\alpha|x^{2}|\alpha\rangle = \frac{\hbar}{2m\omega}\langle\alpha|(a+a^{+})(a+a^{+})|\alpha\rangle = \frac{\hbar}{2m\omega}\langle\alpha|a^{2}+aa^{+}+a^{+}a+a^{+2}|\alpha\rangle = \frac{\hbar}{2m\omega}\langle\alpha|a^{2}+2a^{+}a+1+a^{+2}|\alpha\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\alpha|a^{2}+\underbrace{aa^{+}+a^{+}a}+a^{+2}|\alpha\rangle$$

$$1 = [a^{+},a] = a^{+}a - aa^{+}$$

$$\Rightarrow aa^{+} = 1 + a^{+}a$$

$$= \frac{\hbar}{2m\omega}\left(\alpha^{2}+2|\alpha|^{2}+1+\alpha^{*2}\right)$$

$$\langle\alpha|p^{2}|\alpha\rangle = \frac{-\hbar m\omega}{2}\langle\alpha|(a-a^{+})(a-a^{+})|\alpha\rangle = \frac{-\hbar m\omega}{2}\langle\alpha|a^{2}-aa^{+}-a^{+}a+a^{+2}|\alpha\rangle$$

$$= \frac{-\hbar m\omega}{2}\langle\alpha|a^{2}-2a^{+}a-1+a^{+2}|\alpha\rangle = \frac{-\hbar m\omega}{2}(\alpha^{2}-2|\alpha|^{2}-1+\alpha^{*2})$$

(f)

$$\langle \alpha r | H | \alpha, r\rangle = \langle \alpha | S^{\dagger}(r) H S(r) | \alpha\rangle = \langle \alpha | S^{\dagger}(r) \left(\frac{p^2}{2m} + \frac{m\omega^2}{2} x^2\right) S(r) | \alpha\rangle$$

tomo Hamiltoniano del oscilador armónico $H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2$

$$= \frac{1}{2m} \langle \alpha | S^{\dagger}(r) p^2 S(r) | \alpha\rangle + \frac{m\omega^2}{2} \langle \alpha | S^{\dagger}(r) x^2 S(r) | \alpha\rangle$$

$$= \frac{1}{2m} \langle \alpha | e^{2r} p^2 | \alpha\rangle + \frac{m\omega^2}{2} \langle \alpha | e^{-2r} x^2 | \alpha\rangle$$

$$= \frac{e^{2r}}{2m} \langle \alpha | p^2 | \alpha\rangle + \frac{m\omega^2}{2} e^{-2r} \langle \alpha | x^2 | \alpha\rangle$$

$$= e^{2r} \cdot \frac{1}{2m} \cdot \frac{\hbar m\omega}{2} (\alpha^2 + \alpha^{*2} - 2|\alpha|^2 - 1) + e^{-2r} \frac{m\omega^2}{2} \frac{\hbar}{2m\omega} (\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1)$$

~~$= (\alpha^2 + \alpha^{*2}) \left(\frac{\hbar\omega}{4} e^{2r} + \frac{\hbar\omega}{4} e^{-2r} \right) +$~~

$$= + \frac{\hbar\omega}{4} (\alpha^2 + \alpha^{*2}) (e^{2r} + e^{-2r}) + \frac{\hbar\omega}{4} (2|\alpha|^2 + 1) (e^{2r} + e^{-2r})$$

$$= i \frac{\hbar\omega}{2} (\alpha^2 + \alpha^{*2}) \operatorname{sen}(2r) + \frac{\hbar\omega}{2} (2|\alpha|^2 + 1) \cos(2r)$$

(g) Para el caso $|0, r\rangle$ tengo:

$$\langle x(t)\rangle = \langle 0, r | x\cos(\omega t) + \frac{p}{m\omega} \operatorname{sen}\omega t | 0, r\rangle$$

$$= \cos(\omega t) \langle 0, r | x | 0, r\rangle + \frac{\operatorname{sen}\omega t}{m\omega} \langle 0, r | p | 0, r\rangle$$

$$= 0$$

del inciso (e) $\langle x\rangle\big|_{\alpha=0} = \langle p\rangle\big|_{\alpha=0} = 0$

$\langle p(t)\rangle = 0$ analogamente.

$$\langle x(t)^2\rangle = \langle 0,r|\left(x\cos\omega t + \frac{p}{m\omega}\operatorname{sen}\omega t\right)\left(x\cos\omega t+\frac{p}{m\omega}\operatorname{sen}\omega t\right)|0,r\rangle$$

$$= \langle 0,r|\, x^2\cos^2\omega t + \frac{p^2}{m^2\omega^2}\operatorname{sen}^2\omega t + (xp+px)\frac{\cos\omega t\operatorname{sen}\omega t}{m\omega}\,|0,r\rangle$$

$$= \langle 0,r|x^2|0,r\rangle\cos^2\omega t + \langle 0,r|p^2|0,r\rangle\frac{\operatorname{sen}^2\omega t}{m^2\omega^2} +$$

$$+ \quad \langle 0,r|xp+px|0,r\rangle\frac{\cos\omega t\operatorname{sen}\omega t}{m\omega}$$

Calculo auxiliar:

- $\langle 0,r|x^2|0,r\rangle = \frac{\hbar}{2m\omega}e^{-2r}$

  (caso (e) caso $\alpha=0$)

- $\langle 0,r|p^2|0,r\rangle = \frac{\hbar m\omega}{2}e^{2r}$

- $\langle 0,r|\underbrace{xp+px}|0,r\rangle = \langle 0,r|2xp + i\hbar|0,r\rangle$

  $i\hbar=[x,p]=xp-px \Rightarrow px = xp - i\hbar$

  $= 2\langle 0,r|xp|0,r\rangle + i\hbar = \hbar(1+i)$

- $\langle\alpha r|xp|\alpha r\rangle = \cancel{\langle\alpha|S^\dagger(r)\frac{\hbar}{2}(a^2-a^{\dagger 2}+aa^\dagger+a^\dagger a)S(r)|\alpha\rangle}$ $[a,a^\dagger]=1$

  $\cancel{=\frac{\hbar}{2}\langle\alpha|S^\dagger(r)(a^2-a^{\dagger 2}-1)S(r)|\alpha\rangle}$

  $= \langle\alpha|S^\dagger(r)\,x\,p\,S(r)|\alpha\rangle$

  $= \langle\alpha|\underbrace{S^\dagger(r)xS(r)}_{e^{-r}x}\underbrace{S^\dagger(r)pS(r)}_{e^{r}p}|\alpha\rangle = \langle\alpha|xp|\alpha\rangle$

  $= \frac{\hbar}{2}\langle\alpha|\,a^2-a^{\dagger 2}+\underbrace{a^\dagger a - aa^\dagger}_{=[a^\dagger,a]=1}|\alpha\rangle$

  $= \frac{\hbar}{2}(\alpha^2-\alpha^{*2}+1)$

$$\langle x^2(t)\rangle = \frac{\hbar}{2m\omega}e^{-2r}\cos^2\omega t + e^{2r}\cdot\frac{\hbar m\omega}{2}\frac{\operatorname{sen}^2\omega t}{m^2\omega^2} + \frac{\hbar}{m\omega}\cos\omega t\operatorname{sen}\omega t\,(\alpha^2-\alpha^{*2}+1+i)$$

$$\langle x^2(t)\rangle = \frac{\hbar}{2m\omega}\left(e^{-2r}\cos^2\omega t + e^{2r}\sin^2\omega t\right) + \frac{\hbar}{2m\omega}\,\text{sen}(2\omega t)\,(1+i)$$

$$\cancel{\langle x^2(t)\rangle = \frac{\hbar}{2m\omega}\left[e\right.}$$

$$\langle p^2(t)\rangle = \langle 0,r|\; p^2\cos^2\omega t + m^2\omega^2 x^2\,\text{sen}^2\omega t - (xp+px)\frac{m\omega}{2}\,\text{sen}(2\omega t)\;|0,r\rangle$$

$$= \langle 0,r|p^2|0,r\rangle\cos^2\omega t + m^2\omega^2\,\text{sen}^2\omega t\,\langle 0,r|x^2|0,r\rangle - \frac{m\omega}{2}\,\text{sen}(2\omega t)\,\langle 0,r|xp+px|0,r\rangle$$

$$= e^{2r}\frac{\hbar m\omega}{2}\cos^2\omega t + m^2\omega_0^2\frac{\hbar}{2m\omega}e^{-2r}\,\text{sen}^2\omega t - \frac{m\omega}{2}\,\text{sen}(2\omega t)\,\hbar(1+i)$$

$$= \frac{\hbar m\omega}{2}\left[e^{2r}\cos^2\omega t + e^{-2r}\,\text{sen}^2\omega t - \text{sen}(2\omega t)(1+i)\right]$$

$$\text{Var}[x(t)] = \frac{\hbar}{2m\omega}\left[e^{-2r}\cos^2\omega t + e^{2r}\,\text{sen}^2\omega t + \text{sen}(2\omega t)(1+i)\right]$$

y

$$\text{Var}[p(t)] = \frac{\hbar m\omega}{2}\left[e^{2r}\cos^2\omega t + e^{-2r}\,\text{sen}^2\omega t - \text{sen}(2\omega t)(1+i)\right]$$

$$\text{Var}(x(t))\,\text{Var}(p(t)) = \frac{\hbar^2}{4}\Big[\cos^4\omega t + \text{sen}^4\omega t - \text{sen}^2(2\omega t)(2i^0) + \cos^2\omega t\,\text{sen}^2\omega t\,(e^{-4r}+e^{4r}) + \text{sen}(2\omega t)(1+i)\cos^2\omega t\,(e^{2r}-e^{-2r}) - \text{sen}(2\omega t)(1+i)\,\text{sen}^2\omega t\,(e^{2r}-e^{-2r})\Big]$$

$$Var(x(t))\,Var(p(t)) = \frac{\hbar^2}{4}\Big[\cos^4\omega t + \sin^4\omega t - i\,\sin^2(2\omega t)\left(1 \pm i\cos(4r)\right)$$

$$+ i\,\sin(2\omega t)\,\sin(2r)\,(1+i)\left(\cos^2\omega t - \sin^2\omega t\right)\Big]$$

$$= \frac{\hbar^2}{4}\Big[\cos^4(\omega t) + \sin^4(\omega t) + \sin^2(2\omega t)\left(\cos(4r) - i\right)$$

$$+ \sin(2\omega t)\,\sin(2r)\,\frac{(i-1)}{2}\,\cos(2\omega t)\Big]$$

Si $t=0$

$$Var(x)\,Var(p) = \frac{\hbar^2}{4}$$

Si $t=\frac{\pi}{2\omega}$ $\qquad Var(x)\,Var(p) = \frac{\hbar^2}{4}\left[\, 1 + \right] = \frac{\hbar^2}{4}$

Si $t=\frac{\pi}{\omega}$ $\qquad Var(x)\,Var(p) = 1\cdot\frac{\hbar^2}{4}$

$\Rightarrow$ para $t=0$, $\frac{\pi}{2\omega}$ y $t=\frac{\pi}{\omega}$ el producto de las varianzas es mínima y esperamos que al tener incerteza mínima tenga la forma de una gaussiana.

Dado que el estado squeezed general $|\alpha, r\rangle$ es una traslación en el espacio de fases de $|0, r\rangle$ espero que evolucione temporalmente de la misma forma con una diferente frecuencia.