Guía 7: Rotaciones y Momento Angular

P15 $\{T^{(k)}_{-k}, T^{(k)}_{-k+1}, \dots T^{(k)}_{k}\}$ $2k+1$ operadores

$T^{(k)}_q$: componente q de tensor de rango k.

$$\mathcal{D}(R)\, T^{(k)}_q\, \mathcal{D}^\dagger(R) = \sum_{q'} \mathcal{D}_{q'q}(R)\, T^{(k)}_{q'}$$

ó

$$\begin{cases} [J_z, T^{(k)}_q] = \hbar q\, T^{(k)}_q \\ [J_\pm, T^{(k)}_q] = \hbar\sqrt{k(k+1) - q(q\pm 1)}\; T^{(k)}_{q\pm 1} \end{cases}$$

(a) $K=1$ $\underline{q=0}$: $V^{(1)}_0 = V_z$ $\stackrel{si}{=}$ $[J_z, V_z] = \underbrace{0}_{q=0}$

Se cumple.

$\underline{q=\pm 1}$: $[J_\pm, V^{(1)}_0] = \hbar\sqrt{2}\, V^{(1)}_{\pm 1}$

$$\Rightarrow \quad V^{(1)}_{\pm 1} = \frac{1}{\sqrt{2}} [J_x \pm i J_y, V_z]$$

$$V^{(1)}_{\pm 1} = \frac{1}{\sqrt{2}\hbar}(-i\hbar V_y \pm \underbrace{i^2}_{-1}\hbar V_x)$$

$$\therefore \quad \boxed{V^{(1)}_{\pm 1} = \mp \frac{1}{\sqrt{2}} (V_x \pm i V_y)}$$

(b) $Y^{m}_{l=1}(x,y,z)$

$$\begin{cases} Y^0_1 = \frac{1}{r}\sqrt{\frac{3}{4\pi}}\, z \\ Y^1_1 = -\frac{1}{r}\sqrt{\frac{3}{8\pi}}\, (x+iy) \\ Y^{-1}_1 = \frac{1}{r}\sqrt{\frac{3}{8\pi}}\, (x-iy) \end{cases}$$

$$V_q^{(1)} = r Y_1^q (V_x, V_y, V_z)$$

$$\begin{cases} V_0^{(1)} = \sqrt{\frac{3}{4\pi}} V_z \\ V_{\pm 1}^{(1)} = \mp \sqrt{\frac{3}{4\pi}} \left(\frac{V_x \pm i V_y}{\sqrt{2}} \right) \end{cases}$$

Comparado con (a) hay un factor común que no es relevante para satisfacer las relaciones de conmutación.

$\therefore$ $V_q^{(1)} = r Y_1^q (V_x, V_y, V_z)$

es tensor esférico irreducible de rango 1, componente q.

(c)

$$\mathcal{D}(R)|lm\rangle = \sum_{m'} |lm'\rangle\langle lm'| \mathcal{D}(R)|lm\rangle$$
$$= \sum_{m'} |lm'\rangle \mathcal{D}_{m'm}^{(l)}(R)$$

proyectamos en posición ($\langle \hat{r}|$):

$$\langle \hat{r}|\mathcal{D}(R)|lm\rangle = \sum_{m'} \langle \hat{r}|lm'\rangle \mathcal{D}_{m'm}^{(l)}(R)$$

$$Y_l^m(R^{-1}\hat{r}) = (Y_l^m)' = \sum_{m'} \mathcal{D}_{m'm}^{(l)}(R)\, Y_l^{m'}$$

$\mathcal{D}(R)\vec{V}\mathcal{D}^\dagger(R)$

$$V_q^{(k)} = r^k Y_k^q (V_x, V_y, V_z)$$

ante rotaciones:

$$r^k \mathcal{D}^\dagger(R)\, Y_k^q (V_x, V_y, V_z) \mathcal{D}(R)$$

$\therefore$ $\mathcal{D}(R) V_q^{(k)} \mathcal{D}^\dagger(R) = r^k \sum_{q'} \mathcal{D}_{q'q}^{(k)}(R) Y_k^{q'}(V_x, V_y, V_z)$

$$= \sum_{m'} \mathcal{D}_{m'm}^{(k)}(R)\, V_q^{(k)}$$

es tensor esférico irreducible rango k, comp. q.

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(c) $\vec{V}$ y $\vec{W}$

por el problema anterior:

$$V_0^{(1)} = V_z \qquad V_{\pm 1}^{(1)} = \mp \frac{V_x \pm i V_y}{\sqrt{2}}$$

$$W_0^{(1)} = W_z \qquad W_{\pm 1}^{(1)} = \mp \frac{W_x \pm i W_y}{\sqrt{2}}$$

- $k=0$ (escalar): Usemos la tabla de CG

$$T_0^{(0)} = V_1^{(1)} W_{-1}^{(1)} \underbrace{\langle 11; 1 -1 | 00\rangle}_{\sqrt{1/3}} + V_0^{(1)} W_0^{(1)} \underbrace{\langle 11, 00 | 00\rangle}_{-\sqrt{1/3}} + V_{-1}^{(1)} W_1^{(1)} \underbrace{\langle 11, -1, 1 | 00\rangle}_{\sqrt{1/3}}$$

$$T_0^{(0)} = -\sqrt{\frac{1}{3}} \left[\frac{1}{2}(V_x + iV_y)(W_x - iW_y) + V_z W_z + \frac{1}{2}(V_x - iV_y)(W_x + iW_y)\right]$$

$$T_0^{(0)} = -\sqrt{\frac{1}{3}} \left(V_x W_x + V_y W_y + V_z W_z \right) = -\sqrt{\frac{1}{3}}\, \vec{V} \cdot \vec{W}$$

- $k=1$ (vector):

i) $$T_0^{(1)} = V_1^{(1)} W_{-1}^{(1)} \underbrace{\langle 11; 1 -1 | 10\rangle}_{\sqrt{1/2}} + V_0^{(1)} W_{-1}^{(1)} \underbrace{\langle 11, 00 | 10\rangle}_{0} + V_{-1}^{(1)} W_1^{(1)} \underbrace{\langle 11, -1\, 1 | 10\rangle}_{-\sqrt{1/2}}$$

$$T_0^{(1)} = -\sqrt{\frac{1}{2}}\left[\frac{1}{2}(V_x + iV_y)(W_x - iW_y) - \frac{1}{2}(V_x - iV_y)(W_x + iW_y)\right]$$

$$\therefore\ T_0^{(1)} = -\frac{1}{\sqrt{2}}\left[-iV_xW_y + iV_yW_x\right] = \frac{i}{\sqrt{2}}(\vec{V}\times\vec{W})_z$$

ii) $T_1^{(1)} = V_1^{(1)} W_0^{(1)} \underbrace{\langle 11,10|11\rangle}_{\sqrt{\frac{1}{2}}} + V_0^{(1)} W_1^{(1)} \underbrace{\langle 11,01|11\rangle}_{-\sqrt{\frac{1}{2}}}$

$$T_1^{(1)} = \sqrt{\frac{1}{2}}\cdot\frac{1}{\sqrt{2}}\left[(-)(V_x + iV_y)(W_z) + (V_z)(W_x + iW_y)\right]$$

$$= \frac{i}{\sqrt{2}}\left[\frac{-(\vec{V}\times\vec{W})_x + i(\vec{V}\times\vec{W})_y}{\sqrt{2}}\right]$$

iii) $T_{-1}^{(1)} = V_{-1}^{(1)} W_0^{(1)} \underbrace{\langle 11-10|1-1\rangle}_{-\sqrt{\frac{1}{2}}} + V_0^{(1)} W_{-1}^{(1)} \underbrace{\langle 110-1|1-1\rangle}_{\sqrt{\frac{1}{2}}}$

$$T_{-1}^{(1)} = \frac{i}{\sqrt{2}}\left[\frac{(\vec{V}\times\vec{W})_x - i(\vec{V}\times\vec{W})_y}{\sqrt{2}}\right]$$

$\therefore$ $T_q^{(1)}$ está asociado al vector $\vec{V}\times\vec{W}$ (pseudovector)

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(a) $$T_q^{(k)} = \sum_{q_1 q_2} V_{q_1}^{(k_1)} W_{q_2}^{(k_2)} \langle k_1 k_2; q_1 q_2 | k q \rangle \quad (1)$$

Probar que si $V_{q_1}^{(k_1)}$ y $W_{q_2}^{(k_2)}$ son tensores esféricos irreducibles $\Rightarrow$ $T_q^{(k)}$ también lo es.

Queremos ver si:

$$\mathcal{D}(R)\, T_q^{(k)}\, \mathcal{D}^\dagger(R) = \sum_{q'} \mathcal{D}_{q'q}^{(k)}(R)\, T_{q'}^{(k)} \quad (2)$$

$$\sum_{q_1 q_2} \mathcal{D}(R) V_{q_1}^{(k_1)} \underbrace{\mathcal{D}^\dagger(R) \mathcal{D}(R)}_{\mathbb{1}} W_{q_2}^{(k_2)} \mathcal{D}(R) \langle k_1 k_2; q_1 q_2 | k q \rangle$$

$$= \sum_{q_1 q_2} \sum_{q_1' q_2'} V_{q_1'}^{(k_1)} W_{q_2'}^{(k_2)} \underbrace{\mathcal{D}_{q_1' q_1}^{(k_1)}(R)\, \mathcal{D}_{q_2' q_2}^{(k_2)}(R)} \langle k_1 k_2; q_1 q_2 | k q \rangle \quad (3)$$

- Cálculo auxiliar: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

Escribamos el operador de rotaciones en $\mathcal{H}$ en función de operadores de rotación en $\mathcal{H}_{1,2}$.

$$\mathcal{D}(\hat{n},\theta) = e^{-i\frac{\vec{J}\cdot\hat{n}\theta}{\hbar}} = e^{-i\frac{(\vec{J}_1+\vec{J}_2)\cdot\hat{n}\theta}{\hbar}}$$

$$= e^{-i\frac{\vec{J}_1\cdot\hat{n}\theta}{\hbar}} e^{-i\frac{\vec{J}_2\cdot\hat{n}\theta}{\hbar}} = \mathcal{D}_1(\hat{n},\theta)\,\mathcal{D}_2(\hat{n},\theta)$$

Actuando sobre un estado de la base producto:

$$\mathcal{D}(R)\,|j_1 j_2, m_1 m_2\rangle = (\mathcal{D}_1(R)|j_1 m_1\rangle)(\mathcal{D}_2(R)|j_2 m_2\rangle)$$

$$= \sum_{m_1' m_2'} |j_1 j_2\, m_1' m_2'\rangle\, \mathcal{D}_{m_1' m_1}^{j_1}\, \mathcal{D}_{m_2' m_2}^{j_2}$$

$\sum_{jm} |jm'\rangle\langle jm|$ $\sum_{jm} |jm\rangle\langle jm|$

$$\sum_{jm} \sum_{m'} |jm'\rangle \mathcal{D}_{m'm}^{j}(R) \langle jm | j_1 j_2\, m_1 m_2\rangle$$

proyectando sobre: $\langle j_1 j_2, m_1'' m_2'' |$:

$$\mathcal{D}^{j_1}_{m_1 m_1'}(R)\, D^{j_2}_{m_2 m_2'}(R) = \sum_{J m m'} \langle j_1 j_2\, m_1 m_2 | j m \rangle\, \mathcal{D}^{j}_{m m'}(R) \langle j m' | j_1 j_2 m_1' m_2' \rangle$$

Alternativamente:

$$\mathcal{D}^{j}_{m m'}(R) = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \langle j m | j_1 j_2 m_1 m_2 \rangle D^{j_1}_{m_1 m_1'}(R)\, D^{j_2}_{m_2 m_2'}(R) \langle j_1 j_2 m_1' m_2' | j m' \rangle$$

$j \to K \quad m \to Q \quad , m' \to Q'$

Usando esta relación tenemos:

$$\mathcal{D}^{(k_1)}_{q_1' q_1}(R)\, D^{(k_2)}_{q_2' q_2}(R) = \sum_{KQQ'} \langle k_1 k_2\, q_1' q_2' | K Q' \rangle D^{K}_{Q' Q}(R) \langle K Q | k_1 k_2 q_1 q_2 \rangle$$

reemplazando en (3); se suma usando (4) sobre q_1', q_2'

$$\sum_{q_1' q_2'} V^{k_1}_{q_1'} W^{k_2}_{q_2'} \langle k_1 k_2 q_1' q_2' | K Q' \rangle = T^{K}_{Q'}$$

La suma sobre $q_1 q_2$ se hace usando la ortogonalidad de los coeficientes de Clebsch-Gordan:

$$\sum_{q_1 q_2} \langle K Q | k_1 k_2 q_1 q_2 \rangle \langle k_1 k_2 q_1 q_2 | k q \rangle = \delta_{Kk}\, \delta_{Qq}$$

Finalmente en (2):

$$\mathcal{D}(R)\, T^{k}_{q}\, \mathcal{D}^{\dagger}(R) = \sum_{KQQ'} T_{Q'}^{K} D^{K}_{Q'Q}(R)\, \delta_{Kk}\, \delta_{Qq}$$

$$\Rightarrow \quad \boxed{\mathcal{D}(R)\, T^{(k)}_{q}\, D^{\dagger}(R) = \sum_{q'} T^{k}_{q'} D^{k}_{q' q}(R)}$$

(b) Si $k_1 = 1$, $k_2 = 1$ $\quad V^1_{q_1} W^1_{q_2}$ son productos de componentes de vectores

$$|k_1 - k_2| \le K \le k_1 + k_2 \Rightarrow K = \{0, 1, 2\}$$

Rango 0 (escalar): $\bar{V}\cdot\bar{W}$ 1
Rango 1 (vector): $\bar{V}\times\bar{W}$ 3
Rango 2 (tensor):   5
Simétrico y Traza nula

} total = 9 componentes independientes.

5 condiciones.

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Con [P17] c)

- $K=2$ (tensor simétrico traza nula)

i) $T_0^{(2)} = V_1^{(1)} W_{-1}^{(1)} \underbrace{\langle 1\,1\,1\,{-1} | 2\,0\rangle}_{\sqrt{\frac{1}{6}}}$

$+ V_0^{(1)} W_0^{(1)} \underbrace{\langle 1\,1\,0\,0 | 2\,0\rangle}_{\sqrt{\frac{2}{3}}}$

$+ V_{-1}^{(1)} W_1^{(1)} \underbrace{\langle 1\,1\,{-1}\,1 | 2\,0\rangle}_{+\sqrt{\frac{1}{6}}}$

$$T_0^{(2)} = \sqrt{\frac{1}{6}} \cdot \left[-\frac{(V_x + iV_y)(W_x - iW_y)}{2} + 2V_z W_z - \frac{(V_x - iV_y)(W_x + iW_y)}{2} \right]$$

$$T_0^{(2)} = \frac{1}{\sqrt{6}} \left[2V_z W_z - V_x W_x - V_y W_y \right]$$

Si $\bar{V} = \bar{W} = \bar{R}$

$$\boxed{T_0^{(2)} = \frac{1}{\sqrt{6}} (3z^2 - r^2)}$$

ii) $T_{\pm 1}^{(2)} = V_{\pm 1}^{(1)} W_0^{(1)} \underbrace{\langle 1\,1\,1\,0 | 2\,1\rangle}_{\sqrt{\frac{1}{2}}} + V_0^{(1)} W_{\pm 1}^{(1)} \underbrace{\langle 1\,1\,0\,1 | 2\,1\rangle}_{\sqrt{\frac{1}{2}}}$

$$T_{\pm 1}^{(2)} = \mp \sqrt{\frac{1}{2}} \frac{1}{\sqrt{2}} \left[(V_x \pm iV_y) W_z + V_z (W_x \pm iW_y) \right]$$

Si $\bar{V} = \bar{W} = \bar{R}$

$$\boxed{T_{\pm 1}^{(2)} = \mp (xz \pm iyz)}$$

iii) $T_{\pm 2}^{(2)} = V_{\pm 1}^{(1)} W_{\pm 1}^{(1)}$

$$T_{\pm 2}^{(2)} = \frac{1}{2} (V_x \pm iV_y)(W_x \pm iW_y) = \frac{1}{2} (V_x W_x - V_y W_y \pm 2i V_x W_y)$$

Si $\bar{V} = \bar{W} = \bar{R}$

$$T^{(2)}_{\pm 2} = \frac{1}{2}(x^2 - y^2 \pm 2i\,xy)$$