Guía 7: Rotaciones y Momento Angular

P3 Usaremos: $\begin{cases} S_i = \frac{\hbar}{2}\sigma_i \\ \sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k \end{cases}$

$$\mathcal{D}^{1/2}(\hat{z},\varphi) = e^{-i\frac{S_z\varphi}{\hbar}} = \cos\frac{\varphi}{2}\mathbb{1} - i\sigma_z\sin\frac{\varphi}{2}$$

- $\langle\psi'|S_z|\psi'\rangle = \langle\psi|e^{\frac{iS_z\varphi}{\hbar}} S_z e^{-\frac{iS_z\varphi}{\hbar}}|\psi\rangle$

$$\langle\psi'|S_z|\psi'\rangle = \langle\psi|S_z|\psi\rangle$$

- $\langle\psi'|S_x|\psi'\rangle = \langle\psi|\underbrace{e^{\frac{i\sigma_z\varphi}{2}}\sigma_x e^{-\frac{i\sigma_z\varphi}{2}}}_{\sigma_x'}|\psi\rangle\frac{\hbar}{2}$

$$\begin{aligned}
\sigma_x' &= \left(\cos\frac{\varphi}{2}\mathbb{1} + i\sin\frac{\varphi}{2}\sigma_z\right)\sigma_x\left(\cos\frac{\varphi}{2}\mathbb{1} - i\sin\frac{\varphi}{2}\sigma_z\right) \\
&= \left(\cos\frac{\varphi}{2}\mathbb{1} + i\sin\frac{\varphi}{2}\sigma_z\right)\left(\cos\frac{\varphi}{2}\sigma_x - \sin\frac{\varphi}{2}\sigma_y\right) \\
&= \cos^2\frac{\varphi}{2}\sigma_x - \sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\sigma_y + i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}(i\sigma_y) \\
&\qquad - i\sin^2\frac{\varphi}{2}(-i\sigma_x) \\
&= \underbrace{\left(\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right)}_{\cos\varphi}\sigma_x - \underbrace{2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}_{\sin\varphi}\sigma_y
\end{aligned}$$

$$\therefore \langle\psi'|S_x|\psi'\rangle = \cos\varphi\langle\psi|S_x|\psi\rangle - \sin\varphi\langle\psi|S_y|\psi\rangle$$

$$\circ \langle \psi' | S_y | \psi' \rangle = \langle \psi | \underbrace{e^{i\sigma_z \frac{\varphi}{2}} \sigma_y e^{-i\sigma_z \frac{\varphi}{2}}}_{O_y} | \psi \rangle \left(\frac{\hbar}{2}\right)$$

$$O_y = \left(\cos\frac{\varphi}{2}\mathbb{1} + i\sin\frac{\varphi}{2}\sigma_z\right)\sigma_y\left(\cos\frac{\varphi}{2}\mathbb{1} - i\sin\frac{\varphi}{2}\sigma_z\right)$$

$$= \left(\cos\frac{\varphi}{2}\mathbb{1} + i\sin\frac{\varphi}{2}\sigma_z\right)\left(\cos\frac{\varphi}{2}\sigma_y + \sin\frac{\varphi}{2}\sigma_x\right)$$

$$= \cos^2\frac{\varphi}{2}\sigma_y + \cos\frac{\varphi}{2}\sin\frac{\varphi}{2}\sigma_x + i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}(-i\sigma_x)$$

$$+ i\sin^2\frac{\varphi}{2}(i\sigma_y)$$

$$= \underbrace{\left(\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right)}_{\cos\varphi}\sigma_y + \left(\underbrace{2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}_{\sin\varphi}\right)\sigma_x$$

$$\therefore \langle \psi' | S_y | \psi' \rangle = \cos\varphi \langle \psi | S_y | \psi \rangle + \sin\varphi \langle \psi | S_x | \psi \rangle$$

Los valores medios entre estados rotados equivalen a una rotación del vector de valores medios original.

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$\boxed{P5}$ $J=1$ base: $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$ $(\{|jm\rangle\})$

$$J_\pm |j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\,|j,m\pm1\rangle$$

i) J_z: $J_z = \hbar \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

ii) J^2: $J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$

$$J^2 = 2\hbar^2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

iii) J_x y J_y:

$$\begin{aligned} J_+ &= J_x + iJ_y \\ J_- &= J_x - iJ_y \end{aligned} \Rightarrow \begin{cases} J_x = \frac{1}{2}(J_+ + J_-) \\ J_y = \frac{1}{2i}(J_+ - J_-) \end{cases}$$

$J_+|1,-1\rangle = \hbar\sqrt{2-0}\,|1,0\rangle$ $\qquad J_-|1,-1\rangle = 0$

$J_+|1,0\rangle = \hbar\sqrt{2-0}\,|1,1\rangle$ $\qquad J_-|1,0\rangle = \hbar\sqrt{2}\,|1,-1\rangle$

$J_+|1,1\rangle = 0$ $\qquad J_-|1,1\rangle = \hbar\sqrt{2}\,|1,0\rangle$

$$J_+ = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad J_- = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$\therefore$ $J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\qquad J_y = \frac{\hbar}{\sqrt{2}\,i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

iv) $J_x J_y = \frac{\hbar^2}{2i}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$J_y J_x = \frac{\hbar^2}{2i}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}$

$[J_x, J_y] = J_x J_y - J_y J_x = \hbar^2 i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i\hbar J_z$ ✓

(b) Diagonalizar J_y : ó usar proyectores (entrega 2)

autovalores $\{\hbar, 0, -\hbar\}$

- $J_y \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} \Rightarrow \frac{\hbar}{\sqrt{2}\, i}\begin{pmatrix} a \\ -1+b \\ -a \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$

$a = \sqrt{2}\, i \qquad b = -1$

$\therefore \quad |1, 1_y\rangle = \frac{1}{2}\left(|1,1\rangle + \sqrt{2}\, i\, |1,0\rangle - |1,-1\rangle\right)$

- $J_y \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 0 \Rightarrow \frac{\hbar}{\sqrt{2}\, i}\begin{pmatrix} a \\ -1+b \\ -a \end{pmatrix} = 0$

$a = 0 \qquad b = 1.$

$\therefore \quad |1, 0_y\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle + |1,-1\rangle\right)$

- $J_y \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = -\hbar \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} \Rightarrow \frac{\hbar}{\sqrt{2}\, i}\begin{pmatrix} a \\ -1+b \\ -a \end{pmatrix} = -\hbar \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$

$a = -\sqrt{2}\, i \qquad b = -1$

$\therefore \quad |1, 0_y\rangle = \frac{1}{2}\left(|1,1\rangle - \sqrt{2}\, i\, |1,0\rangle - |1,-1\rangle\right)$

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Cont $\boxed{P5}$

(c) Como $\{\hbar, 0, -\hbar\}$ son los autovalores:

- $$J_z(J_z+\hbar)(J_z-\hbar)\begin{cases}|1,1\rangle\\|1,0\rangle\\|1,-1\rangle\end{cases}=0$$

$\therefore$ $J_z(J_z+\hbar)(J_z-\hbar)=0$ (operador nulo)

- $$J_y(J_y+\hbar)(J_y-\hbar)\begin{cases}|1,1_y\rangle\\|1,0_y\rangle\\|1,-1_y\rangle\end{cases}=0$$

$\therefore$ $J_y(J_y+\hbar)(J_y-\hbar)=0$

(d) Usando esta relación:

$$J_y(J_y^2-\hbar^2)=J_y^3-\hbar^2J_y=0$$

$$\therefore \boxed{\left(\frac{J_y}{\hbar}\right)^3=\left(\frac{J_y}{\hbar}\right)}$$

(e) Usamos $\boxed{P13}$ (b) $\quad B=\frac{J_y}{\hbar} \qquad B^3=B$

$$D^{(1)}(\hat{y},\beta)=e^{-i\frac{J_y\beta}{\hbar}}=1-i\left(\frac{J_y}{\hbar}\right)\sin\beta-\left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta)$$

$$\left(\frac{J_y}{\hbar}\right)^2=-\frac{1}{2}\begin{pmatrix}-1&0&1\\0&-2&0\\1&0&-1\end{pmatrix}$$

$$d^{(1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{\sin\beta}{\sqrt{2}} & \frac{1}{2}(1-\cos\beta) \\ +\frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} \\ \frac{1}{2}(1-\cos\beta) & \frac{\sin\beta}{\sqrt{2}} & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

α, β, γ: angulos de Euler:

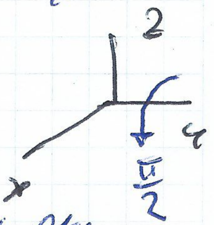
$$\mathcal{D}^{(j)}(\alpha,\beta,\gamma) = e^{-\frac{iJ_z\alpha}{\hbar}} \underbrace{e^{-\frac{iJ_y\beta}{\hbar}}}_{d^{(j)}(\beta)} e^{-\frac{iJ_z\gamma}{\hbar}}$$

Elementos de matriz del operador de Rotaciones (Matriz $\mathcal{D}$ de Wigner)

$$\mathcal{D}^{(j)}_{m',m}(\alpha,\beta,\gamma) = \langle j m' | \mathcal{D}^{(j)}_{(\alpha,\beta,\gamma)} | j m \rangle$$
$$= e^{-im'\alpha} \underbrace{d^{j}_{m'm}(\beta)}_{\text{matriz d de Wigner.}} e^{-im\gamma}$$

Obtención de: $|1, 1_x\rangle$, $|1, 0_x\rangle$, $|1, -1_x\rangle$

Usando rotaciones:



- $$|1, 1_x\rangle = \mathcal{D}^{(1)}(\hat{y}, \tfrac{\pi}{2}) |1, 1\rangle$$
$$= \frac{1}{2}\left(|1,1\rangle + \sqrt{2}\,|1,0\rangle + |1,-1\rangle\right)$$ (1era columna)

- $$|1, 0_x\rangle = \mathcal{D}^{(1)}(\hat{y}, \tfrac{\pi}{2}) |1 0\rangle$$
$$= \frac{1}{\sqrt{2}}\left(-|1,1\rangle + |1,-1\rangle\right)$$

- $$|1, -1_x\rangle = \mathcal{D}^{(1)}(\hat{y}, \tfrac{\pi}{2}) |1,-1\rangle$$
$$= \frac{1}{2}\left(|1,1\rangle - \sqrt{2}\,|1,0\rangle + |1,-1\rangle\right)$$

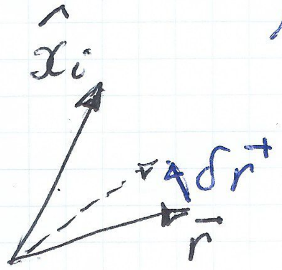
Momento angular en representación de coordenadas

Rotación en el eje $\hat{x}_i$, ángulo η:

$$\langle \vec{r} | e^{-i\frac{\hat{L}_i \eta}{\hbar}} | \psi \rangle$$

$$\underbrace{\langle \vec{r} - \delta\vec{r} | \psi \rangle}_{\psi(\vec{r} - \delta\vec{r})} = \langle \vec{r} | 1 - i\frac{L_i \eta}{\hbar} | \psi \rangle$$

$$= \psi(\vec{r}) - \frac{i\eta}{\hbar} \langle \vec{r} | L_i | \psi \rangle$$



$$\delta\vec{r} = \hat{x}_i \times \vec{r}\,\eta$$
$$= \epsilon_{ijk}\, x_j\, \hat{x}_k\, \eta$$

$\hat{x}$: $\delta\vec{r} = (y\hat{z} - z\hat{y})\eta$

$\hat{y}$: $\delta\vec{r} = (z\hat{x} - x\hat{z})\eta$

$\hat{z}$: $\delta\vec{r} = (x\hat{y} - y\hat{x})\eta$

1) Cartesianas:

$$\psi(\vec{r} - \delta\vec{r}) = \psi(\vec{r}) - \nabla\psi_{(\vec{r})}\,\delta\vec{r}$$
$$= \psi(\vec{r}) - \epsilon_{ijk}\, x_j \frac{\partial \psi(\vec{r})}{\partial x_k}\,\eta$$

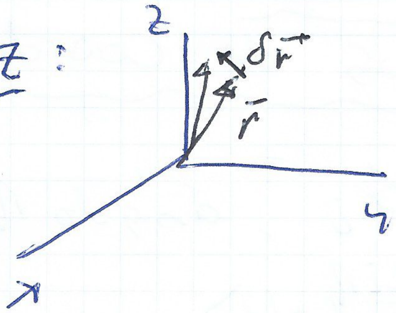
Igualando:

$$\langle \vec{r} | L_i | \psi \rangle = \epsilon_{ijk}\, x_j\, \hat{p}_k\, \psi(\vec{r})$$
$$= (\vec{r} \times \vec{p})_i\, \psi(\vec{r}).$$

con $p_i \longleftrightarrow -i\frac{\partial}{\partial x_i}$

Esféricas :

Lz :



$$\delta\vec r = \begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\psi(\vec r) - \psi(\vec r - \delta\vec r)$$

$$\psi(\vec r-\delta\vec r) \simeq \psi(\vec r) - \frac{\partial\psi(\vec r)}{\partial\phi}\underbrace{\delta\phi}_{n}$$

$$\therefore \langle \vec r|L_z|\psi\rangle = -i\hbar\frac{\partial}{\partial\phi}\psi(\vec r)$$

Lx:

$$\psi(\vec r-\delta\vec r) \simeq \psi(\vec r) - \frac{\partial\psi}{\partial\phi}\delta\phi - \frac{\partial\psi}{\partial\theta}\delta\theta$$

$$\delta\vec r = \frac{\partial y}{\partial\phi}\delta\phi\,\hat y + \frac{\partial z}{\partial\phi}\delta\phi\,\hat z + \frac{\partial y}{\partial\theta}\delta\theta\,\hat y + \frac{\partial z}{\partial\theta}\delta\theta\,\hat z$$

$$= y n\,\hat z - z n\,\hat y$$

$$\begin{cases} \hat z: & -r\sin\theta\,\delta\theta = \underbrace{r\sin\theta\sin\phi}_{y}\,n \\ \hat y: & r\sin\theta\cos\phi\,\delta\phi + r\cos\theta\sin\phi\,\delta\theta = -\underbrace{r\cos\theta}_{z}\,n \end{cases}$$

$$\begin{cases} \delta\theta = -\sin\phi\, n \\ \delta\phi = -\dfrac{\cos\theta}{\sin\theta}\,n\cos\phi \end{cases}$$

$$\therefore \langle \vec r|L_x|\psi\rangle = i\hbar\left[\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right]$$

Cont Esféricas:

L_y : $\psi(\vec{r}-\delta\vec{r}) = \psi(\vec{r}) - \frac{\partial\psi}{\partial\phi}\delta\phi - \frac{\partial\psi}{\partial\theta}\delta\theta$

$$\delta\vec{r} = \frac{\partial x}{\partial\phi}\delta\phi\,\hat{x} + \frac{\partial z}{\partial\phi}\delta\phi\,\hat{z}$$

$$+ \frac{\partial x}{\partial\theta}\delta\theta\,\hat{x} + \frac{\partial z}{\partial\theta}\delta\theta\,\hat{z}$$

$$= z\eta\,\hat{x} - x\eta\,\hat{z}$$

$$\begin{cases} \hat{z}: & -r\sin\theta\,\delta\theta = -r\sin\theta\cos\phi\,\eta \\ \hat{x}: & -r\sin\theta\sin\phi\,\delta\phi + r\cos\theta\cos\phi\,\delta\theta \\ & = r\cos\theta\,\eta \end{cases}$$

$$\begin{cases} \delta\theta = \cos\phi\,\eta \\ \delta\phi = -\frac{\sin\phi}{\sin\theta}\,\eta\cos\theta \end{cases}$$

$\therefore$ $\langle\vec{r}|L_y|\psi\rangle = -i\hbar\left[\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right]\psi(\vec{r})$

$$\begin{cases} \langle\vec{r}|L_+|\psi\rangle = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\psi(\vec{r}) \\ \langle\vec{r}|L_-|\psi\rangle = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\psi(\vec{r}) \end{cases}$$

$\langle \hat{r} | L_+ L_- | \psi \rangle$

- $L_+ L_- = (L_x + iL_y)(L_x - iL_y)$

$$= L_x^2 + L_y^2 - i\underbrace{[L_x, L_y]}_{i\hbar L_z} = L^2 - L_z^2 + \hbar L_z$$

$$\therefore \quad L^2 = L_+ L_- + L_z^2 - \hbar L_z$$

- $\langle \hat{r} | L_+ L_- | \psi \rangle = \hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right)$

$$\cdot\, e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\psi(\bar{r})$$

$$= \Big[-\hbar^2 \frac{\partial^2}{\partial\theta^2} + i\hbar^2\left(-\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi} + \cancel{i\cot\theta\frac{\partial^2}{\partial\theta\partial\phi}}\right)$$

$$+ \hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} - \cancel{i\cot\theta\frac{\partial^2}{\partial\phi\partial\theta}}\right)$$

$$+ \hbar^2\left(+i\cot^2\theta\frac{\partial}{\partial\phi} - \cot^2\theta\frac{\partial^2}{\partial\phi^2}\right)\Big]\psi(\bar{r})$$

$$= \Big[\hbar L_z - L_z^2 - \underbrace{(1+\cot^2\theta)}_{1/\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$- \hbar^2 \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\Big]\psi(\bar{r})$$

$$= \Big[\underbrace{-\hbar^2\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{\hbar^2}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}_{L^2}$$

$$- L_z^2 + \hbar L_z \Big]\psi(\bar{r})$$

$$\therefore \quad \langle \hat{r} | L^2 | \psi \rangle = \underbrace{-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]}_{L^2}\psi(\bar{r})$$

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P7 Armónicos esféricos. $Y_l^{(m)}(\theta,\phi) = \langle \hat{r} | lm \rangle$

- $\langle \hat{r} | L_+ | l l \rangle = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right) \underbrace{\langle \hat{r} | l l \rangle}_{Y_l^{(l)}(\theta,\phi)} = 0$ ①

- $\langle \hat{r} | L_z | l m \rangle = i\hbar \frac{\partial}{\partial\phi} Y_l^{(m)}(\theta,\phi) = \underbrace{m\hbar}_{1} Y_l^{(m)}(\theta,\phi)$ ②

$$Y_l^{(m)}(\theta,\phi) = f(\theta) e^{im\phi} \quad ③$$

Si $\phi = 2\pi$, función univaluada
$\Rightarrow$ m es entero.

③ en ①:
(con $m=l$)

$$\frac{df(\theta)}{d\theta} = -i\cot\theta (il) f(\theta)$$

$$\frac{df(\theta)}{f(\theta)} = l \frac{d(\sin\theta)}{\sin\theta}$$

$$\ln f(\theta) = \ln(\sin\theta)^l + \underbrace{C}_{\text{constante de integración}}$$

$$\therefore \quad f(\theta) = \underbrace{N}_{e^c} \sin\theta^l$$

$$\therefore \quad Y_l^{(l)}(\theta,\phi) = N \sin\theta^l e^{im\phi}$$

Cálculo de N: $\langle l l | \underset{\iint d\theta \sin\theta d\phi |\theta,\phi\rangle\langle\theta,\phi|}{\hat{1}} | l l \rangle = 1$

$$\int_0^{\pi}\int_0^{2\pi} d\theta \sin\theta \, d\phi \; Y_\ell^{(\ell)*}(\theta,\phi)\, Y_\ell^{(\ell)}(\theta,\phi) = 1$$

$$N^2 \int_0^{\pi} d\theta \sin\theta (\sin\theta)^\ell (\sin\theta)^\ell \times 2\pi = 1$$

i) $n=0$:

$$N^2 \int_0^{\pi} d\theta \sin\theta \cdot 2\pi = 1$$

$$N = \frac{1}{\sqrt{4\pi}}$$

ii) $n=1$:

$$N^2 \int_{-1}^{1} dx (1-x^2) \cdot 2\pi = 1$$

$$N^2 \left(x - \frac{x^3}{3}\right)_{-1}^{1} 2\pi = 1$$

$$N = \sqrt{\frac{3}{8\pi}}$$

$$\therefore \; Y_1^{(1)}(\theta,\phi) = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}$$

$Y_0^{(1)}$:

$$\langle \hat{r} | L_- | 1 1 \rangle = \hbar e^{-i\phi}\left(\frac{-\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right) N \sin\theta \, e^{i\phi}$$

$$= \hbar\sqrt{2}\, Y_1^{(0)}(\theta,\phi)$$

$$\therefore \; Y_1^{(0)}(\theta,\phi) = \frac{N}{\sqrt{2}}(-\cos\theta - \cos\theta) = -\sqrt{\frac{3}{4\pi}}\cos\theta$$

$Y_1^{(-1)}$: $$\langle \hat{r} | L_- | 1 0 \rangle = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right)\sqrt{\frac{3}{4\pi}}\cos\theta(-1)$$

$$= \hbar\sqrt{2}\, Y_1^{(-1)}(\theta,\phi)$$

$$\therefore \; Y_1^{(-1)} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi}$$

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Cont $\boxed{P7}$

de $\boxed{P5}$:

$$|1, 1_y\rangle = \frac{1}{2}\left(|1,1\rangle + \sqrt{2}\, i\, |1,0\rangle - |1,-1\rangle\right)$$

$$\langle \hat{r} | 1, 1_y\rangle = \frac{1}{2}\left(Y_1^{(1)}(\theta,\phi) + \sqrt{2}\, i\, Y_1^{(0)}(\theta,\phi) - Y_1^{(-1)}(\theta,\phi)\right)$$

$$\langle \hat{r} | L_y |1, 1_y\rangle = \frac{1}{2}(-i\hbar)\left(\cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi}\right)$$
$$\left[\left(\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi} + \sqrt{2}\, i\left(-\sqrt{\frac{3}{4\pi}}\cos\theta\right)\right.\right.$$
$$\left. + \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}\right]$$

$$= \sqrt{\frac{3}{8\pi}}\, \hbar \left(\cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi}\right)$$
$$\left(\sin\theta\cos\phi - i\cos\theta\right)(-i)$$

$$\langle \hat{r} | L_y |1, 1_y\rangle = \hbar \sqrt{\frac{3}{8\pi}}\left[\cos^2\phi\cos\theta + i\cos\phi\sin\theta \right.$$
$$\left. + \sin^2\phi \cos\theta \quad [illegible] \right](-i)$$

$$= \hbar\, \underbrace{\sqrt{\frac{3}{8\pi}}\left[-i\cos\theta + \cos\phi\sin\theta\right]}_{\langle \hat{r} | 1, 1_y\rangle}$$