Clase 11

3. OSCILADOR ARMÓNICO

3.1. Repaso

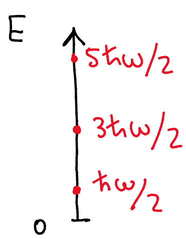
$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \xrightarrow[\text{canónica}]{\text{cuantización}} \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

$$\{x, p\}_{\text{Poisson}} = 1 \qquad\qquad [\hat{x}, \hat{p}] = i\hbar$$

OSC. ARM. CLÁSICO $\qquad\qquad$ OSC. ARM. CUÁNTICO

$$\left.\begin{array}{l} \hat{x} \to \text{op. multiplicación} \quad (\hat{x} f(x) = x f(x)) \\ \hat{p} \to -i\hbar \dfrac{d}{dx} \quad (\hat{p} f(x) = -i\hbar f'(x)) \end{array}\right\} \to \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2$$

$$\hat{H}\varphi(x) = E\varphi(x) \Rightarrow \dots \Rightarrow \begin{cases} \varphi_n(x) = A_n \left(\dfrac{x}{x_0^2} - \dfrac{d}{dx}\right)^n e^{-x^2/(2x_0^2)} & \left(x_0 \equiv \sqrt{\dfrac{\hbar}{m\omega}}\right) \\ E_n = \hbar\omega\left(n + \frac{1}{2}\right) & (n = 0, 1, 2, \dots) \end{cases}$$



$\leftarrow$ Espectro $\quad , \quad \varphi_n(x) = \langle x | \underbrace{n}\rangle$

↳ autoestado de cierto operador $\hat{N}$ ("número")

3.2. Resolución algebraica

$$[\hat{x}, \hat{p}] = i\hbar \quad , \quad \hat{x} = \hat{x}^\dagger \quad , \quad \hat{p} = \hat{p}^\dagger$$

$$\hat{x}' \equiv \left(\frac{m\omega}{\hbar}\right)^{1/2} \hat{x} = \frac{\hat{x}}{x_0} \qquad \left(x_0 \equiv (\hbar/(m\omega))^{1/2}\right)$$

$$\hat{p}' \equiv \left(\frac{1}{m\hbar\omega}\right)^{1/2} \hat{p} = \frac{\hat{p}}{p_0} \qquad \left(p_0 \equiv (m\hbar\omega)^{1/2}\right)$$

$$[\hat{x}', \hat{p}'] = \frac{1}{\hbar}[\hat{x}, \hat{p}] = i$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 = \dots = \frac{\hbar\omega}{2} \left(\hat{p}'^2 + \hat{x}'^2 \right)$$

$$\hat{a} \equiv \frac{\hat{x}' + i\hat{p}'}{\sqrt{2}}$$ (operador de bajada)

$$\hat{a}^+ \equiv \frac{\hat{x}' - i\hat{p}'}{\sqrt{2}}$$ (operador de subida)

$$[\hat{a}, \hat{a}^+] = \dots = 1$$

usando $[\hat{a}, \hat{a}^+] = 1$

$$\hat{H} = \frac{\hbar\omega}{2} \left(\hat{p}'^2 + \hat{x}'^2 \right) = \dots = \frac{\hbar\omega}{2} \left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a} \right) = \hbar\omega \Big(\underbrace{\hat{a}^+\hat{a}}_{\hat{N} \equiv} + \frac{1}{2} \Big)$$

operador de número

es hermítico

$\therefore$ El problema de hallar el espectro de $\hat{H}$ se reduce a hallar el espectro de $\hat{N}$

Propiedades:

$$* \ [\hat{N}, \hat{a}] = [\hat{a}^+\hat{a}, \hat{a}] = \hat{a}^+ \underbrace{[\hat{a}, \hat{a}]}_{=0} + \underbrace{[\hat{a}^+, \hat{a}]}_{-1} \hat{a} = -\hat{a}$$

$$* \ [\hat{N}, \hat{a}^+] = [\hat{a}^+\hat{a}, \hat{a}^+] = \hat{a}^+$$

Espectro de $\hat{N}$:

$$\hat{N} |\nu\rangle = \nu |\nu\rangle$$

$$\hat{N} \left(\hat{a}|\nu\rangle \right) = \hat{N}\hat{a} |\nu\rangle = \Big(\hat{a}\hat{N} + \underbrace{[\hat{N}, \hat{a}]}_{-\hat{a}} \Big) |\nu\rangle = \hat{a}\underbrace{\hat{N}|\nu\rangle}_{\nu|\nu\rangle} - \hat{a}|\nu\rangle =$$

$$= \hat{a}\left(\nu|\nu\rangle \right) - \hat{a}|\nu\rangle = (\nu - 1)\, \hat{a}|\nu\rangle$$

$\therefore$ $\hat{a}|\nu\rangle$ es autovector de $\hat{N}$ con autovalor $(\nu - 1)$ (de aquí el nombre de op. de bajada)

(si $\hat{a}|\nu\rangle \neq 0$)

Veamos que el espectro de $\hat{N}$ es no negativo:

$$\underbrace{\| \hat{a}|\nu\rangle \|^2}_{\geq 0} = \langle \nu | \hat{a}^\dagger \hat{a} | \nu \rangle = \langle \nu | \underbrace{\hat{N} | \nu \rangle}_{\nu|\nu\rangle} = \nu \cdot \underbrace{\langle \nu | \nu \rangle}_{> 0}$$

$\therefore \quad \nu \geq 0$

$\therefore$ Espectro de $\hat{N}$ es no negativo $(*)$

Notemos que aplicando $\hat{a}$ sucesivas veces sobre un autoestado $|\nu\rangle$ de $\hat{N}$ conseguiríamos autoestados de $\hat{N}$ de autovalores $(\nu-1)$, $(\nu-2), (\nu-3), \ldots$, llegando eventualmente a tener autovalores negativos. Esto contradice $(*)$.

La única solución a esto es llegar a tener el estado $|0\rangle$ de autovalor 0 para $\hat{N}$, en cuyo caso:

$$\| \hat{a}|0\rangle \|^2 = \langle 0 | \hat{a}^\dagger \hat{a} | 0 \rangle = \langle 0 | \overbrace{\hat{N} | 0 \rangle}^{0 \cdot |0\rangle} = 0 \Rightarrow \hat{a}|0\rangle = 0$$

$\therefore \quad \hat{a}|0\rangle = 0$ no es autoestado de $\hat{N}$.

$\therefore$ Espectro de $\hat{N} \subseteq \{0, 1, 2, \ldots\} = \mathbb{N}_0$.

Veamos que en realidad $\underset{\nearrow \text{ espectro}}{\text{spec}}(\hat{N}) = \mathbb{N}_0$:

$$\hat{N}\,(\hat{a}^\dagger |\nu\rangle) = (\hat{a}^\dagger \hat{N} + \underbrace{[\hat{N}, \hat{a}^\dagger]}_{\hat{a}^\dagger})\,|\nu\rangle = \hat{a}^\dagger \overbrace{\hat{N}|\nu\rangle}^{\nu|\nu\rangle} + \hat{a}^\dagger |\nu\rangle =$$

$$= (\nu + 1)\, \hat{a}^\dagger |\nu\rangle$$

$= (\nu + 1)\, \hat{a}^\dagger |\nu\rangle$

$\therefore\ \hat{a}^\dagger |\nu\rangle$ es autovector de $\hat{N}$ con autovalor $(\nu+1)$. (de aquí el nombre de op. de subida)

(si $\hat{a}^\dagger |\nu\rangle \neq 0$)

$$\| \hat{a}^\dagger |\nu\rangle \|^2 = \langle \nu | \hat{a}\, \hat{a}^\dagger | \nu \rangle = \dots = \langle \nu | \hat{N} + 1 | \nu \rangle = \underbrace{(\overset{\geq 0}{\nu} + 1)}_{>0} \underbrace{\langle \nu | \nu \rangle}_{>0} > 0$$

$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger \hat{a} + [\hat{a}, \hat{a}^\dagger]$

$\therefore\ \hat{a}^\dagger |\nu\rangle \neq 0$

$\therefore\ \mathrm{spec}(\hat{N}) = \mathbb{N}_0$.

autovalores de $\hat{N}$ (de aquí el nombre de op. número)

$$\hat{N} |n\rangle = n |n\rangle \quad , \quad n = 0, 1, 2, \dots$$

↳ autoestados de $\hat{N}$

- Normalización:

¿Cuál es la norma de $\hat{a}^\dagger |n\rangle$?

$$\| \hat{a}^\dagger |n\rangle \| = \sqrt{\langle n | \hat{a}\, \hat{a}^\dagger | n \rangle} = \sqrt{n+1}\, \langle n | n \rangle$$

Si yo impongo que la norma de $|n\rangle$ sea 1: $\| \hat{a}^\dagger |n\rangle \| = \sqrt{n+1}$

$$\hat{a}^\dagger |n\rangle = c_n \cdot |n+1\rangle \implies |c_n| = \sqrt{n+1}$$

Elijo c_n real $\implies c_n = \sqrt{n+1}$

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle \qquad (\blacktriangle)$$

$$\underbrace{\hat{a}^\dagger |n\rangle}_{\text{normalizado}} = \sqrt{n+1}\ \underbrace{|n+1\rangle}_{\text{normalizado}} \qquad (\blacktriangle)$$

Vale: $\langle n | n' \rangle = \delta_{nn'}$.

De $(\blacktriangle)$: $\quad |n\rangle = \dfrac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$

$$\underbrace{\hat{a}\,|n\rangle}_{\text{normalizado}} = \sqrt{n} \cdot \underbrace{|n-1\rangle}_{\text{normalizado}}$$

~~~ o ~~~

$$\hat{H} = \hbar\omega \left(\hat{N} + \frac{1}{2} \right)$$

$$\hat{H}|n\rangle = \hbar\omega \left(\overbrace{\hat{N}|n\rangle}^{n|n\rangle} + \frac{1}{2}|n\rangle \right) = \overbrace{\hbar\omega \left(n + \frac{1}{2} \right)}^{E_n} |n\rangle \,, \quad \text{con } n \in \mathbb{N}_0$$

$\therefore$ $|n\rangle$ es autoestado de $\hat{H}$ con autovalor $E_n \equiv \hbar\omega\left(n + \frac{1}{2}\right)$, con $n \in \mathbb{N}_0$.

$$\varphi_n(x) = \langle x | n \rangle$$

<u>Representación matricial de $\hat{a}$ y $\hat{a}^\dagger$</u>

$$\langle n' | \underbrace{\hat{a}\,| n \rangle}_{\sqrt{n}|n-1\rangle} = \sqrt{n}\,\langle n' | n-1 \rangle = \sqrt{n}\,\delta_{n',n-1}$$

$$\langle n' | \underbrace{\hat{a}^\dagger | n \rangle}_{\sqrt{n+1}|n+1\rangle} = \sqrt{n+1}\,\delta_{n',n+1}$$

$$[\hat{a}]_{\{|n\rangle\}} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \end{pmatrix} \qquad \text{(matriz infinita)}$$
~~~

$$[\hat{a}]_{\{|n\rangle\}} = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$ (matriz infinita)

En particular, recordar que $\langle n|\hat{a}|n\rangle = 0$.

$\langle n|\hat{a}^{\dagger}|n\rangle = 0$.

Ejercicio (P3E40): Queremos ver que $(\Delta x)(\Delta p) = \left(n+\frac{1}{2}\right)\hbar$ en los autoestados de $\hat{H}$ (osc. armónico).

$$(\Delta x) \equiv \sqrt{\langle \hat{x}^2\rangle - \langle \hat{x}\rangle^2}, \quad (\Delta p) \equiv \sqrt{\langle \hat{p}^2\rangle - \langle \hat{p}\rangle^2}$$

$$\begin{cases} \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^{\dagger}+\hat{a}) \\ \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^{\dagger}-\hat{a}) \end{cases}$$

$$\langle n|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\underbrace{\langle n|\hat{a}^{\dagger}|n\rangle}_{=0} + \underbrace{\langle n|\hat{a}|n\rangle}_{=0}\right) = 0$$

$$\langle n|\hat{p}|n\rangle \propto \underbrace{\langle n|\hat{a}^{\dagger}|n\rangle}_{=0} - \underbrace{\langle n|\hat{a}|n\rangle}_{=0} = 0$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}(\hat{a}^{\dagger}+\hat{a})^2 = \frac{\hbar}{2m\omega}\left[(\hat{a}^{\dagger})^2 + \hat{a}^2 + \underbrace{\hat{a}^{\dagger}\hat{a}}_{\hat{N}} + \overbrace{\hat{a}\hat{a}^{\dagger}}^{\hat{N}+1}\right] =$$

$$= \frac{\hbar}{2m\omega}\cdot\left[(\hat{a}^{\dagger})^2 + \hat{a}^2 + 2\hat{N} + 1\right]$$

Usando que $\langle n|n'\rangle = \delta_{nn'}$

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\left(\langle n|\underbrace{(\hat{a}^{\dagger})^2|n\rangle}_{\propto|n+2\rangle} + \langle n|\underbrace{\hat{a}^2|n\rangle}_{\propto|n-2\rangle} + \langle n|2\hat{N}+1|n\rangle\right) =$$

$$= \frac{\hbar}{2m\omega}\left[0 + 0 + (2n+1)\right] = \frac{\hbar}{2m\omega}(2n+1)$$

$$\langle n|\hat{p}^2|n\rangle = -\frac{\hbar m\omega}{}\left(\langle n|(\hat{a}^{\dagger})^2|n\rangle + \langle n|\hat{a}^2|n\rangle - \langle n|2\hat{N}+1|n\rangle\right) =$$

$$\langle n|\hat{p}^2|n\rangle = -\frac{\hbar m\omega}{2}\left(\underbrace{\langle n|(\hat{a}^{\dagger})^2|n\rangle}_{=0} + \underbrace{\langle n|\hat{a}^2|n\rangle}_{=0} - \langle n|2\hat{N}+1|n\rangle\right) =$$

$$= \frac{\hbar m\omega}{2}(2n+1)$$

$$(\Delta x)^2(\Delta p)^2 = \frac{\hbar}{2m\omega}(2n+1) \times \frac{\hbar m\omega}{2}(2n+1) = \left(\frac{\hbar}{2}\right)^2(2n+1)^2 = \hbar^2\left(n+\frac{1}{2}\right)^2$$

$$\therefore \quad \Delta x\,\Delta p = \hbar\left(n+\frac{1}{2}\right).$$

Si $n=0$, el producto de dispersiones es mínimo: $\Delta x\,\Delta p = \frac{\hbar}{2}$

↳ SATURA LA RELACIÓN DE INCERTEZA

$|0\rangle$ es Gaussiano

($\langle x|0\rangle$ es una Gaussiana)

RELACIONES IMPORTANTES DE LA CLASE DE HOY

$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{n}+\frac{1}{2}\right)$, donde $\hat{n} \equiv \hat{a}^{\dagger}\hat{a}$, con

$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$ y $\hat{a}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$.

Recuerden que: $[\hat{a},\hat{a}^{\dagger}] = 1$.

Si $|n\rangle$ / $\hat{N}|n\rangle = n|n\rangle$ $(n=0,1,2,\dots)$, $\langle n|n'\rangle = \delta_{nn'}$

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$\hat{H}|n\rangle = E_n|n\rangle$, con $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$ $(n=0,1,2,\dots)$