Simetrías Discretas.

Vimos simetrías continuas: $X \to X + dx$

$\sigma \to \sigma + d\sigma$

$t \to t + dt$

$$T(\bar{\alpha}) = e^{-i\bar{P}\cdot\bar{\alpha}/\hbar}$$
$$R(\bar{\varphi}) = e^{-i\bar{J}\cdot\bar{\varphi}/\hbar}$$
$$U_t(\tau) = e^{-iH\tau/\hbar}$$

Si H es invariante ante traslaciones

$\Rightarrow [H, T(\bar{\alpha})] = 0 \Rightarrow [H, \bar{P}] = 0$

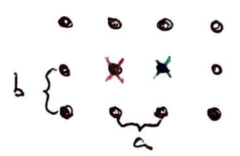
$\frac{d\bar{P}}{dt} = 0 \to$ Cantidad conservada

¿Cuándo hay invariancia ante traslaciones?

$$[P, H] = 0 : \left[P, \frac{P^2}{2m} + V(x)\right] = [P, V(x)] = -i\hbar \frac{\partial V}{\partial x} = 0$$

$\Rightarrow V(x) = cte$

• Traslaciones discretas:



Cristal: Arreglo periódico de átomos, Si me paro en X veo exactamente lo mismo que si me paro en X

$\Rightarrow$ El sistema tiene una invariancia traslacional discreta: $[H, T(a\hat{x})] = 0$ $\quad H(\bar{r}) = H(\bar{r} + a\hat{x})$

def: vectores primitivos $\{\bar{a}_i\}$ / $\bar{R} = \sum_i n_i \bar{a}_i$, $n_i \in \mathbb{Z}$ $\forall \bar{R} \in$ red

Las traslaciones que mantienen invariante al sistema son de la forma $T(\bar{R}_n)$, con $\bar{R}_n = n_1\bar{a}_1 + n_2\bar{a}_2 + n_3\bar{a}_3$

En nuestra red $\bar{a}_1 = a\hat{x}$ y $\bar{a}_2 = b\hat{x}$

83)

Teorema de Bloch

$R_n = \sum_i n_i a_i \qquad [H, T(R_n)] = 0 \qquad T(R_n) = e^{-i\bar{P}\cdot\bar{R}_n/\hbar}$

$\psi(\bar{r} - \bar{R}_n) = \exp[-i\bar{k}\cdot\bar{R}_n]\,\psi(\bar{r})$ ← No es periódica

a) Autovalores de $T(R_n)$ tienen la forma $C(R_n) = e^{-i\bar{k}\cdot\bar{R}_n}$ $k \in \mathbb{R}^3$.

dem: $T(R_n)$ es unitario $\qquad U v = \lambda v$

$v^+ U^+ = \lambda^* v^+$

$v^+ \underbrace{U^+ U}_{\mathbb{1}} v = v^+ \lambda^* \lambda v \Rightarrow |\lambda|^2 = 1 \rightarrow |\lambda| = 1$

$\Rightarrow \boxed{C(R_n) = e^{i\alpha(\bar{R}_n)}} \qquad \alpha : \mathbb{R}^3 \to \mathbb{R}$

$T(\bar{R}'_n)\,T(\bar{R}_n)\,|\bar{x}\rangle = T(\bar{R}'_n)\,|\bar{x} + \bar{R}_n\rangle = |\bar{x} + \bar{R}_n + \bar{R}'_n\rangle$

$\Rightarrow T(\bar{R}'_n)\,T(\bar{R}_n) = T(\bar{R}'_n + \bar{R}_n)$

$\boxed{C(\bar{R}'_n)\,C(\bar{R}_n) = C(\bar{R}'_n + \bar{R}_n)}$

$e^{i\alpha(\bar{R}'_n)}\,e^{i\alpha(\bar{R}_n)} = e^{i\alpha(\bar{R}'_n + \bar{R}_n)}$

$\Rightarrow e^{i\bar{k}\cdot\bar{R}'_n}\,e^{i\bar{k}\cdot\bar{R}_n} = e^{i\bar{k}\cdot(\bar{R}'_n + \bar{R}_n)} \Rightarrow \boxed{C(\bar{R}_n) = e^{i\bar{k}\cdot\bar{R}_n} \quad \bar{k} \in \mathbb{R}^3}$

b) $[H, T(\bar{R}_n)] = 0 \Rightarrow \exists \{\psi_n(\bar{r})\} /\ H\psi_n = \epsilon_n \psi_n$

$T(\bar{R}_n)\psi_n = e^{i\bar{k}\cdot\bar{R}_n}\psi_n$

$T(\bar{R}_n)\,\psi(\bar{r}) = e^{i\bar{k}\cdot\bar{R}_n}\,\psi(\bar{r})$

$$\boxed{\psi(\bar{r} + \bar{R}_n) = e^{i\bar{k}\cdot\bar{R}_n}\,\psi(\bar{r})}$$

c) Expandir en ondas planas y mostrar que los momentos están discretizados.

$$\psi(\bar{r}-\bar{R}_n) = e^{-i\bar{u}\cdot\bar{R}_n}\psi(\bar{r})$$

$$\psi(\bar{r}) = e^{+i\bar{u}\cdot\bar{r}}\,u(\bar{r}) \quad \text{con} \quad u(\bar{r}+\bar{R}_n) = u(\bar{r})$$

$$\psi(\bar{r}-\bar{R}_n) = e^{i\bar{u}\cdot(\bar{r}-\bar{R}_n)}\,u(\bar{r}-\bar{R}_n) = e^{-i\bar{u}\cdot\bar{R}_n}\underbrace{e^{i\bar{u}\cdot\bar{r}}\,u(\bar{r})}_{\psi(\bar{r})}$$

$$\psi(\bar{r}) = e^{i\bar{k}\cdot\bar{r}}\,u(\bar{r}) \quad \text{expandimos } u(\bar{r})$$

$$\psi(\bar{r}) = e^{i\bar{u}\cdot\bar{r}}\sum_{\bar{k}} c_{\bar{h}}\, e^{i\bar{h}\cdot\bar{r}} \qquad \tilde{h} = 2\pi/L$$

$$\psi(\bar{r}) = \sum_{\bar{h}} c_h\, e^{i(\bar{h}+\bar{u})\cdot\bar{r}} \qquad \bar{p} = \hbar(\bar{h}+\bar{k}) \text{ discretos}$$

Operador Paridad

$$\Pi: \Pi^{+}\Pi = \mathbb{1} \quad |\alpha\rangle \to \Pi|\alpha\rangle \qquad \langle\alpha|\Pi^{+}x\Pi|\alpha\rangle = -\langle\alpha|x|\alpha\rangle$$

$$\Pi^{+}x\Pi = -x \to x\Pi = -\Pi x \Rightarrow \boxed{\{x,\Pi\}=0}$$

$$\hat{x}\Pi|x\rangle = -\Pi\hat{x}|x\rangle = -\Pi x|x\rangle = -x\,\Pi|x\rangle \to \boxed{\Pi|x\rangle = |-x\rangle}$$

$$\Pi|-x\rangle = |x\rangle \qquad \Pi|-x\rangle = \Pi(\Pi|x\rangle) = |x\rangle \Rightarrow \Pi^2 = \mathbb{1}$$

$$\Pi\Pi = \mathbb{1} = \Pi^{+}\Pi \Rightarrow \boxed{\Pi = \Pi^{+}} \qquad \Pi \text{ es unitario y hermítico}$$

Autovalores de Π: $\Pi|\alpha\rangle = \varepsilon_\alpha|\alpha\rangle$

$$\underbrace{\Pi^2}_{\mathbb{1}}|\alpha\rangle = \varepsilon_\alpha^2|\alpha\rangle = |\alpha\rangle \Rightarrow \varepsilon_\alpha^2 = 1$$

$$\Rightarrow \boxed{\varepsilon_\alpha = \pm 1}$$

Π y la función de onda

$\psi(x) = \langle x|\psi\rangle \quad \langle x|\Pi|\psi\rangle = \langle x|\Pi^{+}|\psi\rangle = \langle -x|\psi\rangle = \psi(-x)$

84) $\Pi|\alpha\rangle = \varepsilon_\alpha|\alpha\rangle \quad \Pi|\beta\rangle = \varepsilon_\beta|\beta\rangle \quad \varepsilon_\alpha, \varepsilon_\beta = \pm 1$

Mostrar que $\langle\beta|x|\alpha\rangle \neq 0 \iff \varepsilon_\alpha = -\varepsilon_\beta$

$\langle\beta|x|\alpha\rangle = \langle\beta|\Pi^{+}\Pi\, x\, \Pi^{+}\Pi|\alpha\rangle = \varepsilon_\alpha\varepsilon_\beta\langle\beta|\Pi x \Pi^{+}|\alpha\rangle = -\varepsilon_\alpha\varepsilon_\beta\langle\beta|x|\alpha\rangle$

$\langle\beta|x|\alpha\rangle = -\varepsilon_\alpha\varepsilon_\beta\langle\beta|x|\alpha\rangle \Rightarrow \varepsilon_\alpha\varepsilon_\beta = -1 \quad \boxed{\varepsilon_\alpha = -\varepsilon_\beta}$

Por ejemplo $H = H_0 + eEz$ Solo induce transiciones entre estados con paridad opuesta.

$\langle\beta|x|\alpha\rangle = \int d^3x\, \phi_\beta^{*}\, x\, \phi_\alpha$ $\quad x$ es impar

$\phi_\alpha\phi_\beta$ tiene que ser par $\begin{cases} \phi_\alpha \text{ par } \phi_\beta \text{ impar} \\ \phi_\alpha \text{ impar } \phi_\beta \text{ par} \end{cases}$

¿$\langle\beta|\bar{p}|\alpha\rangle$?

$T_a|x\rangle = |x+a\rangle \quad \Pi T_a|x\rangle = \Pi|x+a\rangle = |-x-a\rangle$

$T_{-a}\Pi|x\rangle = T_{-a}|-x\rangle = |-x-a\rangle$

$\Rightarrow \Pi T_a = T_{-a}\Pi \rightarrow \Pi T_a \Pi = T_{-a}$

Traslaciones infinitesimales

$T_{\bar{\varepsilon}} = \mathbb{1} - \frac{i}{\hbar}\bar{p}\cdot\bar{\varepsilon} \quad T_{-\bar{\varepsilon}} = \mathbb{1} + \frac{i}{\hbar}\bar{p}\cdot\bar{\varepsilon} \quad \Pi^{+}T_{\bar{\varepsilon}}\Pi = T_{-\bar{\varepsilon}}$

$\Pi^{+}(\mathbb{1} - \frac{i}{\hbar}\bar{p}\cdot\bar{\varepsilon})\Pi = \mathbb{1} + \frac{i}{\hbar}\bar{p}\cdot\bar{\varepsilon} \quad -\frac{i}{\hbar}\Pi^{+}\bar{p}\cdot\bar{\varepsilon}\,\Pi = \frac{i}{\hbar}\bar{p}\cdot\bar{\varepsilon}$

$\Rightarrow \Pi^{+}\bar{p}\cdot\bar{\varepsilon}\,\Pi = -\bar{p}\cdot\bar{\varepsilon} \quad \boxed{\Pi^{+}\bar{p}\,\Pi = -\bar{p}}$ Actúa igual que con $\bar{x}$.

$\boxed{\{\Pi, \bar{p}\} = 0}$ $\quad \bar{L} = \bar{x}\times\bar{p} \Rightarrow \boxed{\Pi^{+}\bar{L}\,\Pi = \bar{L}}$ idem con $\bar{S}$, $\bar{J}$

85) $|\beta\rangle = e^{-|\beta|^2/2} \sum_n \frac{\beta^n}{\sqrt{n!}} |n\rangle$

$H = \frac{P^2}{2m} + \frac{m\omega^2}{2} x^2 \qquad \Pi H \Pi^+ = \Pi \frac{P^2}{2m} \Pi^+ + \frac{m\omega^2}{2} \Pi x^2 \Pi^+ = \frac{\Pi P \Pi^+ \Pi P \Pi^+}{2m} + \frac{m\omega^2}{2} \Pi x \Pi^+ \Pi x \Pi^+$

$\Rightarrow \Pi H \Pi^+ = H \Rightarrow \boxed{[\Pi, H] = 0}$ Los autoestados de H tienen paridad definida

$|0\rangle$ es par (gaussiano) $\quad a^+ \sim x - iP \Rightarrow \Pi a^+ \Pi^+ = -a^+$

$\Rightarrow |1\rangle = a^+|0\rangle \Rightarrow$ impar $\quad |n\rangle = \frac{a^{+n}}{\sqrt{n!}} |0\rangle$ tiene paridad $(-1)^n$

$|\beta\rangle = e^{-|\beta|^2/2} \sum_n \frac{\beta^n}{\sqrt{n!}} |n\rangle$ Medimos $\Pi = +1$

$\Rightarrow |\tilde{\beta}\rangle = \underbrace{\frac{e^{-|\beta|^2/2}}{N}}_{\text{Normalización}} \sum_n \frac{\beta^{2n}}{\sqrt{(2n)!}} |2n\rangle$ sólo quedan los pares

$\langle\tilde{\beta}|\tilde{\beta}\rangle = \frac{e^{-|\beta|^2}}{N^2} \sum_{nm} \frac{\beta^{2n} \beta^{*2m}}{\sqrt{(2n)!}\sqrt{(2m)!}} \overbrace{\langle 2m|2n\rangle}^{\delta_{mn}} = \frac{e^{-|\beta|^2}}{N^2} \underbrace{\sum_n \frac{|\beta|^{4n}}{(2n)!}}_{\cosh(|\beta|^2)} = 1$

$\Rightarrow N^2 = e^{-|\beta|^2} \cosh(|\beta|^2)$

$\Rightarrow \boxed{|\tilde{\beta}\rangle = \frac{1}{(\cosh(|\beta|^2))^{1/2}} \sum_n \frac{\beta^{2n}}{\sqrt{(2n)!}} |2n\rangle}$ $\quad P(E = \hbar\omega(m + \frac{1}{2})) = \langle m|\tilde{\beta}\rangle$

Si m es impar $\Rightarrow P(E = \hbar\omega(m + \frac{1}{2})) = 0$

Si m es par $\langle m|\tilde{\beta}\rangle = \frac{1}{(\cosh(|\beta|^2))^{1/2}} \sum_n \frac{\beta^{2n}}{\sqrt{(2n)!}} \langle m|2n\rangle = \frac{1}{(\cosh(|\beta|^2))^{1/2}} \frac{\beta^m}{\sqrt{m!}}$