# Oscilador armónico

$$\begin{cases} \hat{H} = \dfrac{\hat{p}^2}{2m} + \dfrac{1}{2} m\omega^2 \hat{x}^2 \\ [\hat{x}, \hat{p}] = i\hbar \end{cases}$$

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega\hat{x} + i\hat{p}) \quad , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega\hat{x} - i\hat{p})$$

De analizar $\hat{a}\hat{a}^\dagger \Longrightarrow \hat{H} = \hbar\omega \left( \underbrace{\hat{a}^\dagger \hat{a}}_{} + 1/2 \right)$

$\hat{N} \equiv$ (operador de número)

$\therefore$ El problema de hallar el espectro de $\hat{H}$ se reduce al problema de hallar el espectro de $\hat{N}$

$$\boxed{\bullet\ [\hat{a}, \hat{a}^\dagger] = 1}$$

## Propiedades de $\hat{N}$

1) $[\hat{N}, \hat{a}] = [a^\dagger a, a] = a^\dagger \underbrace{[a, a]}_{=0} + \underbrace{[a^\dagger, a]}_{-1} a = -\hat{a}$

2) $[\hat{N}, \hat{a}^\dagger] = [a^\dagger a, a^\dagger] = \hat{a}^\dagger$

## Espectro de $\hat{N}$

Sea $|\nu\rangle$ / $\hat{N}|\nu\rangle = \nu|\nu\rangle$

$$\hat{N}(\hat{a}|\nu\rangle) = \hat{N}\hat{a}|\nu\rangle = (\hat{a}\hat{N} + \underbrace{[\hat{N}, \hat{a}]}_{-\hat{a}})|\nu\rangle = \hat{a}\hat{N}|\nu\rangle - \hat{a}|\nu\rangle =$$

$$= \hat{a}\,(\nu - 1)\,|\nu\rangle = (\nu - 1)\,(\hat{a}\,|\nu\rangle)$$

$\therefore$ $\hat{a}\,|\nu\rangle$ es autoestado de $\hat{N}$ con autovalor $(\nu - 1)$ (si $\hat{a}\,|\nu\rangle \neq 0$)

Veamos que el espectro de $\hat{N}$ es no negativo:

$$\underbrace{\|\hat{a}\,|\nu\rangle\|^2}_{\geq 0} = \langle\nu|\,\hat{a}^\dagger\,\hat{a}\,|\nu\rangle = \langle\nu|\,\underbrace{\hat{N}\,|\nu\rangle}_{\nu|\nu\rangle} = \nu\,\underbrace{\langle\nu|\nu\rangle}_{>0}$$

$|\psi\rangle = A\,|\phi\rangle$

$\langle\psi| = \langle\phi|\,A^\dagger$

$\langle\psi|\psi\rangle = \||\psi\rangle\|^2$

$$\Longrightarrow \boxed{\nu \geq 0} \quad (3)$$

$\therefore$ $\nu \geq 0$

Si aplicara sucesivamente el operador $\hat{a}$ a un autoestado $|\nu\rangle$ de $\hat{N}$, conseguiría otros autoestados con autovalores $(\nu-1)$, $(\nu-2)$, ..., llegando eventualmente a tener autovalores negativos.

La única manera de no contradecir a (3), es tener un autoestado de $\hat{N}$ con autovalor $0$, $|0\rangle$:

$$\|\hat{a}\,|0\rangle\|^2 = \langle 0|\,a^\dagger\,a\,|0\rangle = \langle 0|\,\underbrace{\hat{N}\,|0\rangle}_{0\,|0\rangle} = 0 \Longrightarrow \hat{a}\,|0\rangle = 0$$

$\therefore$ $\hat{a}\,|0\rangle$ no es autoestado de $\hat{N}$.

$\therefore$ El espectro de $\hat{N} \subseteq \{0, 1, 2, ...\} = \mathbb{N}_0$

Veamos que no hay un autovalor máximo y entonces el espectro de $\hat{N}$ es efectivamente $\mathbb{N}_0$.

$$\hat{N}\,(\hat{a}^\dagger|\nu\rangle) = \hat{N}\,\hat{a}^\dagger|\nu\rangle = (\hat{a}^\dagger\hat{N} + \overbrace{[\hat{N}, \hat{a}^\dagger]}^{\hat{a}^\dagger})\,|\nu\rangle = (\hat{a}^\dagger\hat{N} + \hat{a}^\dagger)\,|\nu\rangle =$$

$$= \hat{a}^\dagger(\hat{N}+1)|\nu\rangle = (\nu+1)(\hat{a}^\dagger|\nu\rangle)$$

$\therefore$ $a^\dagger|\nu\rangle$ es autoestado de $\hat{H}$ con autovalor $(\nu+1)$

$$\| a^\dagger|\nu\rangle\|^2 = \langle\nu|\hat{a}\hat{a}^\dagger|\nu\rangle = \langle\nu|\hat{a}^\dagger a + \underbrace{[\hat{a},\hat{a}^\dagger]}_{=1}|\nu\rangle =$$

$$= \langle\nu|\hat{N}+1|\nu\rangle = \underbrace{(\nu+1)}_{>0}^{\geq 0}\underbrace{\langle\nu|\nu\rangle}_{>0} > 0$$

$\therefore$ $a^\dagger|\nu\rangle \neq 0$

$\therefore$ Espectro de $\hat{N}$ es $\mathbb{N}_0$

$\therefore$ $|\nu\rangle \longrightarrow |n\rangle$, $\hat{N}|n\rangle = n|n\rangle$

$n$ ↳ natural

## Normalización

¿Cuál es la norma $\hat{a}^\dagger|n\rangle$?

Asumo que $|n\rangle$ está normalizado, $\langle n|n\rangle = 1$

$$\|a^\dagger|n\rangle\| = \sqrt{\langle n|\hat{a}\hat{a}^\dagger|n\rangle} = \sqrt{n+1}\langle n|n\rangle \Longrightarrow \|\hat{a}^\dagger|n\rangle\| = \sqrt{n+1}$$

$\hat{a}\hat{a}^\dagger = \hat{N}+1$

$$\hat{a}^\dagger|n\rangle = c_n|n+1\rangle, \quad |c_n| = \sqrt{n+1}$$

Si elijo $c_n$ real $\longrightarrow$ $c_n = \sqrt{n+1}$

$$\boxed{\therefore \quad a^\dagger\underbrace{|n\rangle} = \sqrt{n+1}\,\underbrace{|n+1\rangle}}$$

Normalizados

Partiendo del $|0\rangle$, $\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle}$

$$\therefore \quad \boxed{\hat{a}\,|n\rangle = \sqrt{n}\;|n-1\rangle}$$

$$\hat{H} = \hbar\omega\,(\hat{N} + 1/2)$$

$$\hat{H}\,|n\rangle = \hbar\omega\,(\underbrace{\hat{N}\,|n\rangle}_{n\,|n\rangle} + 1/2\,|n\rangle) = \overbrace{\hbar\omega\,(n + 1/2)}^{E_n}\,|n\rangle\,, \quad \text{con } n \in \mathbb{N}_0$$

$$\varphi_n(x) = \langle x\,|\,n\rangle$$

## Representación matricial de $\hat{a}$ y $\hat{a}^{\dagger}$

En la base $\{|n\rangle\}$,

$$\underbrace{\langle n'|\,\hat{a}\,|n\rangle}_{\hat{a}_{n'n}} = \langle n'|\,\sqrt{n}\,|n-1\rangle = \sqrt{n}\,\langle n'|n-1\rangle = \sqrt{n}\,\delta_{n',\,n-1}$$

$$\langle n'|\,\underbrace{\hat{a}^{\dagger}\,|n\rangle}_{\sqrt{n+1}\,|n+1\rangle} = \sqrt{n+1}\;\delta_{n',\,n+1}$$

$$[\hat{a}]_{\{|n\rangle\}} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ 0 & 0 & 0 & \ddots & \end{pmatrix} \quad \text{(matriz infinita)}$$

P40) $(\Delta X)(\Delta P) = (n+1/2)\hbar$ en los autoestados de $\hat{H}$

$$\Delta x = \sqrt{\langle \hat{X}^2\rangle - \langle \hat{X}\rangle^2}\ , \quad \Delta P = \sqrt{\langle \hat{p}^2\rangle - \langle \hat{p}\rangle^2}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a})$$

$$\hat{P} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger - \hat{a})$$

$$\langle n|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\underbrace{\langle n|a^\dagger|n\rangle}_{=0} + \underbrace{\langle n|a|n\rangle}_{=0}\right) = 0$$

$$\langle n|\hat{p}|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\langle n|a^\dagger|n\rangle - \langle n|a|n\rangle\right) = 0$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left[(\hat{a}^\dagger)^2 + \hat{a}^2 + \underbrace{\hat{a}^\dagger\hat{a}}_{\hat{N}} + \overbrace{\hat{a}\hat{a}^\dagger}^{\hat{N}+1}\right] =$$

$$= \frac{\hbar}{2m\omega}\left[(\hat{a}^\dagger)^2 + \hat{a}^2 + 2\hat{N} + 1\right]$$

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\left[\langle n|\underbrace{(a^\dagger)^2|n\rangle}_{\propto |n+2\rangle} + \langle n|\underbrace{\hat{a}|n\rangle}_{\propto |n-2\rangle} + \langle n|2\hat{N}+1|n\rangle\right] =$$

$$= \frac{\hbar}{2m\omega}\left[\ 0 + 0 + 2n+1\right] = \frac{\hbar}{2m\omega}(2n+1)$$

$$\langle n|\hat{p}^2|n\rangle = -\frac{\hbar m\omega}{2}\left[\cancel{\langle n|(a^\dagger)^2|n\rangle}_{=0} + \cancel{\langle n|\hat{a}|n\rangle}_{=0} - \langle n|2\hat{N}+1|n\rangle\right] =$$

$$= \frac{\hbar m\omega}{2}(2n+1)$$

$$(\Delta\hat{x})^2(\Delta\hat{P})^2 = \frac{\hbar}{2m\omega}(2n+1)\cdot\frac{\hbar m\omega}{2}(2n+1) = \left(\frac{\hbar}{2}\right)^2(2n+1)^2 = \hbar^2(n+1/2)^2$$

$$\therefore\ \Delta X\,\Delta P = \hbar(n+1/2)$$

Si $\underline{n=0}$ , $\underline{\Delta X \Delta P = \hbar/2}$

El fundamental del oscilador armónico satura la relación de incerteza $\longrightarrow$ $\langle x | 0 \rangle$ es una Gaussiana