Representaciones en el espacio de estados

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Elegir una base

Representación $\longleftrightarrow$ base ortonormal de $\mathcal{E}$

$$|\psi\rangle \rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_n \end{pmatrix} = \begin{bmatrix} \langle u_1|\psi\rangle \\ \langle u_2|\psi\rangle \\ \vdots \\ \langle u_n|\psi\rangle \end{bmatrix} \qquad \hat{A} \rightarrow \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & \cdots \\ \vdots & & \\ \vdots & & \end{pmatrix} = \begin{bmatrix} \langle u_1|A|u_1\rangle & \langle u_1|A|u_2\rangle & \cdots \\ \langle u_2|A|u_1\rangle & \langle u_2|A|u_2\rangle & \cdots \\ \vdots & & \\ \vdots & & \end{bmatrix}$$

La base satisface: ortonormalidad y clausura:

Representación discreta $\{|u_i\rangle\}$ — Continua $\{|w_\alpha\rangle\}$

$$\langle u_i|u_j\rangle = \delta_{ij} \qquad \langle w_\alpha|w_{\alpha'}\rangle = \delta(\alpha-\alpha')$$

$$\sum_i |u_i\rangle\langle u_i| = \mathbb{1} \qquad \int d\alpha\, |w_\alpha\rangle\langle w_\alpha| = \mathbb{1}$$

Con la relación de clausura se expande fácilmente:

$$|\psi\rangle = \mathbb{1}|\psi\rangle = \sum_i |u_i\rangle\langle u_i|\psi\rangle$$

discreto

$$= \sum_i c_i |u_i\rangle$$

$$|\psi\rangle = \int d\alpha\, \underbrace{c(\alpha)}_{\to \langle w_\alpha|\psi\rangle} |w_\alpha\rangle$$

contínuo

Representación de bras:

$$\langle\psi| = \langle\psi|\mathbb{1} = \langle\psi| \sum_i |u_i\rangle\langle u_i|$$

$$= \sum_i \langle\psi|u_i\rangle\langle u_i| =$$

$$= \sum_i c_i^* \langle u_i|$$

Para hacer el producto escalar $\langle\varphi|\psi\rangle$ disponemos:

$$\begin{pmatrix} \langle\varphi|u_1\rangle & \langle\varphi|u_2\rangle & \dots \end{pmatrix} \begin{pmatrix} \langle u_1|\psi\rangle \\ \langle u_2|\psi\rangle \\ \vdots \end{pmatrix}$$

matriz fila y matriz columna

$$= \sum_i \langle\varphi|u_i\rangle\langle u_i|\psi\rangle = \langle\varphi| \overbrace{\sum_i |u_i\rangle\langle u_i|}^{\mathbb{1}} |\psi\rangle$$

$$= \langle\varphi|\psi\rangle \leftarrow$$ ir así, de atrás para adelante

Operadores: matrices cuadradas

$$\langle u_i|A|u_j\rangle = A_{ij} \qquad \begin{vmatrix} A_{11} & A_{12} & A_{13} & \dots \\ A_{21} & A_{22} & \dots & \\ \vdots & & & \end{vmatrix}$$

- Elemento de matriz de AB

$$(AB)_{ij} = \langle u_i|AB|u_j\rangle = \langle u_i|A \sum_k |u_k\rangle\langle u_k| B|u_j\rangle$$

$$= \sum_k \langle u_i|A|u_k\rangle\langle u_k|B|u_j\rangle = \sum_k A_{ik}B_{kj}$$

- Aplicar A a $|\psi\rangle$: $|\psi'\rangle = A|\psi\rangle$

$$\hat{A}|\psi\rangle = \hat{A}\sum_i |u_i\rangle\langle u_i|\psi\rangle = \sum_i c_i \hat{A}|u_i\rangle$$

Si $|\psi'\rangle = \hat{A}|\psi\rangle$ y $|\psi'\rangle = \sum_i c_i'|u_i\rangle$

$$c_i' = \langle u_i|\psi'\rangle = \langle u_i| \sum_j c_j \hat{A}|u_j\rangle = \sum_j c_j \langle u_i|\hat{A}|u_j\rangle$$

$$= \sum_j c_j A_{ij} = \sum_j A_{ij} c_j \leftarrow$$ producto de matriz y vector

$$c_i' = \sum_j A_{ij} c_j \longleftrightarrow |\psi'\rangle = \hat{A}|\psi\rangle$$

$$|\psi\rangle = \sum_i c_i \, |u_i\rangle$$

$$|\varphi\rangle = \sum_i b_i \, |u_i\rangle$$

$$\langle \varphi | A | \psi \rangle = \langle \varphi | \sum_i |u_i\rangle\langle u_i | A \sum_j |u_j\rangle\langle u_j | \psi\rangle$$

$$= \sum_{ij} b_i^* \, A_{ij} \, c_j$$

$$= \begin{pmatrix} b_1^* & b_2^* & \cdots \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & \cdots & \\ \vdots & & & \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

Operadores

$$A = \sum_i |u_i\rangle\langle u_i | A \sum_j |u_j\rangle\langle u_j|$$

$$= \sum_{ij} \langle u_i | A | u_j \rangle \, |u_i\rangle\langle u_j| = \sum_{ij} A_{ij} \, |u_i\rangle\langle u_j|$$

Notar:

$$|\psi\rangle\langle\psi| = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} \begin{pmatrix} c_1^* & c_2^* & c_3^* & \cdots \end{pmatrix} = \begin{pmatrix} c_1 c_1^* & c_1 c_2^* & c_1 c_3^* & \cdots \\ c_2 c_1^* & c_2 c_2^* & \cdots & \\ \cdots & & & \end{pmatrix}$$

operador

$$|u_i\rangle\langle u_j| = \begin{array}{c} \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \vdots & & & \\ 0 & 0 & 1 & \\ \vdots & & & \end{pmatrix} \\ \uparrow \\ j \end{array} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \end{array}$$

Operador adjunto: elemento de matriz

$$(A^\dagger)_{ij} = \langle u_i | A^\dagger | u_j \rangle = \langle u_j | A | u_i \rangle^* = A^*_{ji}$$

Operador Hermítico : $A^{\dagger} = A$

$$A^{*}_{ji} = (A^{\dagger})_{ij} = A_{ij} \Rightarrow A_{ij} = A^{*}_{ji}$$

Notar : $A^{*}_{ii} = A_{ii} \Rightarrow A_{ii} \in \mathbb{R}$

Cambio de representación

Sean bases $\{|u_i\rangle\}$ y $\{|t_k\rangle\} \rightarrow S_{ik} = \langle u_i | t_k \rangle$

$\Rightarrow \langle t_k | u_i \rangle = S^{*}_{ik} = (S^{\dagger})_{ki}$

Tenemos:

$$|\psi\rangle = \sum_i \langle u_i | \psi \rangle \, |u_i\rangle$$

$$|\psi\rangle = \sum_k \langle t_k | \psi \rangle \, |t_k\rangle$$

Cómo se relacionan $\langle u_i | \psi \rangle$ con $\langle t_k | \psi \rangle$

$$\langle t_k | \psi \rangle = \langle t_k | \mathbb{I} | \psi \rangle =$$

$$= \langle t_k | \sum_i |u_i\rangle\langle u_i| \, |\psi\rangle$$

$$= \sum_i \langle t_k | u_i \rangle \langle u_i | \psi \rangle$$

$$= \sum_i (S^{\dagger})_{ki} \langle u_i | \psi \rangle$$

(notar $S^{*}_{ik} = (S^{\dagger})_{ki}$)

donde $S_{ik} = \langle u_i | t_k \rangle$. Y la transformación inversa:

$$\langle u_i | \psi \rangle = \langle u_i | \sum_k |t_k\rangle\langle t_k| \, |\psi\rangle$$

$$= \sum_k \langle u_i | t_k \rangle \langle t_k | \psi \rangle$$

$$= \sum_k S_{ik} \langle t_k | \psi \rangle$$

S es una matriz unitaria:

$$S^{\dagger} S = S S^{\dagger} = \mathbb{I}$$

Es decir: $S^{\dagger} = S^{-1}$

Verificarlo:

$$(S^{\dagger} S)_{k\ell} = \sum_i S^{\dagger}_{ki} S_{i\ell} = \sum_i S^{*}_{ik} S_{i\ell} = \sum_i \langle u_i | t_k \rangle^{*} \langle u_i | t_\ell \rangle =$$

$$= \sum_i \langle t_k | u_i \rangle \langle u_i | t_\ell \rangle$$

$$= \langle t_k | t_\ell \rangle$$

$$= \delta_{k\ell}$$

Transformación de elementos de matriz de un operador

$$\langle t_k | A | t_\ell \rangle \overset{?}{\longleftrightarrow} \langle u_i | A | u_j \rangle$$

Demostrar:

$$A_{k\ell} = \sum_{i,j} S^{\dagger}_{ki} A_{ij} S_{j\ell}$$

$$A_{ij} = \sum_{k,\ell} S_{ik} A_{k\ell} S^{\dagger}_{ij}$$