Momento angular

momento angular orbital $\left(\overset{\text{MAO}}{\text{OAM}}\right)$: $\vec{L}$

" " de espín $\left(\overset{\text{MAS}}{\text{SAM}}\right)$: $\vec{S}$

$\vec{J} = \vec{L} + \vec{S}$ Total angular momentum (TAM) MAT

Notación : mom. ang. arbitrario : $\vec{J}$

OAM

Usando las reglas de cuantización:

$$\left.\begin{aligned} L_x &= Y P_z - Z P_y \\ L_y &= Z P_x - X P_z \\ L_z &= X P_y - Y P_x \end{aligned}\right\} \quad \vec{L} = \vec{R} \times \vec{P}$$

Evaluemos:

$$\left[L_x, L_y\right] = \left[Y P_z - Z P_y,\, Z P_x - X P_z\right]$$

$$= \left[Y P_z, Z P_x\right] - \left[Y P_z, X P_z\right] - \left[Z P_y, Z P_x\right] + \left[Z P_y, X P_z\right]$$

$$= \left[Y P_z, Z P_x\right] + \left[Z P_y, X P_z\right]$$

$$= Y \left[P_z, Z\right] P_x + X \left[Z, P_z\right] P_y$$

$$= -i\hbar\, Y P_x + i\hbar\, X P_y = i\hbar \left(X P_y - Y P_x\right)$$

$$= i\hbar L_z$$

y análogamente $\left[L_y, L_z\right] = i\hbar L_x$ $\quad \left[L_i, L_j\right] = i\hbar \epsilon_{ijk} L_k$

$\left[L_z, L_x\right] = i\hbar L_y$

Def. un mom. ang. general $\vec{J}$ por las reglas de conmutación:

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$$

Introducimos el operador hermítico:

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

Se muestra fácilmente que:

$$[J^2, J_i] = 0 \quad ⊛$$

Conmutan y ∴ son operadores compatibles.

En cambio, J_x, J_y, J_z no.

En problemas con potencial central tomaremos $\{H, L^2, L_z\}$ como CCOC.

⊛ Prueba:

$$[J^2, J_x] = [J_x^2 + J_y^2 + J_z^2, J_x] = [J_y^2, J_x] + [J_z^2, J_x]$$

$$= J_y [J_y, J_x] + [J_y, J_x] J_y + J_z [J_z, J_x] + [J_z, J_x] J_z$$

$$= -i\hbar J_y J_z - i\hbar J_z J_y + i\hbar J_z J_y + i\hbar J_y J_z = 0$$

Ladder operators

$$J_\pm = J_x \pm i J_y$$

$$J_+ J_- = (J_x + i J_y)(J_x - i J_y)$$

$$= J_x^2 + J_y^2 + i J_y J_x - i J_x J_y$$

$$= J_x^2 + J_y^2 - i [J_x, J_y]$$

$$= J_x^2 + J_y^2 + \hbar J_z \quad (*)$$

$$J_- J_+ = J_x^2 + J_y^2 - \hbar J_z \quad (*)$$

$$\Rightarrow [J_+, J_-] = 2\hbar J_z$$

$$[J^2, J_\pm] = [J^2, J_x \pm i J_y] = [J^2, J_x] \pm i [J^2, J_y] = 0$$

$$[J_z, J_\pm] = [J_z, J_x \pm i J_y]$$

$$= [J_z, J_x] \pm i [J_z, J_y]$$

$$= (i\hbar J_y \pm i (-i\hbar) J_x)$$

$$= i\hbar J_y \pm \hbar J_x$$

$$= \pm\hbar (J_x \pm i J_y) = \pm\hbar J_\pm$$

(*) y usando $J^2 = J_x^2 + J_y^2 + J_z^2$

$$\begin{cases} J_+ J_- = J^2 - J_z^2 + \hbar J_z & \Rightarrow J^2 = J_+ J_- + J_z^2 - \hbar J_z \\ J_- J_+ = J^2 - J_z^2 - \hbar J_z & \Rightarrow J^2 = J_- J_+ + J_z^2 + \hbar J_z \end{cases}$$

$$\Rightarrow J^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2$$

Teo. Los autovalores de J^2 son positivos $\left(J^2|\lambda\rangle = \lambda|\lambda\rangle\right)$

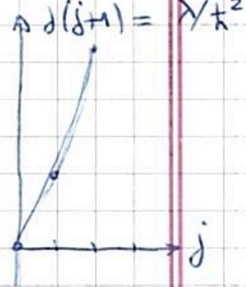
$$\forall|\psi\rangle : \langle\psi|J^2|\psi\rangle = \langle\psi|J_x^2|\psi\rangle + \langle\psi|J_y^2|\psi\rangle + \langle\psi|J_z^2|\psi\rangle$$

$$= \| J_x|\psi\rangle \|^2 + \| J_y|\psi\rangle \|^2 + \| J_z|\psi\rangle \|^2 \geq 0$$

En particular si $|\psi\rangle = |\lambda\rangle$ es autoestado de $J^2|\lambda\rangle = \lambda|\lambda\rangle$

$$\langle\lambda|J^2|\lambda\rangle = \lambda\langle\lambda|\lambda\rangle = \lambda \geq 0$$

Podemos expresarlo : $\lambda = j(j+1)\hbar^2$ (con $j \geq 0$)



Autovalores de J_z : $m\hbar$

Problema: resolver simultáneamente:

$$\begin{cases} J^2|k\,j\,m\rangle = j(j+1)\hbar^2|k\,j\,m\rangle \\ J_z|k\,j\,m\rangle = m\hbar|k\,j\,m\rangle \end{cases}$$

¿Cuáles son los posibles valores de j, m?

~~Lema 1:~~ $\cancel{-j \leq m \leq j \quad (|m| \leq j)}$

~~Lema 1~~

~~(i) si~~ $\cancel{m = -j \Rightarrow J_-|k\,j,-j\rangle = 0}$

~~(ii) si~~ $\cancel{m > -j \Rightarrow J^2 J_-|k\,j\,m\rangle = j(j+1)\hbar^2 |J_-|k\,j\,m\rangle}$

$$\cancel{J_z J_-|k\,j\,m\rangle = (m-1)\hbar|J_-|k\,j\,m\rangle}$$

Autovalores de J^2 y J_z

- Si $|jm\rangle$ es autoestado común de J^2 y J_z:

$$\begin{cases} J^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle \\ J_z |jm\rangle = m\hbar |jm\rangle \end{cases}$$

Entonces $J_\pm |jm\rangle$ también lo es.

<u>Dem.</u>

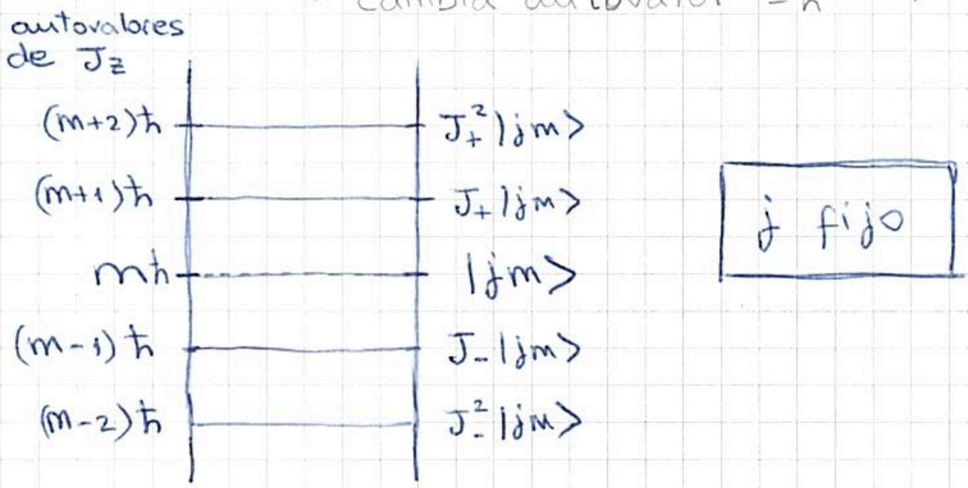
$$\underbrace{J^2 J_\pm |jm\rangle}\ \overset{[J^2, J_\pm] = 0}{=}\ J_\pm J^2 |jm\rangle = \underbrace{j(j+1)\hbar^2}_{\text{mismo autovalor}}\ \underbrace{J_\pm |jm\rangle}$$

$$J_z J_\pm |jm\rangle \overset{[J_z, J_\pm] = \pm\hbar J_\pm}{=} (J_\pm J_z \pm \hbar J_\pm)|jm\rangle =$$

$$= J_\pm (J_z \pm \hbar \mathbb{1})|jm\rangle$$

$$= J_\pm (m\hbar \pm \hbar)|jm\rangle$$

$$= \underbrace{(m \pm 1)\hbar}_{\text{cambia autovalor } \pm\hbar}\ \underline{J_\pm |jm\rangle}$$



- Existe un m_{max} / $J_+ |j\, m_{max}\rangle = 0$

(La componente J_z no puede exceder el J total)

$\Rightarrow$

$$J^2 |j\, m_{max}\rangle = \left(J_- J_+ + J_z^2 + \hbar J_z \right) |j\, m_{max}\rangle$$
$$= \left(0 + m_{max}^2 \hbar^2 + m_{max} \hbar^2 \right) |j\, m_{max}\rangle$$
$$= m_{max} (m_{max} + 1) \hbar^2 |j\, m_{max}\rangle$$
$$= j(j+1) \hbar^2 |j\, m_{max}\rangle$$

- También existe m_{min} / $J_- |j\, m_{min}\rangle = 0 \Rightarrow$

$$J^2 |j\, m_{min}\rangle = \left(J_+ J_- + J_z^2 - \hbar J_z \right) |j\, m_{min}\rangle$$
$$= m_{min} (m_{min} - 1) \hbar^2 |j\, m_{min}\rangle$$
$$= j(j+1) \hbar^2 |j\, m_{min}\rangle$$

Vemos que:

$$m_{max} (m_{max} + 1) = m_{min} (m_{min} - 1)$$

Si $m_{max} = m_{min} - 1$, se satisface, pero $m_{max} < m_{min}$ no sirve

Si $m_{max} = -m_{min}$, se satisface ✓

Tenemos $m_{max} = m_{min} + N = -m_{max} + N$

$\Rightarrow \quad N = 2 m_{max} \Rightarrow m_{max} = \dfrac{N}{2}$

- Además: $j = m_{max}$

Conclusión sobre autovalores:

$$\begin{cases} j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \\ m = -j, -j+1, \dots, j-1, j \end{cases}$$

~~Autovectores comunes de J^2 y J_z~~

Normalization

~~Si $J_+ |jm\rangle = A_{jm} |j, m+1\rangle$~~

~~Hallar A_{jm} que normalice $J_+|jm\rangle$~~

$J_+ |jm\rangle \propto |j, m+1\rangle$

↳ está normalizado?

$$\| J_+ |jm\rangle \|^2 = \langle jm | J_- J_+ | jm \rangle$$

$$= \langle jm | J^2 - J_z^2 - \hbar J_z | jm \rangle$$

$$= j(j+1)\hbar^2 - m(m+1)\hbar^2$$

$$\Rightarrow |j, m\pm1\rangle = \frac{1}{\sqrt{j(j+1) - m(m\pm1)}\,\hbar} J_\pm |jm\rangle$$

En síntesis:

$$J_z |j\, m\rangle = m\hbar\, |j\, m\rangle$$

$$J_\pm |j\, m\rangle = \hbar \sqrt{j(j+1) - m(m\pm 1)}\, |j, m\pm 1\rangle$$

$$J^2 |j m\rangle = j(j+1)\hbar^2\, |j m\rangle$$

Matriz de J^2

Elementos de matriz: $\langle j' m' | J^2 | j m\rangle$

$$\langle j^* m^* | J^2 | j' m'\rangle = j(j+1)\hbar^2 \langle j^* m^* | j' m'\rangle$$

$$= j(j+1)\hbar^2\, \delta_{jj'}\, \delta_{mm'}$$

j'		0	1/2								
j^*	m^* \ m'	0	1/2	-1/2	1	0	-1	3/2	1/2	-1/2	-3/2
0	0	0									
1/2	1/2		$\frac{3}{4}\hbar^2$								
	-1/2			$\frac{3}{4}\hbar^2$							
1	1				$2\hbar$						
	0					$2\hbar$					
	-1						$+2\hbar^2$				
3/2	3/2							$\frac{15}{4}\hbar^2$			
	1/2								$\frac{15}{4}\hbar^2$		
	-1/2									$\frac{15}{4}\hbar^2$	
	-3/2										$\frac{15}{4}\hbar^2$

$\frac{3}{2}\cdot\frac{5}{2} = \frac{15}{4}$

En el pizarrón, borrar los valores y escribir:

$$\langle j^* m^* | J_z | j' m'\rangle = m\hbar\, \delta_{j^* j'}\, \delta_{m^* m'}$$

Elementos de matriz de J_x y J_y

$$\begin{cases} J_+ = J_x + i J_y \\ J_- = J_x - i J_y \end{cases} \qquad \begin{cases} J_x = \frac{1}{2}\left(J_+ + J_-\right) \\ \\ J_y = \frac{1}{2i}\left(J_+ - J_-\right) = \frac{i}{2}\left(J_- - J_+\right) \end{cases}$$

$$\langle jm | J_\pm | j'm' \rangle = \langle jm | \hbar \sqrt{j'(j'+1) - m'(m'\pm 1)}\, | j', m'\pm 1 \rangle$$

$$= \hbar \sqrt{j(j+1) - m'(m'\pm 1)}\; \delta_{jj'}\, \delta_{m, m'\pm 1}$$

Entonces:

$$\langle jm | J_x | j'm' \rangle = \frac{1}{2} \langle jm | J_+ | j'm' \rangle + \frac{1}{2} \langle jm | J_- | j'm' \rangle$$

$$= \frac{\hbar}{2}\delta_{jj'} \left(\sqrt{j(j+1) - m'(m'+1)}\; \delta_{m,m'+1} + \sqrt{j(j+1) - m'(m'-1)}\; \delta_{m,m'-1} \right)$$

$$\langle jm | J_y | j'm' \rangle = \frac{\hbar}{2i} \left(\sqrt{\cdots}\; \delta_{m,m'+1} - \sqrt{\cdots}\; \delta_{m,m'-1} \right) \delta_{jj'}$$

$j = 1/2$

$$(J_x)^{1/2} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (J_y)^{1/2} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$(J_+)^{1/2} = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (J_-)^{1/2} = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$j = 1$

$$(J_+)^1 = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad (J_-)^1 = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\hbar$$

$$(J_x)^1 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (J_y)^1 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$