Juan Laurnagaray

Simetrías

In classical mechanics the meaning of a conservation law is straightforward: the quantity in question is the same before and after some event. Drop a rock, and potential energy is converted into kinetic energy, but the *total* is the same just before it hits the ground as when it was released; collide two billiard balls and momentum is transferred from one to the other, but the total remains unchanged. But in quantum mechanics a system does not in general *have* a definite energy (or momentum) before the process begins (or afterward). What does it *mean*, in that case, to say that the observable Q is (or is not) conserved? Here are two possibilities:

- **First definition**: The *expectation value* $\langle Q \rangle$ is independent of time.
- **Second definition**: The probability of getting any particular value is independent of time.

Under what conditions does each of these conservation laws hold?

Let us stipulate that the observable in question does not depend explicitly on time: $\partial Q/\partial t = 0$. In that case the generalized Ehrenfest theorem (Equation 3.73) tells us that the expectation value of Q is independent of time if *The operator* $\hat{Q}$ *commutes with the Hamiltonian.* It so happens that the same criterion guaranatees conservation by the second definition.

Introduction to QM, Griffiths

Las simetrías van a cumplir que conservan el valor del producto interno

$$|\psi\rangle \xrightarrow{\text{Transf}} |\psi'\rangle \qquad |\phi\rangle \xrightarrow{\text{Transf}} |\phi'\rangle$$

$$|\langle\psi'|\phi'\rangle| = |\langle\psi|\phi\rangle|$$

Queremos que ocurra si la transf es una simetría

Teorema de Wigner: Una transf de un espacio vectorial en sí mismo que cumpla $|\langle\psi'|\phi'\rangle| = |\langle\psi|\phi\rangle|$ está implementado por un operador unitario o antiunitario.

Los casos que a nosotros nos van a interesar son las transf representadas por operadores unitarios U st $UU^\dagger = U^\dagger U = I$. En el aspecto dinámico vamos a pedir que $[H,U]=0 \iff U^\dagger H U = H$. Si H dependiese explícitamente del tiempo deberíamos pedir $[H(t),U]=0 \ \forall t$. Por ejemplo la partícula libre tiene

$$\hat{H} = \frac{\hat{p}^2}{2m} \qquad \hat{p}^2 = \hat{\bar{p}}\cdot\hat{\bar{p}}$$

y sabemos es invariante ante traslaciones, $\hat{T}_a = e^{-\frac{i}{\hbar}\hat{\bar{p}}\cdot\bar{a}}$

$$[\hat{H},\hat{T}_a]=0$$

Tarea

P. 05.09

Supongamos que el espectro de A puede estar degenerado

$A|a_n^i\rangle = a_n|a_n^i\rangle \qquad i=1,\dots,g_n$ → Orden de la degeneración

Dado el estado $|\psi\rangle$, la proba de medir a_n

$$P(a_n) = \sum_{i=1}^{g_n} |\langle a_n^i|\psi\rangle|^2$$

Si a $t=t_0$ el estado es $|\psi(t_0)\rangle = |\psi\rangle$ entonces

$$|\psi(t)\rangle = U(t,t_0)|\psi\rangle$$

Si H no depende de t explícitamente, i.e. $\partial_t H = 0$ entonces

$$U(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}$$

y la prob de medir a_n a lo largo de la evolución

$$P(a_n) = \sum_{i=1}^{g_n} |\langle a_n^i|\psi(t)\rangle|^2$$
$$= \sum_{i=1}^{g_n} |\langle a_n^i|U(t,t_0)|\psi\rangle|^2$$

Por otro lado si $[H,A]=0$ existe una base de autovectores, que vamos a tomar es $|a_n^i\rangle$

$$H|a_n^i\rangle = E_n|a_n^i\rangle$$
$$|\psi\rangle = \sum_n\sum_{i=1}^{g_n} |a_n^i\rangle\langle a_n^i|\psi\rangle = \sum_n\sum_{i=1}^{g} \psi_n^i|a_n^i\rangle$$

$$P(a_n) = \sum_{i=1}^{g_n}\Big|\sum_m\sum_{j=1}^{g_m}\psi_m^j\langle a_n^i|e^{-\frac{i}{\hbar}H(t-t_0)}|a_m\rangle\Big|^2$$
$$= \sum_{i=1}^{g_n}\Big|\sum_m\sum_{j=1}^{g_m} e^{-\frac{i}{\hbar}E_m(t-t_0)}\psi_m^j\underbrace{\langle a_n^i|a_m^j\rangle}_{=\delta_{nm}\delta_{ij}}\Big|^2$$
$$= \sum_{i=1}^{g_n}\Big|\psi_n^i e^{-\frac{i}{\hbar}E_n(t-t_0)}\Big|^2$$
$$= \sum_{i=1}^{g_n}|\psi_n^i|^2$$

P.os.12

$$\rho = \frac{1}{2}(I + P_i\sigma_i) \qquad H = \frac{\omega\hbar}{2}\sigma_z \qquad [\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

$$\rho(t) = e^{-\frac{i\omega}{2}t\sigma_z}\rho\, e^{\frac{i\omega}{2}t\sigma_z} = \frac{1}{2}\left(I + P_i e^{-\frac{i\omega}{2}t\sigma_z}\sigma_i e^{\frac{i\omega}{2}t\sigma_z}\right)$$

$$= \frac{1}{2}\left(I + P_i\sigma_i(t)\right)$$

$$\sigma_z(t) = \sigma_z$$

$$e^{A}Be^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots$$

$$\sigma_x(t) = e^{-\frac{i\omega t}{2}\sigma_z}\sigma_x e^{i\frac{\omega t}{2}\sigma_z}$$

$$= \sigma_x - \frac{i\omega t}{2}\underbrace{[\sigma_z,\sigma_x]}_{2i\sigma_y\epsilon_{zxy}} + \frac{1}{2!}\left(-\frac{i\omega t}{2}\right)^2\underbrace{[\sigma_z,[\sigma_z,\sigma_x]]}_{(2i)^2\epsilon_{zyx}\epsilon_{zxy}} + \frac{1}{3!}\left(-\frac{i\omega t}{2}\right)^3\underbrace{[\sigma_z,[\sigma_z,[\sigma_z,\sigma_x]]]}_{(2i)^3\epsilon_{zxy}\epsilon_{zyx}\epsilon_{zxy}} + \dots$$

$$= \sigma_x + \omega t\sigma_y - \frac{1}{2!}(\omega t)^2\sigma_x - \frac{1}{3!}(\omega t)^3\sigma_y + \dots$$

$$= \sigma_x\cos(\omega t) + \sigma_y\sin(\omega t)$$

$$= \begin{pmatrix} 0 & \cos(\omega t) - i\sin(\omega t) \\ \cos(\omega t) + i\sin(\omega t) & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix}$$

De la misma manera

$$\sigma_y(t) = e^{-\frac{i\omega t}{2}\sigma_z}\sigma_y e^{i\frac{\omega t}{2}\sigma_z}$$

$$= \sigma_y - \frac{i\omega t}{2}\underbrace{[\sigma_z,\sigma_y]}_{2i\sigma_x\epsilon_{zyx}} + \frac{1}{2!}\left(-\frac{i\omega t}{2}\right)^2\underbrace{[\sigma_z,[\sigma_z,\sigma_y]]}_{(2i)^2\epsilon_{zxy}\epsilon_{zyx}\sigma_y} + \frac{1}{3!}\left(-\frac{i\omega t}{2}\right)^3\underbrace{[\sigma_z,[\sigma_z,[\sigma_z,\sigma_y]]]}_{(2i)^3\epsilon_{zyx}\epsilon_{zxy}\epsilon_{zyx}\sigma_x} + \dots$$

$$= \cos(\omega t)\sigma_y - \sigma_x\sin(\omega t)$$

$$\rho(t) = \frac{1}{2}\left(I + P_x\sigma_x\cos\omega t + P_x\sigma_y\sin\omega t - P_y\sigma_x\sin\omega t + P_y\sigma_y\cos\omega t + P_z\sigma_z\right)$$

$$= \frac{1}{2}\left(I + P_i(t)\cdot\sigma_i\right)$$

$$P_x(t) = P_x\cos\omega t - P_y\sin\omega t$$

$$P_y(t) = P_x\sin\omega t + P_y\cos\omega t$$

$$P_z(t) = P_z$$

Recordemos que en $\rho = \frac{1}{2}(I + \bar{P}\cdot\bar{\sigma})$, $\bar{P}$ lo llamamos vector de polarización y representa los valores medios

$$\langle \sigma_i \rangle = \mathrm{Tr}(\rho\,\sigma_i) = P_i .$$

Podríamos haber usado directamente

$$e^{-i\bar{\sigma}\cdot\hat{n}\alpha} = \cos\alpha - i\bar{\sigma}\cdot\hat{n}\sin\alpha, \quad \hat{n}\cdot\hat{n} = 1$$ → Sakurai equat (3.2.44) p166

y evaluar directamente en $\hat{n} = \hat{z}$.

Gracias Daniel por el comentario.