Oscilador Armónico.

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 \qquad a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$$

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

$$a^+ a = \frac{m\omega}{2\hbar}\left(x^2 + \frac{i}{m\omega}[x,p] + \frac{p^2}{m^2\omega^2}\right) = \frac{p^2}{2m\hbar\omega} + \frac{m\omega^2}{2}\frac{x^2}{\hbar\omega} - \frac{1}{2}$$

$$\boxed{\hbar\omega\left(a^+ a + 1/2\right) = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2}$$

$$[a, a^+] = 1 \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a^+ a|n\rangle = a^+\sqrt{n}|n-1\rangle = n|n\rangle \Rightarrow a^+a = N \rightarrow \text{operador Número}$$

$$H = \hbar\omega(N + 1/2) \Rightarrow H|n\rangle = \underbrace{\hbar\omega(n + 1/2)}_{E_n}|n\rangle$$

<u>Estados coherentes</u>: Probleme 4

$$a|\alpha\rangle = \alpha|\alpha\rangle \qquad \alpha \in \mathbb{C}$$

$$\langle\alpha|a^+ = \alpha^*\langle\alpha|$$

a) $\langle\alpha|a|\alpha\rangle = \alpha\langle\alpha|\alpha\rangle = \alpha$ $\qquad \langle\alpha|a^+|\alpha\rangle = \alpha^*\langle\alpha|\alpha\rangle = \alpha^*$

$\langle\alpha|a^2|\alpha\rangle = \alpha^2$ $\qquad \langle\alpha|a^{+2}|\alpha\rangle = \alpha^{*2}$

b) $\langle\alpha|N|\alpha\rangle = \langle\alpha|a^+a|\alpha\rangle = \alpha\alpha^*\langle\alpha|\alpha\rangle = |\alpha|^2$

$\langle\alpha|N^2|\alpha\rangle = \langle\alpha|a^+a\,a^+a|\alpha\rangle = \langle\alpha|a^+(1+a^+a)a|\alpha\rangle$

$aa^+ = 1+a^+a$ $\qquad = \langle\alpha|a^+a + a^{+2}a^2|\alpha\rangle = |\alpha|^2 + |\alpha|^4$

$(\Delta N)^2 = \langle N^2\rangle - \langle N\rangle^2 = |\alpha|^4 + |\alpha|^2 - |\alpha|^4 = |\alpha|^2$

c) $\langle\alpha|H|\alpha\rangle = \hbar\omega\langle\alpha|a^+a + \frac{1}{2}|\alpha\rangle = \hbar\omega(|\alpha|^2 + 1/2)$

$\langle\alpha|H^2|\alpha\rangle = \hbar^2\omega^2\langle\alpha|N^2 + N + \frac{1}{4}|\alpha\rangle = \hbar^2\omega^2(|\alpha|^4 + 2|\alpha|^2 + \frac{1}{4})$

$(\Delta H)^2 = \hbar^2\omega^2(|\alpha|^4 + 2|\alpha|^2 + \frac{1}{4}) - \hbar^2\omega^2(|\alpha|^4 + |\alpha|^2 + 1/4)$

$$\boxed{(\Delta H)^2 = \hbar^2\omega^2|\alpha|^2}$$

d) $\langle\alpha|x|\alpha\rangle$? $\qquad x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^+)$

$\langle\alpha|x|\alpha\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\langle\alpha|a + a^+|\alpha\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\alpha + \alpha^*) = \left(\frac{2\hbar}{m\omega}\right)^{1/2}\mathrm{Re}(\alpha)$

$\langle\alpha|x^2|\alpha\rangle = \frac{\hbar}{2m\omega}\langle\alpha|a^2 + \underset{\substack{\downarrow \\ 1+a^+a}}{aa^+} + a^+a + a^{+2}|\alpha\rangle$

$\langle\alpha|x^2|\alpha\rangle = \frac{\hbar}{2m\omega}(\alpha^2 + 1 + 2|\alpha|^2 + \alpha^{*2})$

$$\langle x\rangle^2 = \frac{\hbar}{2m\omega}\left(\alpha^2 + \alpha^{*2} + 2|\alpha|^2\right)$$

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle = \frac{\hbar}{2m\omega}$$

e) $\langle p\rangle$? $\quad p = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}(a^\dagger - a)$

$$\langle p\rangle = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}(\alpha^* - \alpha) = (2\hbar m\omega)^{1/2}\, I_m(\alpha)$$

$$(\Delta p)^2 = \frac{\hbar m\omega}{2}$$ (Hagan la cuenta)

Para estados coherentes $\Delta x \Delta p = \frac{\hbar}{2} \Rightarrow$ Son gaussianos.

n) $\langle H\rangle = \hbar\omega\left(|\alpha|^2 + 1/2\right)$ $\quad \langle x\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\alpha + \alpha^*)$

$$\langle p\rangle = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}(\alpha^* - \alpha)$$

clásico: $\langle H\rangle = \frac{\langle p\rangle^2}{2m} + \frac{m\omega^2}{2}\langle x\rangle^2$

$$\langle x\rangle^2 = \frac{\hbar}{2m\omega}\left(\alpha^2 + \alpha^{*2} + 2|\alpha|^2\right) \qquad \langle p\rangle^2 = -\frac{\hbar m\omega}{2}\left(\alpha^2 + \alpha^{*2} - 2|\alpha|^2\right)$$

$$\frac{\langle p\rangle^2}{2m} = -\frac{\hbar\omega}{4}\left(\alpha^2 + \alpha^{*2} - 2|\alpha|^2\right); \quad \frac{m\omega^2}{2}\langle x\rangle^2 = \frac{\hbar\omega}{4}\left(\alpha^2 + \alpha^{*2} + 2|\alpha|^2\right)$$

$$\boxed{\frac{\langle p\rangle^2}{2m} + \frac{m\omega^2}{2}\langle x\rangle^2 = \hbar\omega|\alpha|^2} \qquad \langle H\rangle = \hbar\omega\left(|\alpha|^2 + 1/2\right)$$

Si $|\alpha| >> 1 \Rightarrow \langle H \rangle \simeq \frac{\langle p \rangle^2}{2m} + \frac{m\omega^2}{2} \langle x \rangle^2$

Recuperamos lo clásico.

h) $|\alpha\rangle = \sum_n c_n |n\rangle \qquad a|\alpha\rangle = \alpha|\alpha\rangle$

$$\sum_n c_n a |n\rangle = \sum_n c_n \alpha |n\rangle$$

$$\sum_n c_n \sqrt{n} |n-1\rangle = \sum_n c_n \alpha |n\rangle$$

$$\Rightarrow \boxed{c_n \sqrt{n} = \alpha c_{n-1}}$$

$$c_1 = \alpha c_0 \qquad c_2 \sqrt{2} = \alpha c_1 = \alpha^2 c_0$$

$$c_3 \sqrt{3} = \alpha c_2 = \alpha^3 \frac{c_0}{\sqrt{1}\sqrt{2}} \qquad c_3 = \frac{\alpha^3 c_0}{\sqrt{1}\cdot\sqrt{2}\cdot\sqrt{3}}$$

$$c_n = c_0 \frac{\alpha^n}{\sqrt{n!}}$$

$$|\alpha\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} c_0 |n\rangle$$

$$\langle \alpha | \alpha \rangle = \sum_{nm} \frac{\alpha^{*m}}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} |c_0|^2 \langle n | m \rangle$$

$$\langle \alpha | \alpha \rangle = |c_0|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{\alpha^2} = 1$$

$$|c_0|^2 = e^{-|\alpha|^2} \Rightarrow c_0 = e^{-|\alpha|^2/2}$$

$$\Rightarrow \boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

Problema 5:

a) $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = \sum_n \frac{(\alpha a^\dagger - \alpha^* a)^n}{n!}$

$$D^\dagger(\alpha) = \sum_n \frac{(\alpha a^\dagger - \alpha^* a)^{\dagger n}}{n!} = \sum_n \frac{(\alpha^* a - \alpha a^\dagger)^n}{n!} = e^{-(\alpha a^\dagger - \alpha^* a)}$$

$$\rightarrow D(\alpha)\, D^\dagger(\alpha) = e^{\alpha a^\dagger - \alpha^* a}\, e^{-(\alpha a^\dagger - \alpha^* a)} = \mathbb{1} \Rightarrow \text{unitario}$$

Además $D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha)$

c) $D^\dagger(\alpha)\, a\, D(\alpha) = e^{-(\alpha a^\dagger - \alpha^* a)}\, a\, e^{\alpha a^\dagger - \alpha^* a}$

$$= a - [\alpha a^\dagger - \cancel{\alpha^* a}^{\,0}, a] + \cancel{\cdots}^{\,0}$$

$$= a + \alpha$$

$$\boxed{D^\dagger(\alpha)\, a\, D(\alpha) = a + \alpha}$$

$$D^\dagger(\alpha)\, a^\dagger D(\alpha) = a + \alpha^*$$

d) $|\alpha\rangle = D(\alpha)|0\rangle$ es autoestado de a

$$D^\dagger(\alpha)\, a\, D(\alpha)|0\rangle = (a+\alpha)|0\rangle = \alpha|0\rangle$$

$$D^\dagger(\alpha)\, a\, D(\alpha)\,|0\rangle = \alpha|0\rangle \qquad D(\alpha)D^\dagger(\alpha) = \mathbb{1}$$

$$a\, D(\alpha)|0\rangle = D(\alpha)\,\alpha|0\rangle$$

$$a\,\underbrace{D(\alpha)|0\rangle}_{|\alpha\rangle} = \alpha\,\underbrace{D(\alpha)|0\rangle}_{|\alpha\rangle}$$

f) $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$

Sup. $\alpha \in \mathbb{R}$ $\qquad D(\alpha) = e^{\alpha(a^\dagger - a)}$

$$P = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}(a^\dagger - a) \qquad a^\dagger - a = -iP\left(\frac{2}{m\omega\hbar}\right)^{1/2}$$

$D(\alpha) = e^{-i\tilde{\alpha}P}$ $\Rightarrow$ Traslación en x.

Sup $\alpha \in \mathbb{I}$ $\qquad D(\alpha) = e^{\alpha(a^\dagger + a)}$ $\qquad \alpha = i\tilde{\alpha}$

$\Rightarrow D(\alpha) \sim e^{i\tilde{\alpha}x}$ $\Rightarrow$ traslación en P.