The calculation of the geometrical series is quite analogous to the procedure of Example 8.1. Thus we obtain the result

$$I = \frac{4}{\beta} \cos\left(\frac{l\pi}{\alpha}\epsilon_f - \frac{\pi}{4}\right) \frac{\frac{l\pi^2}{\alpha\beta}}{\sinh\left(\frac{l\pi^2}{\alpha\beta}\right)}$$

Example 14.11: Ultrarelativistic Fermi gas

We want to study the thermodynamic properties of an ultrarelativistic ideal Fermi gas. Ultrarelativistic particles have the energy–momentum relationship $\epsilon = |\vec{p}|c$, which follows from the general formula $\epsilon = (p^2c^2 + m^2c^4)^{1/2}$ for vanishing rest mass.

While there are certain bosons with this energy–momentum relation (e.g., photons, phonons, and plasmons), the number of fermions with vanishing rest mass seems to be rather small. It is still not clear whether there are any fermions with vanishing rest mass. For instance, one can assert only an upper bound for the rest mass of the neutrino which has relatively large measurement errors, $m_\nu < 8$ eV. On the other hand, the ultrarelativistic Fermi gas can be used as a model system for a hot gas of fermions with nonvanishing rest mass, if the average momenta in the gas are large compared to mc; i.e., if the average thermal energy kT is large compared to the rest mass mc^2.

From relativistic quantum mechanics it is known that one can create pairs of particles and antiparticles (e.g., e^- and e^+) out of the vacuum at the expense of the energy $2mc^2$. These creation (and annihilation) processes will play a major role in an ultrarelativistic Fermi gas ($kT \gg mc^2$). Therefore, we must not consider a gas of Fermi particles alone; rather, we have to add the corresponding antiparticles. The vacuum represents the particle reservoir of the grand canonical ensemble, and particles and antiparticles are always exchanged with this reservoir via creation and annihilation processes.

Thus, we deal with a mixture of two ideal Fermi gases, between which "chemical" reactions are possible. In the case of the ultrarelativistic Bose gas it was not necessary to consider the antiparticles explicitly, since particles and antiparticles are identical in the more important applications (photons and phonons).

As a concrete example, we consider a hot gas of electrons and positrons. The logarithm of the grand partition function consists of two parts,

$$\ln \mathcal{Z}(T, V, z_+, z_-) = \sum_{\epsilon_+} \ln(1 + z_+ \exp\{-\beta\epsilon_+\}) + \sum_{\epsilon_-} \ln(1 + z_- \exp\{-\beta\epsilon_-\})$$

The sums run over the one-particle states of free electrons and positrons. The term $\ln \mathcal{Z}$ depends now on two fugacities z_+ and z_- or two chemical potentials μ_+ and μ_-, respectively, which are related to the mean particle numbers N_+ and N_- of particles and antiparticles via

$$N_+ = \sum_{\epsilon_+} \frac{1}{z_+^{-1} \exp\{\beta\epsilon_+\} + 1} \qquad N_- = \sum_{\epsilon_-} \frac{1}{z_-^{-1} \exp\{\beta\epsilon_-\} + 1} \tag{14.128}$$

Physically, it would not be sensible to fix all particle numbers N_+ and N_- separately and then determine the chemical potentials μ_+ and μ_-. In thermodynamic equilibrium the mean particle numbers will change via the continuously occurring creation and annihilation processes. Moreover, they may strongly fluctuate.

The changes dN_+ and dN_- of the two particle numbers are related by the equation

$$dN_+ = dN_-$$

If we write the reaction equation (for electrons and positrons) in the form

$$e^+ + e^- \rightleftharpoons \text{reaction products} + \Delta E \tag{14.129}$$

we observe that an antiparticle is also created and annihilated with each particle. Here the reaction products (e.g., photons) play no role, as long as we do not explicitly take them into account in the gas. From Equation (14.129) it follows that the chemical potentials of particles and antiparticles have to be equal (with opposite sign, cf. Chapter 3), since reaction products like photons do not carry a chemical potential,

$$\mu_+ + \mu_- = 0, \qquad z_+ z_- = 1 \tag{14.130}$$

The particle numbers N_+ and N_- are indeed not independent of each other, and there are not two independent fugacities, but actually only one. However, instead of N_+ and N_- one can fix the difference $N = N_+ - N_-$, the particle surplus, since it is not influenced by the creation and annihilation processes:

$$N = N_+ - N_- = \sum_{\epsilon_+ > 0} \frac{1}{z_+^{-1} \exp\{\beta \epsilon_+\} + 1} - \sum_{\epsilon_- > 0} \frac{1}{z_-^{-1} \exp\{\beta \epsilon_-\} + 1} \tag{14.131}$$

From this equation one has to determine the fugacity z, taking into account Equation (14.130). We can provide the system with a certain surplus N of particles, which does not change via pair creation or annihilation, but the mean particle numbers N_+ and N_- cannot be controlled.

We want to expand the result (14.130) and simultaneously explain it quantum mechanically.

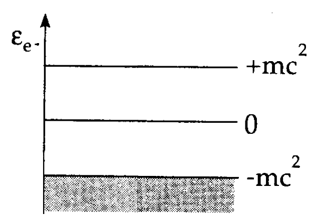

Figure 14.15. Energy spectrum of the free Dirac equation.

To this end, we consider the energy spectrum of the free Dirac equation (see Figure 14.15). In the ultrarelativistic case we must of course let $m \to 0$. As one knows, in this spectrum there are also states of negative energy $\epsilon \leq -mc^2$ besides the states of positive energy $\epsilon \geq mc^2$. One can now describe particles and antiparticles in the spectrum simultaneously, if one assumes that in the vacuum, without particles, all states of the negative energy continuum are occupied by (unobservable) electrons.

In this picture missing electrons in the negative continuum (holes) are to be interpreted as positrons (antiparticles). Let us now consider the general expression for the mean occupation number for Fermi particles:

$$\langle n_\epsilon \rangle = \frac{1}{\exp\{\beta(\epsilon - \mu)\} + 1} \tag{14.132}$$

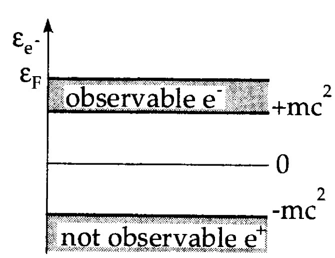

Figure 14.16. Spectrum of an electron gas at $T = 0$.

In the derivation of this occupation number no restriction for the allowed one-particle energies was made, and thus we can expect that Equation (14.132) should correctly reproduce the occupation of all electron states. For $T = 0$ and $\mu = +mc^2$ we have exactly a distribution as shown in Figure 14.16, since then expression (14.132) has the form $\Theta(mc^2 - \epsilon)$. The free Dirac equation has no solutions in the range $-mc^2 \leq \epsilon \leq +mc^2$, and thus there are no occupied states inside the interval. The minimum energy an observable (real) electron has to have is thus $\epsilon = \mu = +mc^2$.

If the Dirac equation with an external potential has bound solutions in the interval $-mc^2 \leq \epsilon \leq +mc^2$, the bound states above the lower continuum are successively filled with the observable electrons. The chemical potential at $T = 0$ is just equal to the energy of the highest occupied state.

Even then, Equation (14.132) correctly describes the physical situation. If there are further unoccupied states above the highest occupied state, a free electron without kinetic energy ($\epsilon = +mc^2$) can be captured by the system, because $\mu < mc^2$. The energy difference $mc^2 - \mu$ is released.

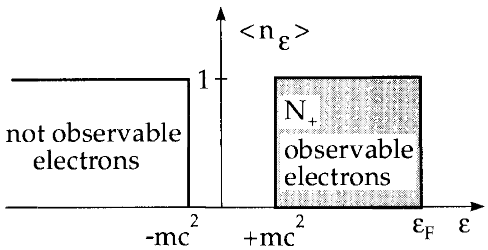

Figure 14.17. $\langle n_\epsilon \rangle$ for an electron gas at $T = 0$.

For an electron gas with N_+ electrons and a Fermi energy $\epsilon_f > mc^2$ we have at $T = 0$ the situation shown in Figure 14.17. This occupation of the electron states is correctly described by Equation (14.132), if we replace μ by the Fermi energy ϵ_f of the electrons.

If we now increase the temperature of the electron gas, at first electrons near the Fermi energy are excited into higher states $\epsilon > \epsilon_f$. This occurs in an energy range of approximate width kT around the Fermi energy.

However, if the temperature is of the order $2mc^2$, more and more electrons from the lower continuum can be excited into free states $\epsilon > \epsilon_f$. These electrons leave holes in the lower continuum, which represent observable positrons. The number of observable electrons has also increased. The difference $N_+ - N_-$, however, is the same as before. The negative energy of the holes $\epsilon_{\text{holes}} < -mc^2$ is simply related to the positive energy of the corresponding positron via $\epsilon_{e^+} = -\epsilon_{\text{hole}}$.

The number of observable electrons and positrons can be calculated as follows:

$$N_+ = \sum_{\epsilon > 0} \langle n_\epsilon \rangle\,, \qquad N_- = \sum_{\epsilon < 0} (1 - \langle n_\epsilon \rangle) \tag{14.133}$$

with $\langle n_\epsilon \rangle$ given by Equation (14.132) and $\mu = \mu_+$ as the chemical potential of the electrons (particles). As one observes, only the chemical potential of the electrons (particles) appears in this interpretation of particles and antiparticles, which is due to Dirac. On the other hand, comparing Equations (14.133) and (14.129) we can establish a connection with the picture of two different Fermi gases, between which chemical reactions are possible. Obviously, the positive electron states correspond exactly to the free electron states ϵ_+ in Equation (14.129). The unoccupied electron states of negative energy $\epsilon < 0$ have to be identified with the occupied positron states with positive energy $\epsilon_- > 0$. The expression for N_- can now be transformed:

$$\begin{aligned} N_- &= \sum_{\epsilon<0} \left(1 - \frac{1}{z^{-1}\exp\{\beta\epsilon\} + 1}\right) \\ &= \sum_{\epsilon<0} \frac{z^{-1}\exp\{\beta\epsilon\}}{z^{-1}\exp\{\beta\epsilon\} + 1} \\ &= \sum_{\epsilon<0} \frac{1}{z\exp\{-\beta\epsilon\} + 1} \end{aligned} \tag{14.134}$$

Furthermore, the energy spectrum of the free Dirac equation is symmetric around $\epsilon = 0$. Thus, one may substitute $\epsilon \to -\epsilon_-$ in Equation (14.134) and instead of counting electrons with negative energy one counts present positrons with positive energy,

$$N_- = \sum_{\epsilon_- > 0} \frac{1}{z\exp\{\beta\epsilon_-\} + 1}$$

A comparison with Equation (14.129) now yields in fact $z = z_-^{-1}$; i.e., $\mu_+ = -\mu_-$, in agreement with Equation (14.131). Both interpretations yield the same results, but in some cases Dirac's particle–hole picture is more convenient. For instance, the particle excess in this picture is simply

$$N = N_+ - N_- = \sum_{\epsilon>0} \langle n_\epsilon \rangle - \sum_{\epsilon<0} (1 - \langle n_\epsilon \rangle)$$
$$= \sum_{\epsilon} \langle n_\epsilon \rangle - \sum_{\epsilon<0} 1 = \sum_{\epsilon} \langle n_\epsilon \rangle - \sum_{\epsilon} \langle n_\epsilon \rangle^{\text{vac}}$$

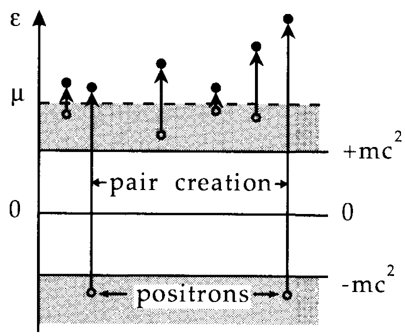

Figure 14.18. Possible processes in the electron gas at $kT \approx 2mc^2$.

The particle excess is thus always given by the difference between the total number of all electrons (observable and unobservable) and the vacuum state. Here the role of the vacuum without observable particles as a reference state becomes especially obvious. Only the deviations from the vacuum state are observable.

Let us add a comment at this place. The whole consideration can also be performed in Dirac's particle–hole picture, if the roles of particles and antiparticles are reversed. For instance, electrons would then have to be identified with holes in the negative energy continuum of the positrons. The reason is the invariance of the free Dirac equation under charge conjugation, as long as there are no electromagnetic fields present. For sake of completeness we explicitly denote both possibilities:

particles=electrons (index+), antiparticles=positrons (index −):

$$N_+ = \sum_{\epsilon>0} \frac{1}{\exp\{\beta(\epsilon - \mu_+)\} + 1}\,,$$
$$N_- = \sum_{\epsilon<0} \left(1 - \frac{1}{\exp\{\beta(\epsilon - \mu_+)\} + 1}\right) \tag{14.135}$$

particles=positrons (index −), antiparticles=electrons (index+):

$$N_+ = \sum_{\epsilon<0} \left(1 - \frac{1}{\exp\{\beta(\epsilon - \mu_-)\} + 1}\right), \qquad N_- = \sum_{\epsilon>0} \frac{1}{\exp\{\beta(\epsilon - \mu_-)\} + 1} \tag{14.136}$$

Of course, Equations (14.135) and (14.136) are identical with $\mu_+ = -\mu_-$.

However, the way of consideration in Dirac's particle–hole picture presented here has a disadvantage, which should be mentioned. By marking the electrons as particles (or the

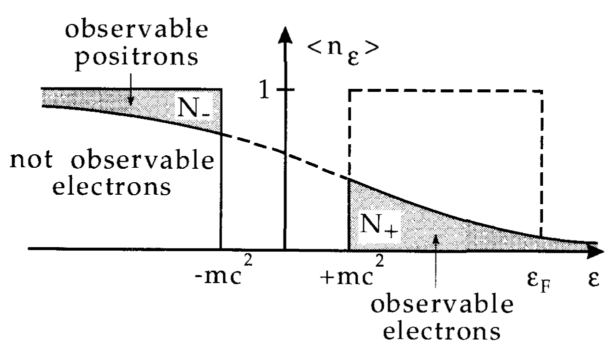

Figure 14.19. $\langle n_k \rangle$ for an electron gas at $kT \approx 2mc^2$.

positrons as particles, respectively) the symmetry of the theory with respect to charge conjugation is somewhat obscured. In the particle–hole picture antiparticles do not appear explicitly, but are replaced by unoccupied states of negative energy. On the other hand, our initial idea of two independent Fermi gases, which react chemically, is completely symmetric in particles and antiparticles.

For fermions it does not matter which representation one uses, as long as one consistently keeps to it. For bosons, which have the same energy spectrum as the fermions, a consistent particle–hole picture is not possible, since for them the Pauli principle is not valid. In the case of fermions (electron gas) the (unobservable) electrons of the negative energy continuum prevent a "falling down" of electrons with positive energy towards infinitely negative energies under steady energy gain. For bosons, however, this process cannot be prevented, and one has to refer to the initial picture of two gases. Equation (14.130) still holds for bosons if the $+1$ in the denominator is replaced by -1.

We now want to proceed with the calculation. At first we rewrite the sums in Equation (14.125) and (14.131) in terms of integrals, for which we need the state density $g(\epsilon)$ of ultrarelativistic particles (see Equation (13.6), with degeneracy factor g),

$$g(\epsilon) = g\frac{4\pi V}{h^3c^3}\epsilon^2$$

$$\ln \mathcal{Z} = g\frac{4\pi V}{h^3c^3}\int_0^\infty \epsilon^2\,d\epsilon\left[\ln\left(1+\exp\{-\beta(\epsilon-\mu)\}\right)+\ln\left(1+\exp\{-\beta(\epsilon+\mu)\}\right)\right]$$

or after integration by parts,

$$\ln \mathcal{Z} = g\frac{4\pi V}{h^3c^3}\frac{\beta}{3}\int_0^\infty \epsilon^3\,d\epsilon\left[\frac{1}{\exp\{\beta(\epsilon-\mu)\}+1}+\frac{1}{\exp\{\beta(\epsilon+\mu)\}+1}\right] \quad \textbf{(14.137)}$$

$$N = N_+ - N_-$$

$$= g\frac{4\pi V}{h^3c^3}\int_0^\infty \epsilon^2\,d\epsilon\left[\frac{1}{\exp\{\beta(\epsilon-\mu)\}+1}-\frac{1}{\exp\{\beta(\epsilon+\mu)\}+1}\right] \quad \textbf{(14.138)}$$

where we simply write μ for the chemical potential μ_+ of the particles and $-\mu$ for μ_- of the antiparticles. The integrals in Equations (14.137) and (14.138) fortunately can be evaluated with analytical means, without using the special functions $f_n(z)$. We substitute $x = \beta(\epsilon-\mu)$ in the first term and $y = \beta(\epsilon+\mu)$ in the second. We then obtain for Equation (14.137)

$$\ln \mathcal{Z} = \frac{g4\pi V}{c^3h^3}\frac{\beta}{3}\left[\beta^{-1}\int_{-\beta\mu}^\infty dx\,\frac{\left(\frac{x}{\beta}+\mu\right)^3}{e^x+1}+\beta^{-1}\int_{\beta\mu}^\infty dy\,\frac{\left(\frac{y}{\beta}-\mu\right)^3}{e^y+1}\right]$$

We now rewrite the integrals so that we can integrate from 0 to ∞,

$$\ln \mathcal{Z} = \frac{g4\pi V}{c^3h^3}\frac{\beta^{-3}}{3}\left[\int_0^\infty dx\,\frac{(x+\beta\mu)^3}{e^x+1}+\int_0^\infty dy\,\frac{(y-\beta\mu)^3}{e^y+1}\right.$$
$$\left.+\int_{-\beta\mu}^0 dx\,\frac{(x+\beta\mu)^3}{e^x+1}-\int_0^{\beta\mu} dy\,\frac{(y-\beta\mu)^3}{e^y+1}\right]$$

The first two integrals can be directly combined, the last two after the substitution $y = -x$.

$$\ln \mathcal{Z} = \frac{g4\pi V}{c^3h^3}\frac{\beta^{-3}}{3}\left[\int_0^\infty dx\,\frac{2x^3+6x(\beta\mu)^2}{e^x+1} + \int_{-\beta\mu}^0 dx(x+\beta\mu)^3\left(\frac{1}{e^x+1}+\frac{1}{e^{-x}+1}\right)\right]$$

If we now consider $(e^x+1)^{-1}+(e^{-x}+1)^{-1}=1$, we find

$$\ln \mathcal{Z} = \frac{g4\pi V}{c^3h^3}\frac{\beta^{-3}}{3}\left[2\int_0^\infty dx\,\frac{x^3}{e^x+1} + 6(\beta\mu)^2\int_0^\infty dx\,\frac{x}{e^x+1} + \int_0^{\beta\mu} dz z^3\right] \tag{14.139}$$

In the last integral we have substituted $z = x + \beta\mu$. Analogously, we now treat Equation (14.138):

$$\begin{aligned} N &= \frac{g4\pi V}{c^3h^3}\left[\beta^{-1}\int_{-\beta\mu}^\infty dx\,\frac{\left(\frac{x}{\beta}+\mu\right)^2}{e^x+1} - \beta^{-1}\int_{\beta\mu}^\infty dy\,\frac{\left(\frac{y}{\beta}-\mu\right)^2}{e^y+1}\right] \\ &= \frac{g4\pi V}{c^3h^3}\beta^{-3}\left[\int_0^\infty dx\,\frac{(x+\beta\mu)^2}{e^x+1} - \int_0^\infty dy\,\frac{(y-\beta\mu)^2}{e^y+1} + \int_{-\beta\mu}^0 dx\,\frac{(x+\beta\mu)^2}{e^x+1} + \int_0^{\beta\mu} dy\,\frac{(y-\beta\mu)^2}{e^y+1}\right] \\ &= \frac{g4\pi V}{c^3h^3}\beta^{-3}\left[4\beta\mu\int_0^\infty dx\,\frac{x}{e^x+1} + \int_0^{\beta\mu} dz z^2\right] \end{aligned} \tag{14.140}$$

In the last line we have again combined the two last integrals from the preceding line and substituted $z = x + \beta\mu$. Note that this can be done only for $N_+ - N_-$, and not for $N_+ + N_-$.

For the total particle number there is no simple analytical solution, as well as for N_+ and N_- separately. One can calculate these quantities with the help of the $f_n(z)$ functions.

The integrals occurring in Equations (14.139) and (14.140) can be expressed with the help of Equation (14.27):

$$\begin{aligned} \int_0^\infty dx\,\frac{x^3}{e^x+1} &= \Gamma(4)f_4(1) = 6\left(1-\frac{1}{2^3}\right)\zeta(4) = \frac{7\pi^4}{120} \\ \int_0^\infty dx\,\frac{x}{e^x+1} &= \Gamma(2)f_2(1) = 1\left(1-\frac{1}{2}\right)\zeta(2) = \frac{\pi^2}{12} \end{aligned} \tag{14.141}$$

Therewith we have the results

$$\begin{aligned} \ln \mathcal{Z}(T,V,\mu) &= \frac{g4\pi V}{h^3c^3}\frac{\beta^{-3}}{3}\left[2\frac{7\pi^4}{120} + 6(\beta\mu)^2\frac{\pi^2}{12} + \frac{1}{4}(\beta\mu)^4\right] \\ &= \frac{gV}{h^3c^3}\frac{4\pi}{3}(kT)^3\left[\frac{7\pi^4}{60} + \left(\frac{\mu}{kT}\right)^2\frac{\pi^2}{2} + \left(\frac{\mu}{kT}\right)^4\frac{1}{4}\right] \\ N(T,V,\mu) &= \frac{g4\pi V}{h^3c^3}\beta^{-3}\left[4\beta\mu\frac{\pi^2}{12} + \frac{1}{3}(\beta\mu)^3\right] \\ &= \frac{g4\pi V}{h^3c^3}(kT)^3\left[\left(\frac{\mu}{kT}\right)\frac{\pi^2}{3} + \frac{1}{3}\left(\frac{\mu}{kT}\right)^3\right] \end{aligned}$$

From Equation (14.141) one can in principle calculate the internal energy via $U = -\partial(\ln \mathcal{Z})/\partial\beta$, but the following consideration is simpler:

$$
\begin{aligned}
U = U_+ + U_- &= \sum_{\epsilon_+} \langle n_\epsilon \rangle_+ \epsilon_+ + \sum_{\epsilon_-} \langle n_\epsilon \rangle_- \epsilon_- \\
&= \frac{g 4\pi V}{h^3 c^3} \int_0^\infty \epsilon^3 \, d\epsilon \left[\frac{1}{\exp\{\beta(\epsilon - \mu)\} + 1} + \frac{1}{\exp\{\beta(\epsilon + \mu)\} + 1} \right]
\end{aligned}
$$

This is identical to Equation (14.137) up to a factor $\beta/3$, so that

$$\ln \mathcal{Z} = \frac{pV}{kT} = \frac{\beta}{3} U$$

or

$$p = \frac{1}{3}\frac{U}{V} = g \frac{(kT)^4}{(\hbar c)^3} \frac{1}{3} \left[\frac{7\pi^2}{120} + \left(\frac{\mu}{kT}\right)^2 \frac{1}{4} + \left(\frac{\mu}{kT}\right)^4 \frac{1}{8\pi^2} \right] \tag{14.142}$$

By the way, the first term in Equation (14.142) is quite similar to the Stefan–Boltzmann law for the ultrarelativistic photon gas ($\mu = 0$).

The density of the particle surplus is

$$\frac{N}{V} = \frac{N_+ - N_-}{V} = \frac{g}{6} \left(\frac{kT}{\hbar c}\right)^3 \left[\left(\frac{\mu}{kT}\right) + \left(\frac{\mu}{kT}\right)^3 \frac{1}{\pi^2} \right]$$

The free energy density follows from $\frac{F}{V} = \frac{N}{V}\mu - p$ as

$$\frac{F}{V} = g \frac{(kT)^4}{(\hbar c)^3} \left[\frac{1}{8\pi^2} \left(\frac{\mu}{kT}\right)^4 + \frac{1}{12} \left(\frac{\mu}{kT}\right)^2 - \frac{1}{3}\frac{7\pi^2}{120} \right]$$

With this one can also calculate the entropy density $\frac{S}{V} = \frac{1}{T}\left(\frac{U}{V} - \frac{F}{V}\right)$:

$$\frac{S}{V} = gk \left(\frac{kT}{\hbar c}\right)^3 \left[\frac{7\pi^2}{90} + \frac{1}{6}\left(\frac{\mu}{kT}\right)^2 \right]$$