Quantal_4b

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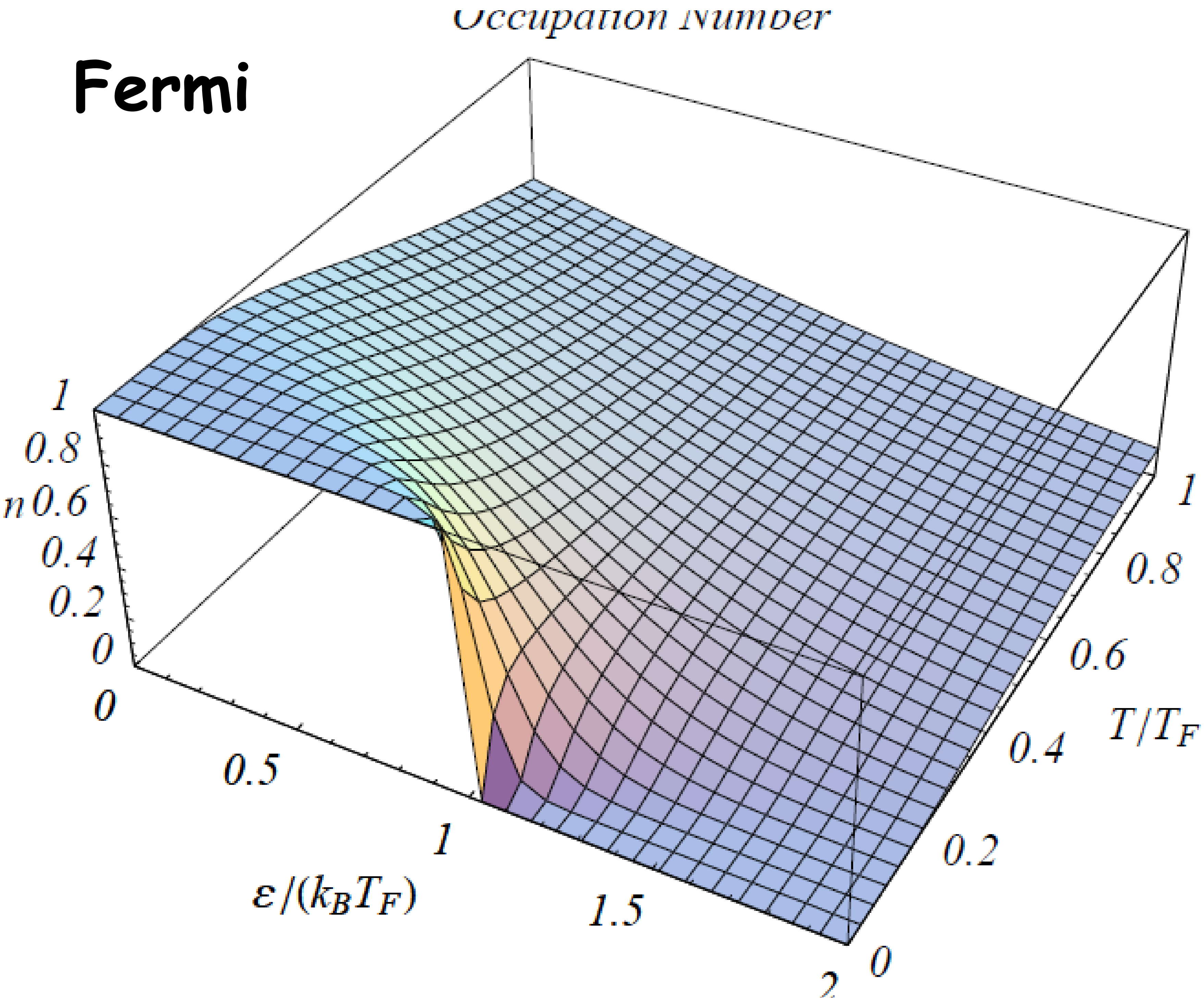
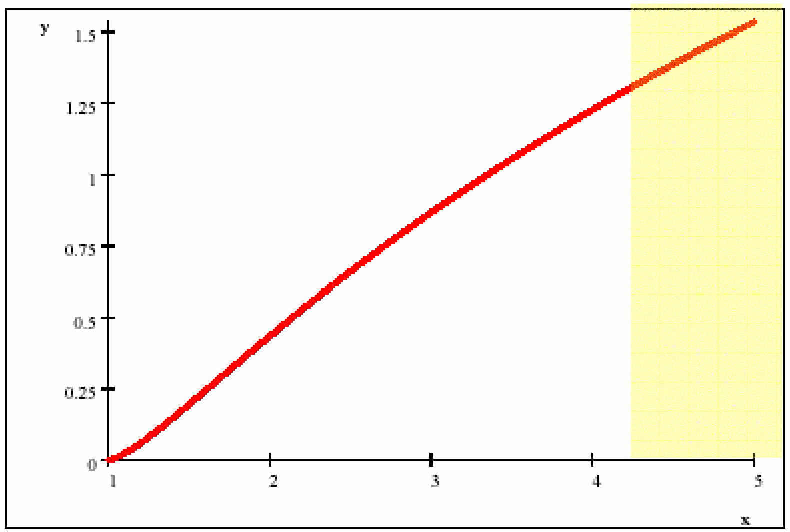

Occupation Number
Fermi
n
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5
2
ε/(k_B T_F)
0
0.2
0.4
0.6
0.8
1
T/T_F

Gas de Fermi

$f_{3/2}(z)=\lambda^3/v$

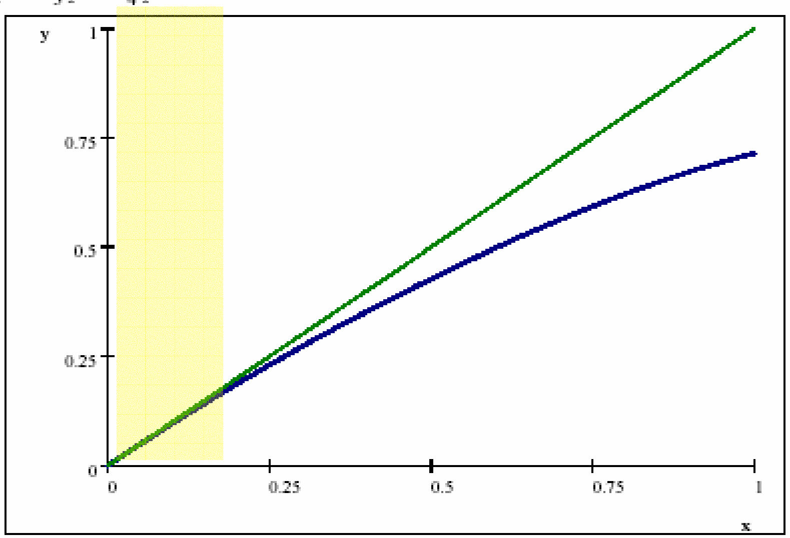
z grande



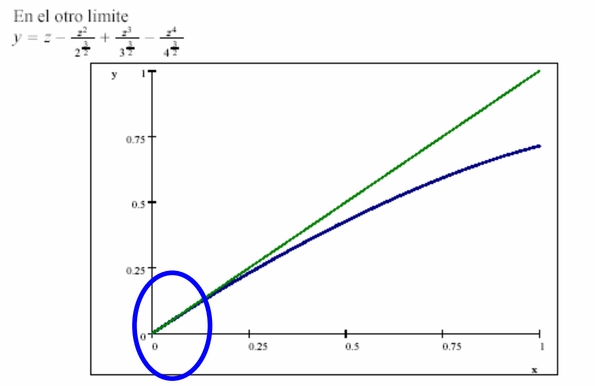
En el otro limite

$y = z - \frac{z^2}{2^{\frac{3}{2}}} + \frac{z^3}{3^{\frac{3}{2}}} - \frac{z^4}{4^{\frac{3}{2}}}$

z pequeño



Que pasaba con el gas de Fermi



1) $\frac{\lambda^3}{v} \ll 1$

Estas condiciones corresponden a z pequeño luego

$$\frac{\lambda^3}{v} \approx z - \frac{z^2}{2^{\frac{3}{2}}}$$

de donde

$$z \approx \frac{\lambda^3}{v} + \frac{1}{2^{\frac{3}{2}}}\left(\frac{\lambda^3}{v}\right)^2$$

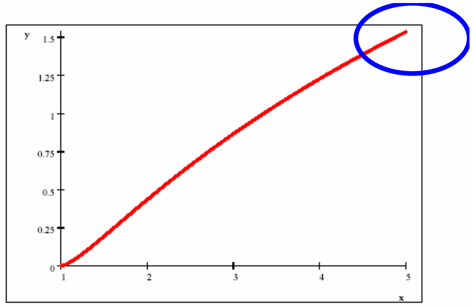
cuando $\frac{\lambda^3}{v} \longrightarrow 0 \Rightarrow z \longrightarrow \frac{\lambda^3}{v}$ que es la solucion para el gas de Boltzmann.

$$\frac{Pv}{kT} = \left(1 + \frac{\lambda^3}{v}\frac{1}{2^{\frac{3}{2}}}\ldots\right)$$

que es el gas ideal mas una correccion de origen cuantico

$$\frac{\lambda^3}{v} \gg 1$$

Para valores altos de $\frac{\lambda^3}{v} \Rightarrow$ valores altos de z



Definimos

$$\epsilon_f = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{v}\right)^{\frac{2}{3}}$$

La población media

$$\langle n_{\mathbf{p}} \rangle = \frac{z e^{-\beta \varepsilon_{\mathbf{p}}}}{1 + z e^{-\beta \varepsilon_{\mathbf{p}}}} = \frac{1}{1 + \exp \beta(\epsilon_{\mathbf{p}} - \epsilon_f)}$$

La energía

$$U = -\left(\frac{\partial}{\partial \beta} \log \Xi\right)_{z,V} = kT^2\left(\frac{\partial}{\partial T} \log \Xi\right) = kT^2\left(\frac{\partial}{\partial T} \frac{PV}{kT}\right)$$

$$= \frac{3}{2} kT \frac{V}{\lambda^3} f_{\frac{5}{2}}(z) = \frac{3}{2} NkT \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} \Rightarrow$$

$$\frac{U}{N} = \frac{3}{2} kT \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} = \frac{3}{2} kT \frac{\frac{8}{15\sqrt{\pi}} (\log z)^{\frac{5}{2}} [1 + \frac{5\pi^2}{8} (\log z)^{-2} + \ldots]}{\frac{4}{3\sqrt{\pi}} (\log z)^{\frac{3}{2}} [1 + \frac{\pi^2}{8} (\log z)^{-2} + \ldots]} \Rightarrow$$

$$\frac{U}{N} = \frac{3}{5}\epsilon_f\left(1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_f}\right)^2 + \ldots\right)$$

Del mismo modo

$$P = \frac{2}{3}\frac{U}{V}$$

Y para C_V

$$C_V = \frac{\pi^2}{2}\frac{kT}{\epsilon_f} + \ldots$$

Entonces para temperaturas bajas $T \ll T_f = \epsilon_f/k$, C_V es lineal en T

Ecuacion de Estado para situaciones no extremas

Hemos visto que ocurre cuando

$\frac{\lambda^3}{v} << 1$

y

$\frac{\lambda^3}{v} >> 1$

Que ocurre si $\frac{\lambda^3}{v} \lesssim 1$ hay que considerar toda la serie.

Proponemosun desarrollo en terminos del virial

$$\frac{PV}{NkT} = \sum_{l=1}^{\infty}(-1)^{l}a_{l}\left(\frac{\lambda^3}{gv}\right)^{l-1}$$

$$\frac{P}{kT} = \frac{g}{\lambda^3}f_{\frac{5}{2}}(z)$$

$$\frac{\lambda^3}{gv} = f_{\frac{3}{2}}(z) \Rightarrow \lambda^3 = g\frac{N}{V}f_{\frac{3}{2}}(z)$$

De donde

$$\frac{PV}{NkT} = \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}$$

Entonces

$$\left[\frac{P}{kT} = \frac{N}{V}gf_{\frac{5}{2}}(z) = \frac{1}{v}gf_{\frac{5}{2}}(z) = \frac{g}{\lambda^3}f_{\frac{5}{2}}(z)\right]$$

$$\sum_{l=1}^{\infty}(-1)^l a_l\left(\frac{\lambda^3}{gv}\right)^{l-1} = \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}$$

Resultando

$$\left[\sum_{l=1}^{\infty}(-1)^{l-1} a_l\left(\frac{\lambda^3}{gv}\right)^{l-1}\right] f_{\frac{3}{2}}(z) = f_{\frac{5}{2}}(z)$$

Reemplazando $f_{\frac{3}{2}}(z)$ y $f_{\frac{5}{2}}(z)$ por sus expresiones

$$f_{\frac{3}{2}}(z) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l^{\frac{3}{2}}} = z - \frac{z^2}{2^{\frac{3}{2}}} + \frac{z^3}{3^{\frac{3}{2}}} - \frac{z^4}{4^{\frac{3}{2}}} \dots$$

$$f_{\frac{5}{2}}(z) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l^{\frac{5}{2}}} = z - \frac{z^2}{2^{\frac{5}{2}}} + \frac{z^3}{3^{\frac{5}{2}}} - \frac{z^4}{4^{\frac{5}{2}}} \dots$$

luego

$$\left[\sum_{l=1}^{\infty}(-1)^{l-1}a_l\left(z-\frac{z^2}{2^{\frac{3}{2}}}+\frac{z^3}{3^{\frac{3}{2}}}-\frac{z^4}{4^{\frac{3}{2}}}\ldots\right)^{l-1}\right]\cdot\left[z-\frac{z^2}{2^{\frac{3}{2}}}+\frac{z^3}{3^{\frac{3}{2}}}-\ldots\right.$$

$$=z-\frac{z^2}{2^{\frac{5}{2}}}+\frac{z^3}{3^{\frac{5}{2}}}-\ldots$$

Cosechando terminos del mismo orden en z

$$a_1 z = z \Rightarrow a_1 = 1$$

$$\left\{[a_1]+\left[-a_2\left(z-\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}-\frac{z^4}{4^{3/2}}\ldots\right)\right]+\left[-a_3\left(z-\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}-\frac{z^4}{4^{3/2}}\right)^2\right]-\ldots\right\}\bullet$$

$$\bullet\left\{z-\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}-\frac{z^4}{4^{3/2}}\ldots\right\}=\left\{z-\frac{z^2}{2^{5/2}}+\frac{z^3}{3^{5/2}}-\frac{z^4}{4^{5/2}}\ldots\right\}$$

$$\Rightarrow$$

$$1)\, a_1 z = z$$

$$2)\, -a_1\frac{z^2}{2^{3/2}}-a_2 z^2=-\frac{z^2}{2^{5/2}}$$

$$3)\, a_1\frac{z^3}{3^{3/2}}+a_2 2\frac{z^3}{2^{3/2}}-a_3 z^3=\frac{z^3}{3^{5/2}}$$

$$a_2 z^2 + a_1 \frac{z^2}{2^{\frac{3}{2}}} = -\frac{z^2}{2^{\frac{5}{2}}} = a_2 z^2 + \frac{z^2}{2^{\frac{3}{2}}} \Rightarrow$$

$$a_2 = -\frac{1}{2^{\frac{3}{2}}} + \frac{1}{2^{\frac{5}{2}}} = -0.17678$$

$$\frac{z^3}{3^{\frac{5}{2}}} = a_1 \frac{z^3}{3^{\frac{3}{2}}} + a_2 2 \frac{z^3}{2^{\frac{3}{2}}} + a_3 z^3 \Rightarrow$$

$$a_3 = -\frac{1}{3^{\frac{3}{2}}} + \frac{1}{3^{\frac{5}{2}}} + 0.17678 \frac{1}{2^{1/2}} = -0.0032977$$

De donde

$$\frac{PV}{NkT} = \sum_{l=1}^{\infty} (-1)^{l-1} a_l \left(\frac{\lambda^3}{gv}\right)^{l-1} = 1 + 0.17678\left(\frac{\lambda^3}{gv}\right) - 0.0033\left(\frac{\lambda^3}{gv}\right)^2$$

Como $P = \frac{2}{3}(U/V) \Rightarrow U = \frac{3}{2}PV \Rightarrow C_V = \left[\frac{\partial}{\partial T}\frac{3}{2}PV\right]$

Que resulta

$$C_V = \left[\frac{\partial}{\partial T}\frac{3}{2}PV\right] = \frac{3}{2}Nk\left[\frac{\partial}{\partial T}\left(T\sum_{l=1}^{\infty}(-1)^l a_l\left(\frac{\lambda^3}{gv}\right)^{l-1}\right)\right]$$

entonces con $\lambda^3 = \left[\frac{2\pi\hbar^2}{mkT}\right]^{3/2} \Rightarrow$

$$\frac{\partial}{\partial T}\lambda^3 = -\frac{3}{2}\left[\frac{2\pi\hbar^2}{mk}\right]^{3/2}\left[\frac{1}{T}\right]^{5/2} = -\frac{3}{2}\frac{1}{T}\lambda^3$$

$$C_V = \frac{3}{2}Nk\left[\sum_{l=1}^{\infty}(-1)^l a_l\left(\frac{\lambda^3}{gv}\right)^{l-1} - \frac{3}{2}\sum_{l=1}^{\infty}(-1)^l a_l(l-1)\left(\frac{\lambda^3}{gv}\right)^{l-1}\right.$$

Entonces

$$\frac{C_V}{Nk} = \frac{3}{2}\left[\sum_{l=1}^{\infty}(-1)^l a_l\left(\frac{\lambda^3}{gv}\right)^{l-1} + \frac{3}{2}\sum_{l=1}^{\infty}(-1)^l a_l(-l+1)\left(\frac{\lambda^3}{gv}\right)^{l-1}\right]$$

$$\frac{C_V}{Nk} = \frac{3}{2}\left[\sum_{l=1}^{\infty}(-1)^l a_l\left(\frac{\lambda^3}{gv}\right)^{l-1}\left(\frac{5}{2} - \frac{3}{2}l\right)\right]$$

$$\frac{C_V}{Nk} = \frac{3}{2}\left[1 - (-0.17678)\left(\frac{5}{2} - \frac{3}{2}2\right)\left(\frac{\lambda^3}{gv}\right) - 2(-0.0033)\left(\frac{\lambda^3}{gv}\right)^2\right]$$

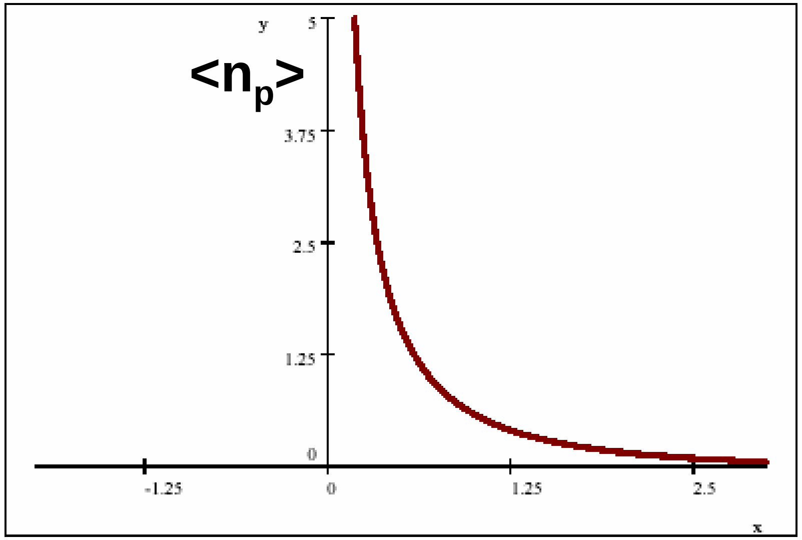
$$\frac{C_V}{Nk} = \frac{3}{2}\left[1 - 0.08839\left(\frac{\lambda^3}{gv}\right) + 0.0066\left(\frac{\lambda^3}{gv}\right)^2 + ..\right]$$

Pero λ va como $1/T^{1/2}$

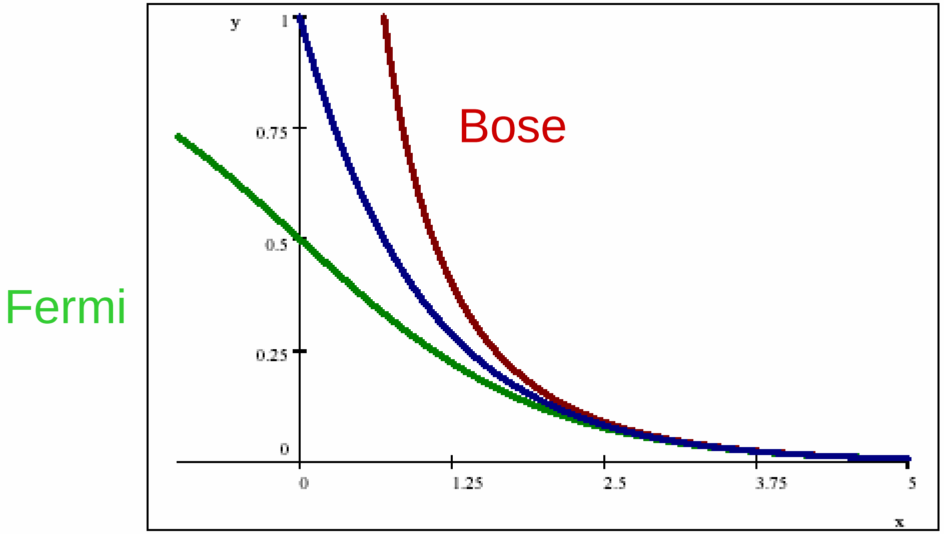
Acerca de los numeros de ocupacion

Para Bose

$$\langle n_{\mathbf{p}} \rangle = \frac{z e^{-\beta \varepsilon_{\mathbf{p}}}}{1 - z e^{-\beta \varepsilon_{\mathbf{p}}}} = \frac{1}{e^{\beta(\varepsilon_{\mathbf{p}} - \mu)} - 1}$$



Comparando numeros de ocupacion



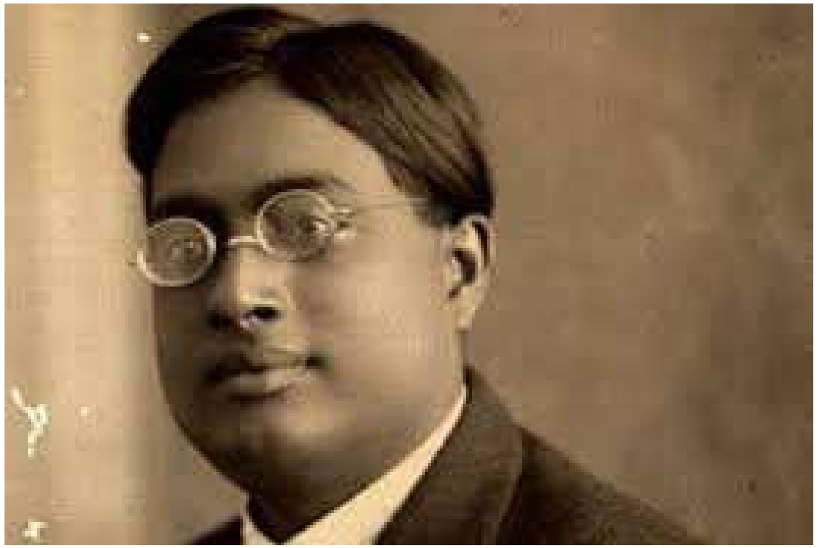
la diferencia entre las estadisticas cuanticas y la clasica se hace imperceptible cuando $e^{\beta(\varepsilon_{\mathbf{p}}-\mu)} \gg 1$

$$e^{\beta(\varepsilon_{\mathbf{p}})}/z \gg 1 \Rightarrow z \ll 1 \Rightarrow e^{\frac{\mu}{kT}} \ll 1 \Rightarrow \mu < 0 \text{ y } |\mu| \gg 1$$

como $0 \geq \mu$ para Bose

y para Fermi **no** habia restriccion $\Rightarrow$ es consistente.

BOSE



Gas de Bose

1)

$$\langle n_{\mathbf{p}} \rangle = \frac{z e^{-\beta \varepsilon_{\mathbf{p}}}}{1 - z e^{-\beta \varepsilon_{\mathbf{p}}}}$$

$$0 \leq \frac{z e^{-\beta \varepsilon_{\mathbf{p}}}}{1 - z e^{-\beta \varepsilon_{\mathbf{p}}}} = \frac{1}{e^{\beta(\varepsilon_{\mathbf{p}} - \mu)} - 1}$$

$$0 \leq e^{\beta(\varepsilon_{\mathbf{p}} - \mu)} - 1 \Rightarrow 1 \leq e^{\beta(\varepsilon_{\mathbf{p}} - \mu)} \Rightarrow 0 \leq (\varepsilon_{\mathbf{p}} - \mu) \Rightarrow \varepsilon_{\mathbf{p}} \geq \mu \Rightarrow$$

para el fundamental $\varepsilon_{\mathbf{p}} = 0 \Rightarrow 0 \geq \mu$ de otra forma el fundamental tendria poblacion negativa

$$\frac{PV}{kT} = -\sum \log(1 - z e^{-\beta \varepsilon_{\mathbf{p}}})$$

$$N = \sum_{\mathbf{p}} \frac{z e^{-\beta \varepsilon_{\mathbf{p}}}}{1 - z e^{-\beta \varepsilon_{\mathbf{p}}}}$$

2)

$$\frac{P}{kT} = -\frac{4\pi}{h^3}\int_0^\infty dp\,p^2 \log\left(1 - z e^{-\beta\frac{p^2}{2m}}\right) - \frac{1}{V}\log(1-z)$$

ademas

$$\frac{1}{v} = \frac{4\pi}{h^3}\int_0^\infty dp\,p^2 \frac{1}{-1 + z^{-1} e^{\beta\frac{p^2}{2m}}} + \frac{1}{V}\frac{z}{1-z}$$

Lo cual puede ser reescrito siguiendo metodo usado para el gas de Fermi

$$\frac{P}{kT} = \frac{1}{\lambda^3} g_{\frac{5}{2}}(z) - \frac{1}{V}\log(1-z)$$

Tenemos $$\langle n_p \rangle = \frac{1}{e^{\beta(e_p-\mu)}-1} \Rightarrow \langle n_0 \rangle = \frac{1}{e^{\beta(-\mu)}-1} = \frac{1}{\frac{1}{z}-1} = \frac{z}{1-z}$$

con $$\frac{P}{kT} = -\frac{4\pi}{h^3}\int_0^\infty dpp^2 \log\left(1 - ze^{-\beta\frac{p^2}{2m}}\right) - \frac{1}{V}\log(1-z)$$

pero $$N_0 = \frac{z}{1-z} \Rightarrow z = \frac{N_0}{N_0+1} \Rightarrow 1-z = 1-\frac{N_0}{N_0+1} = \frac{1}{N_0+1}$$

$$\Rightarrow$$

$$-\frac{1}{V}\log(1-z) \to \frac{1}{V}\log(N_0+1) \approx \frac{1}{N_0}\log(N_0+1)$$

Resulta entonces que este termino se va a 0 con $N_0 \to \infty$

$$\frac{1}{v} = \frac{4\pi}{h^3}\int_0^\infty dpp^2 \frac{1}{-1+z^{-1}e^{\beta\frac{p^2}{2m}}} + \frac{1}{V}\frac{z}{1-z}$$

Este, no

2)

$$\frac{P}{kT} = -\frac{4\pi}{h^3}\int_0^\infty dp p^2 \log\left(1 - z e^{-\beta\frac{p^2}{2m}}\right) - \frac{1}{V}\log(1-z)$$

ademas

$$\frac{1}{v} = \frac{4\pi}{h^3}\int_0^\infty dp p^2 \frac{1}{-1 + z^{-1} e^{\beta\frac{p^2}{2m}}} + \frac{1}{V}\frac{z}{1-z}$$

Lo cual puede ser reescrito siguiendo metodo usado para el gas de Fermi

$$\frac{P}{kT} = \frac{1}{\lambda^3} g_{\frac{5}{2}}(z) - \frac{1}{V}\log(1-z)$$

$$\frac{1}{v} = \frac{1}{\lambda^3} g_{\frac{3}{2}}(z) + \frac{1}{V}\frac{z}{1-z}$$

con

$$g_{\frac{5}{2}}(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^{\frac{5}{2}}}$$

$$g_{\frac{3}{2}}(z) = z\frac{\partial}{\partial z} g_{\frac{5}{2}}(z)$$

Entonces

$\langle n_0 \rangle = \frac{z}{1-z}$; $\frac{1}{v} = \frac{1}{\lambda^3} g_{\frac{3}{2}}(z) + \frac{\langle n_0 \rangle}{V}$ este ultimo termino tendra relevancia en el limite V$\rightarrow \infty$ si…

$$U(z,T) = \frac{-\partial}{\partial\beta}\log\Xi = \frac{-\partial}{\partial\beta}\frac{PV}{kT} = \frac{-\partial}{\partial\beta}\frac{g_{\frac{5}{2}}(z)}{\lambda^3}V \Rightarrow$$

$$\boxed{U(z,T)\frac{1}{V} = \frac{3kT}{2}\frac{g_{\frac{5}{2}}(z)}{\lambda^3}}$$

Propiedades de la $g_{\frac{3}{2}}(z)$

$$g_{\frac{3}{2}}(z) = \sum_{l=1}\frac{z^l}{l^{3/2}}$$

$z = e^{\beta\mu}$, como $\mu \leq 0 \Rightarrow 0 \leq z \leq 1$

Para Bose $0 \le z \le 1$ **(μ≤0)**

(recordemos que para Fermi se cumple $0 \le z \le \infty$)

Si z es pequeño

$$g_{\frac{3}{2}}(z) = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \ldots$$

Para $z = 1$

$g_{\frac{3}{2}}(1) = \zeta(1) = 2.612$ (y la derivada diverge)

$$g_{3/2}(z) = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} +$$

$$g'_{3/2}(z) = 1 + \frac{z}{2^{1/2}} + \frac{z^2}{3^{1/2}} +$$

$$g'_{3/2}(1) = 1 + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \ldots = \sum_{1}^{\infty} \frac{1}{n^{1/2}}$$

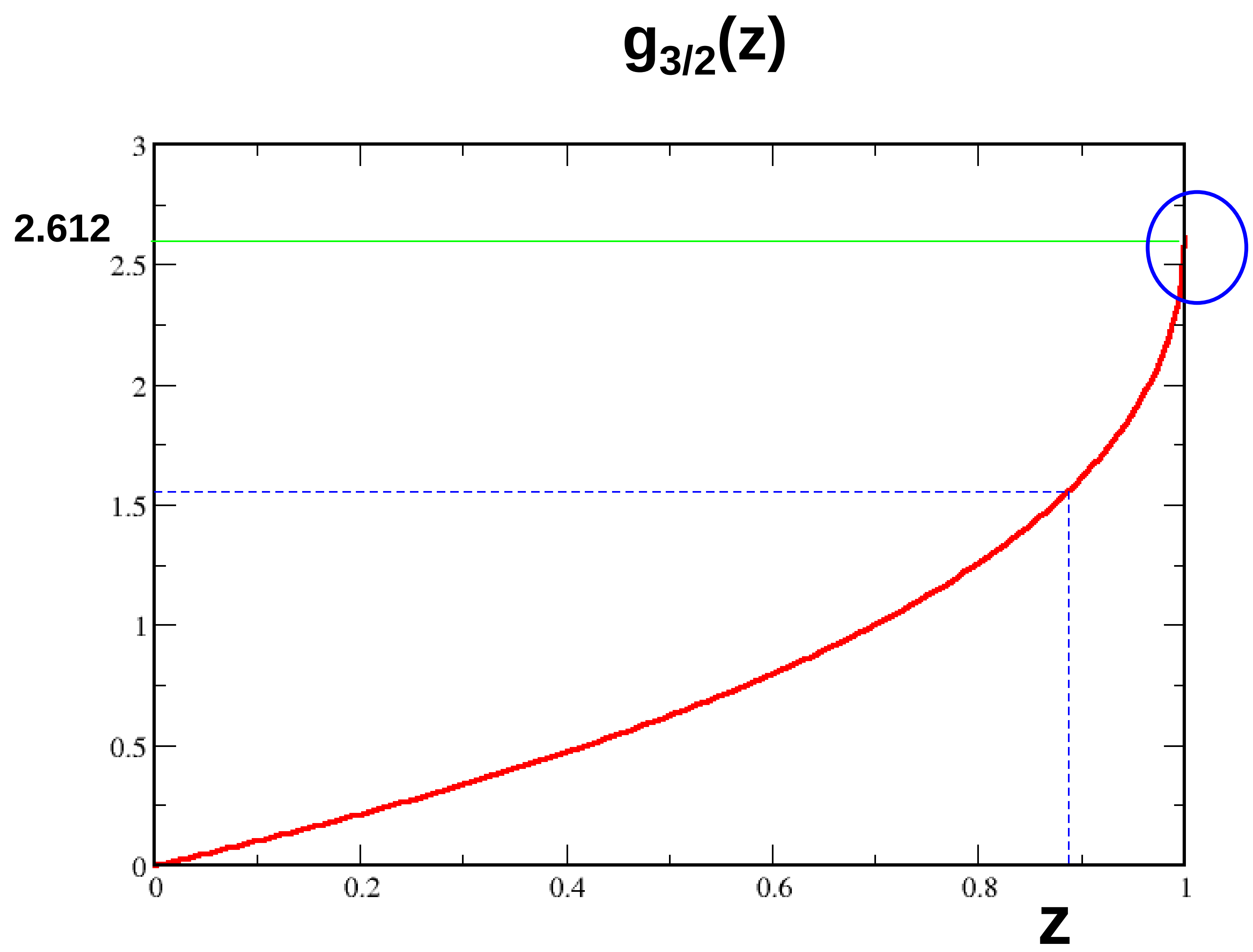
pero

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

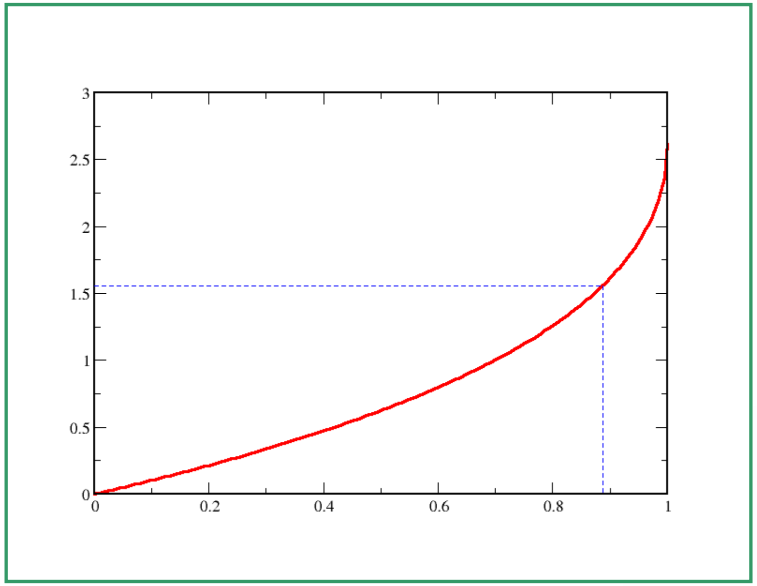
como

$$\frac{1}{n} \leq \frac{1}{n^{1/2}}$$

diverge



g3/2(z)
2.612
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
z



Si z es pequeño

$$g_{\frac{3}{2}}(z) = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \ldots$$

Para $z = 1$

$g_{\frac{3}{2}}(1) = \zeta(1) = 2.612$ (y la derivada diverge)

De donde es claro que si erroneamente hubiesemos asumido que $\frac{\lambda^3}{v} = g_{\frac{3}{2}}(z)$ no podriamos superar

$$\frac{\lambda^3}{v} = 2.612!!!!$$

Podemos escribir

Tomando en cuenta el termino de población del fundamental

$$\frac{\lambda^3}{v} = g_{\frac{3}{2}}(z) + \frac{\lambda^3}{V}\langle n_0 \rangle$$

de donde

$$\lambda^3 \frac{\langle n_0 \rangle}{V} = \frac{\lambda^3}{v} - g_{\frac{3}{2}}(z)$$

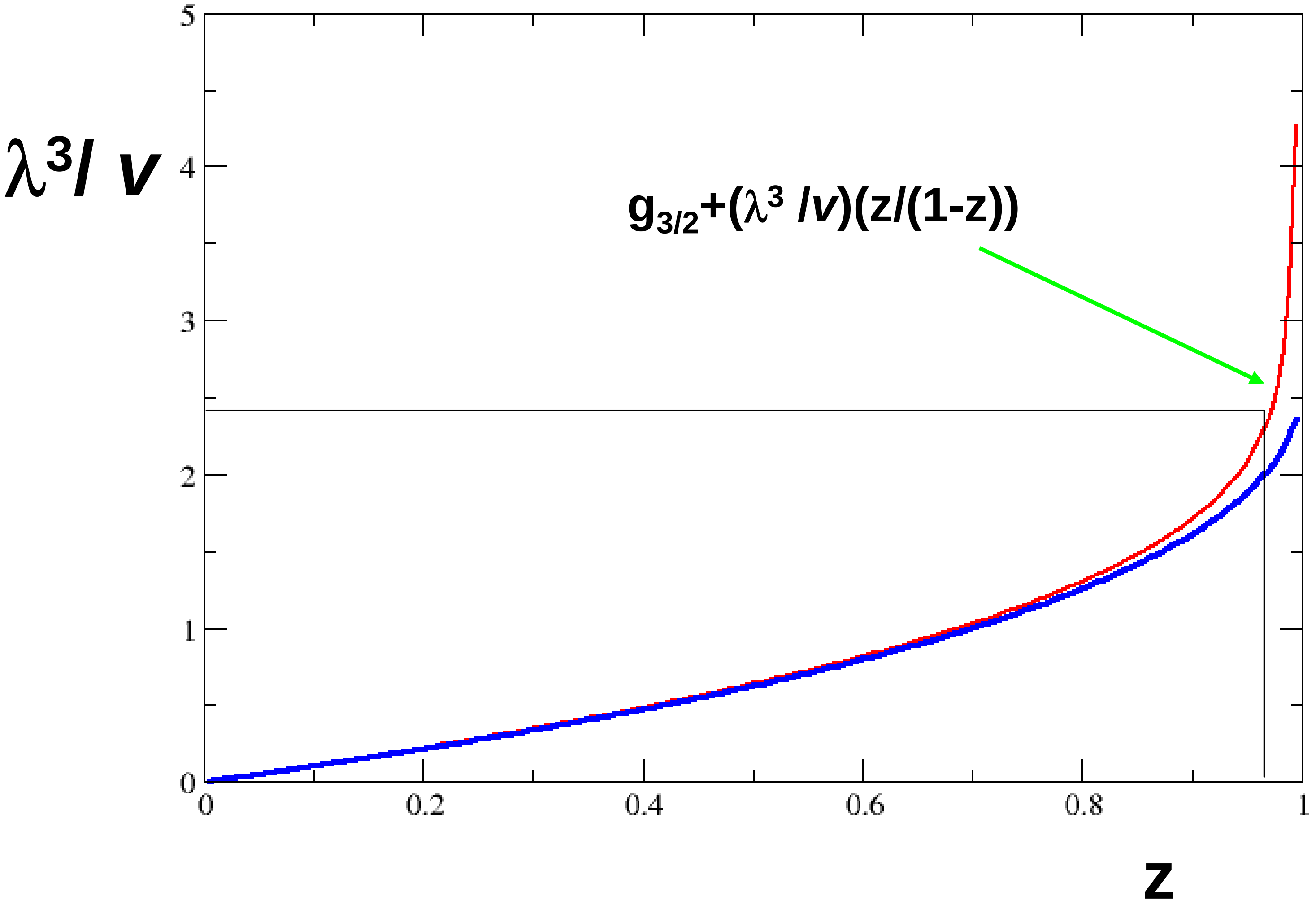
Entonces cuando $\frac{\lambda^3}{v} > g_{\frac{3}{2}}(1)$ se cumple que

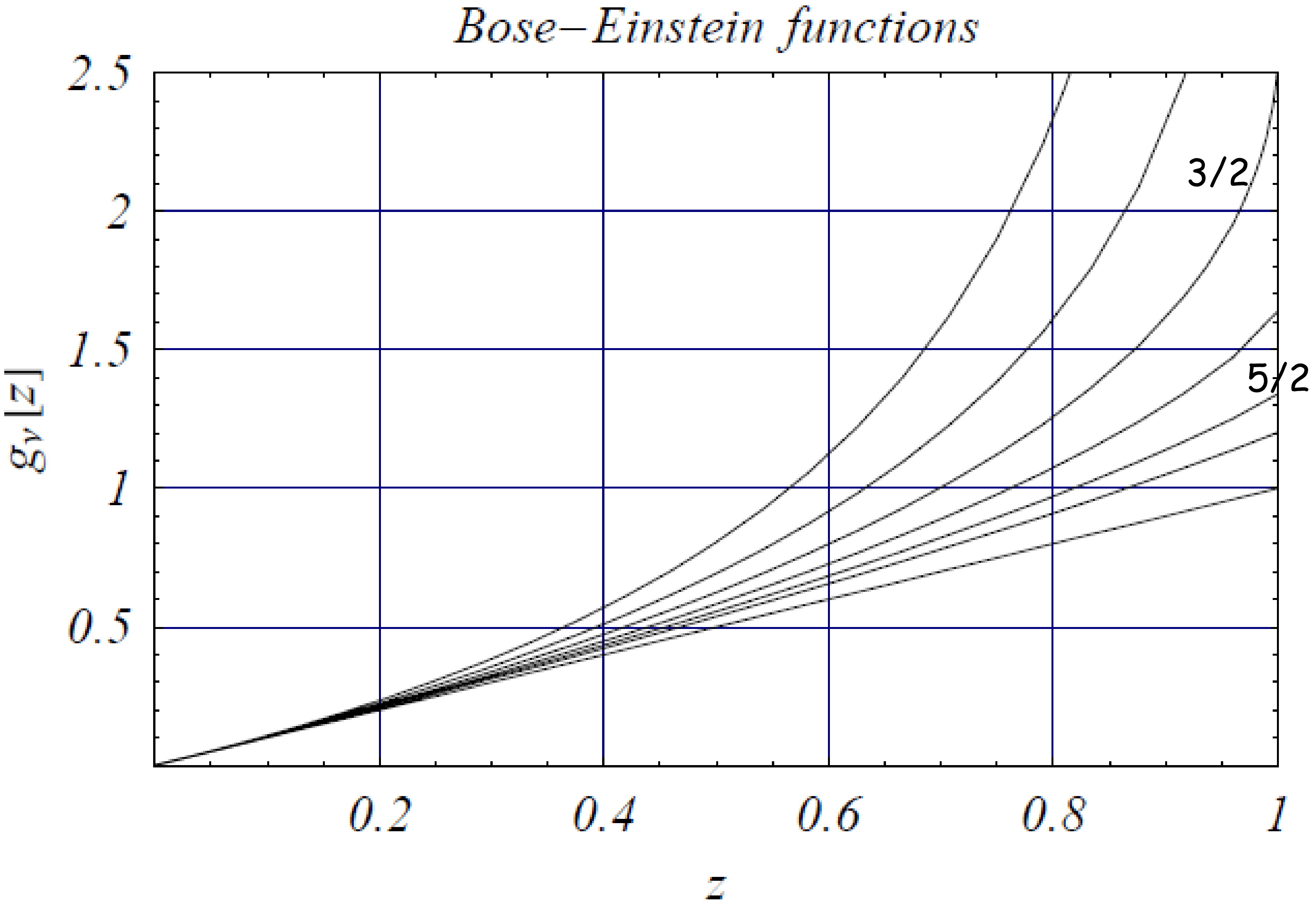
$$\frac{\langle n_0 \rangle}{V} > 0$$

Esto es lo que nos salva, pues de otra forma….

como $V \to \infty \Rightarrow \langle n_0 \rangle \to \infty \Rightarrow$ una fraccion macroscopica ocupa el nivel con $\mathbf{p} = 0$, esto se llama *condensacion de Bose-Einstein*

Tomar en cuenta que estamos considerando la población de un único nivel !!!!!


λ³/ v
g3/2+(λ³ /v)(z/(1-z))
z
0
0.2
0.4
0.6
0.8
1
2
3
4
5



Bose–Einstein functions
2.5
2
1.5
1
0.5
0.2
0.4
0.6
0.8
1
$g_\nu[z]$
z
3/2
5/2

Vemos que pasa con los niveles de energia con $\mathbf{p} \neq 0$

$$\frac{\langle n_1 \rangle}{V} = \frac{1}{V}\,\frac{1}{\frac{1}{z}\exp(\beta\epsilon_1)-1} \leq \frac{1}{V}\,\frac{1}{\exp(\beta\epsilon_1)-1}$$

$$\left[\frac{1}{\beta\varepsilon_1} = \frac{2mV^{2/3}}{\beta(h)^2}\right]$$

Dejamos de lado el fundamental $\Rightarrow$

como $2m\epsilon_i = \left(2\pi\hbar\right)^2 \frac{l_i}{V^{2/3}}$, con l_i el la suma de los cuadrados de eneteros no todos 0 , pues $\mathbf{p} = \frac{2\pi\hbar\,\mathbf{n}}{V^{1/3}}$

Entonces desarrollando en serie con $l_1 = 1$

$$\frac{\langle n_1 \rangle}{V} \leq \frac{1}{V}\,\frac{2mV^{2/3}}{\left(2\pi\hbar\right)^2\beta} = \frac{2mV^{-1/3}}{\left(2\pi\hbar\right)^2\beta} \rightarrow 0 \text{ si } V \rightarrow \infty$$

$$\exp(x) \approx 1 + x + \frac{1}{2}x^2 + \ldots$$

Las poblaciones parciales de los niveles distintos del fund. son muy diluidas
Luego es apropiado hablar del condensado por un lado y “el resto” por otro

La condicion $\frac{\lambda^3}{v} > g_{\frac{3}{2}}(1)$ define una region en el diagrama de fases $P - v - T$,en esta region podemos pensar en la coexistencia de dos fluidos ($\mathbf{p} = 0$ y $\mathbf{p} \neq 0$)

De esta forma, si fijamos v , podemos definir una temperatura critica T_c

$$\frac{\lambda_c^3}{v} = g_{3/2}(1)$$

como $\lambda = \sqrt{2\pi\hbar^2/mkT} \Rightarrow$

$$\lambda_c^3 = \left[\sqrt{2\pi\hbar^2/mkT_c}\right]^3 = v g_{\frac{3}{2}}(1)$$

luego λ_c^3 es del orden del volumen especifico $(g_{3/2}(1) \cong 2)$

entonces

$$(kT_c)^{3/2} = \frac{h^3}{(2\pi m)^{3/2}} \frac{1}{v g_{3/2}(1)} \Rightarrow$$

$$\frac{(2\pi m)^{3/2}}{h^3} v g_{3/2}(1) = \frac{1}{(kT_c)^{3/2}}$$

$$kT_c = \frac{2\pi\hbar^2}{m(v g_{\frac{3}{2}}(1))^{\frac{2}{3}}}$$

del mismo modo para una dada T existe un $v_c = \frac{\lambda^3}{g_{\frac{3}{2}}(1)}$

$\Rightarrow$ el fenomeno de aparicion de la fase macroscopica en **p** $= 0$ ocurre cuando

$$T < T_c \quad \text{o} \quad v < v_c$$

Podemos expresar la coexistencia de fases como:

i) una fase normal consistente en ($\frac{\lambda^3}{v} = g_{\frac{3}{2}}(1)$)

$$N_e = V\frac{(2\pi mkT)^{3/2}}{h^3} g_{\frac{3}{2}}(1) = N(\frac{T}{T_c})^{3/2}$$

(o sea que $g_{3/2}$ describe la parte normal)

ii) una fase condensada compuesta por

$$N_0 = N - N_e = N(1 - (\frac{T}{T_c})^{3/2})$$

acumuladas en $\mathbf{p} = 0$

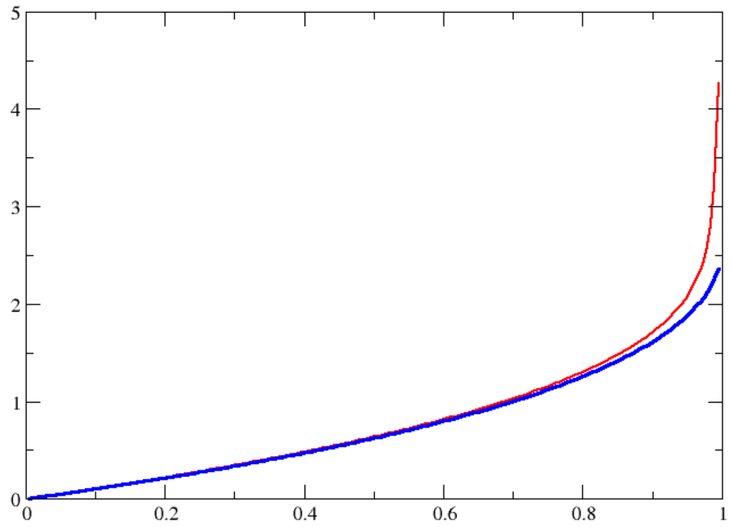
$N_0 = N - N_e = N(1 - (\frac{T}{T_c})^{3/2})$, acumuladas en $\mathbf{p} = 0$

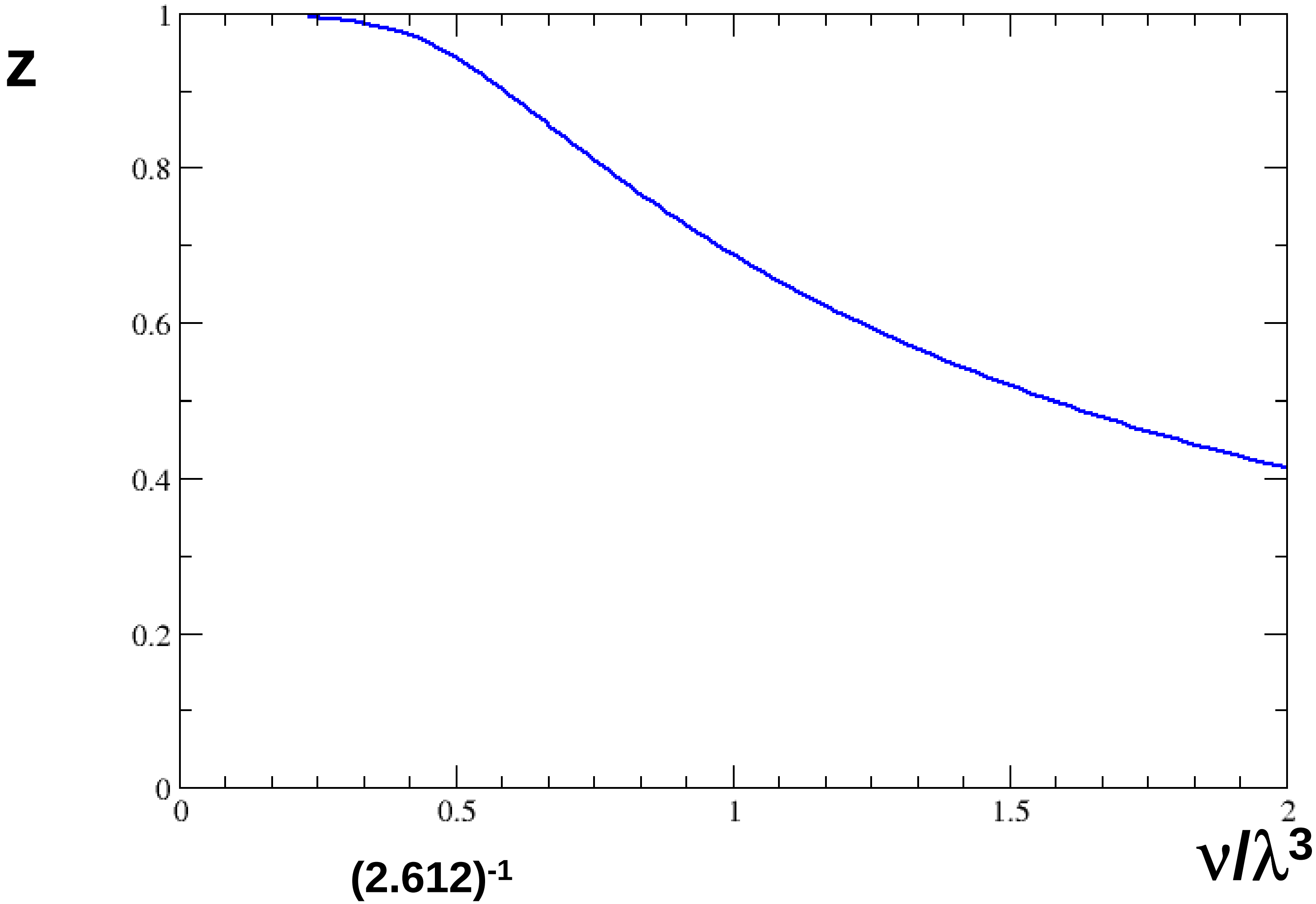
(expresado como diferencia)

Para encontrar z en funcion de T y v resolvemos numericamente

$$\frac{1}{v} = \frac{1}{\lambda^3} g_{3/2}(z) + \frac{1}{V}\frac{z}{1-z}$$

Si V es grande pero finito podemos usar el grafico




z
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
ν/λ³
(2.612)⁻¹

En el limite $V \to \infty$ resulta

$\frac{\lambda^3}{v} \geq g_{3/2}(1)$	$\Rightarrow \quad z = 1$
$\frac{\lambda^3}{v} < g_{3/2}(1)$	$\Rightarrow \quad$ z raiz de $\left(\frac{\lambda^3}{v} = g_{3/2}(z) \right)$

Diagrama P-T del gas de Bose

Estudiamos la variacion de P con T a v cte.

$$\frac{P}{kT} = \frac{1}{\lambda^3} g_{5/2}(z)$$

-Si $T < T_c$ [⇒ z = 1]

$$P(T) = \frac{kT}{\lambda^3} g_{5/2}(1)$$

como $\lambda^3 \propto T^{3/2} \Rightarrow P(T) \propto T^{5/2}$ y ademas independiente de V

Tiende a 0 con T →0

-Si $T = T_c$ (punto de transicion)

$$P(T_c) = \left(\frac{2\pi m}{h^2}\right)^{3/2} (kT_c)^{\frac{5}{2}} g_{5/2}(1)$$

ver

reemplazando $T_c = \frac{h^2}{2\pi mk}\left(\frac{N}{V g_{3/2}(1)}\right)^{2/3}$

$$P(T_c) = \left(\frac{N}{V} kT_c\right) \frac{g_{5/2}(1)}{g_{3/2}(1)} \simeq 0.51\left(\frac{N}{V} kT_c\right)$$

pues $g_{5/2}(1) \simeq 1.34$

$$P(T_c) = \left(\frac{2\pi m}{h^2}\right)^{3/2} (kT_c)^{5/2} g_{5/2}(1) = kT_c \left(\frac{2\pi m}{h^2}\right)^{3/2} (kT_c)^{3/2} g_{5/2}(1)$$

$$T_c = \frac{h^2}{2\pi mk}\left(\frac{N}{V g_{3/2}(1)}\right)^{2/3}$$

$$P(T_c) = kT_c \left(\frac{2\pi m}{h^2}\right)^{3/2} \left(k \frac{h^2}{2\pi mk}\left(\frac{N}{V g_{3/2}(1)}\right)^{2/3}\right)^{3/2} g_{5/2}(1)$$

$$\boxed{P(T_c) = \left(\frac{N}{V} kT_c\right)\left(\frac{g_{5/2}(1)}{g_{3/2}(1)}\right)}$$

resulta entonces que a la temperatura critica la presion del gas de Bose es aprox. la mitad que el de Boltzmann.

-Si $T > T_c$

$$P(T) = \frac{N}{V} kT \frac{g_{5/2}(z)}{g_{3/2}(z)}$$

y esto es lo mas simple que podemos escribir

pero si $T \gg T_c$

z sera pequeňo

N_0 despreciable frente a $N \Rightarrow$

proponemos una expansion del virial

$$\frac{PV}{NkT} = \sum a_l \left(\frac{\lambda^3}{v}\right)^{l-1}$$

Como sabemos tambien que

$$\frac{P}{kT} = \frac{1}{\lambda^3} g_{5/2}(z)$$

y

$$\frac{N}{V} = \frac{N - N_0}{V} = \frac{1}{\lambda^3} g_{3/2}(z)$$

Resulta de las dos ultimas que

$$\frac{PV}{NkT} = \sum a_l \left(\frac{\lambda^3}{v}\right)^{l-1} = \frac{g_{5/2}(z)}{g_{3/2}(z)}$$

de donde usando las expansiones de $g_\tau(z)$

Obtenemos

$$z+\frac{z^2}{2^{5/2}}+\frac{z^3}{3^{5/2}}+\ldots=\left[z+\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}+\ldots\right]*$$
$$*\left[\sum a_l\left(z+\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}+\ldots\right)^{l-1}\right]$$

$$z+\frac{z^2}{2^{5/2}}+\frac{z^3}{3^{5/2}}+\ldots=\left[z+\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}+\ldots\right]*$$
$$*\left[a_1+a_2\left(z+\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}+\ldots\right)+a_3\left(z+\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}+\ldots\right)^2+\right]$$

$$z=za_1$$

$$\frac{z^2}{2^{5/2}}=a_2z^2+a_1\frac{z^2}{2^{3/2}}$$

$$\frac{z^3}{3^{5/2}}=a_3z^3+2a_2\frac{z^3}{2^{3/2}}+a_1\frac{z^3}{3^{3/2}}$$

$$z = z a_1 \Rightarrow \boxed{a_1 = 1}$$

$$\frac{z^2}{2^{5/2}} = a_2 z^2 + a_1 \frac{z^2}{2^{3/2}} \Rightarrow$$

$$\frac{1}{2^{5/2}} = a_2 + \frac{1}{2^{3/2}} \Rightarrow \boxed{a_2 = \frac{1}{2^{5/2}} - \frac{1}{2^{3/2}}}$$

$$\frac{z^3}{3^{5/2}} = a_3 z^3 + a_2 \frac{z^3}{2^{3/2}} + a_1 \frac{z^3}{3^{3/2}} \Rightarrow$$

$$\frac{1}{3^{5/2}} = a_3 + 2\left(\frac{1}{2^{5/2}} - \frac{1}{2^{3/2}}\right)\frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} \Rightarrow$$

$$\boxed{\frac{1}{3^{5/2}} - \left(\frac{1}{2^{5/2}} - \frac{1}{2^{3/2}}\right)\frac{2}{2^{3/2}} - \frac{1}{3^{3/2}} = a_3}$$

Los valores resultantes son (comparar con Fermi)

$$\begin{aligned} a_1 &= 1 \\ a_2 &= -0.17678 \\ a_3 &= -0.00330 \end{aligned}$$

Luego cuando T se hace muy grande se converge al limite clasico.

$$\frac{PV}{NkT} = 1 - 0.17678\frac{\lambda^3}{v} - 0.0033\frac{\lambda^6}{v^2}\ldots$$

$$\frac{PV}{NkT} = 1 - 0.17678\frac{1}{vT^{3/2}}\left[\frac{2\pi\hbar^2}{mk}\right]^{3/2} - 0.0033\frac{1}{v^2T^3}\left[\frac{2\pi\hbar^2}{mk}\right]^3\ldots$$

$$PV = NkT - 0.17678\frac{1}{T^{1/2}}\frac{Nk}{v}\left[\frac{2\pi\hbar^2}{mk}\right]^{3/2} - 0.0033\frac{1}{T^2}[\,]\ldots$$

Entonces para el gas de Bose la presión es menor que la del gas ideal y converge al mismo para $T \gg T_c$

Conocido P

$$U = -\left(\frac{\partial}{\partial \beta}\ln \Xi\right) = kT^2\left[\frac{\partial}{\partial T}\left(\frac{PV}{kT}\right)\right]$$

$$\frac{C_V}{Nk} = \frac{1}{Nk}\left(\frac{\partial U}{\partial T}\right)_{N,V}$$

Si $T < T_c$

$$\frac{C_v}{Nk} = \frac{3}{2}\frac{V}{N}g_{5/2}(1)\frac{d}{dT}\left(\frac{T}{\lambda^3}\right) = \frac{15}{4}g_{5/2}(1)\frac{v}{\lambda^3}$$

$$\left[\frac{T}{\lambda^3} \propto T^{5/2} \Rightarrow \frac{\partial}{\partial T} \rightarrow \frac{5}{2}\frac{1}{\lambda^3}\right]$$

o sea que $\frac{C_v}{Nk} \propto T^{3/2}$

Si $T = T_c$

$$\frac{C_v(T_c)}{Nk} = \frac{15}{4}\frac{g_{5/2}(1)}{g_{3/2}(1)} \simeq 1.925 > 1.5$$

Recordemos que :

$$T < T_c$$

$$P(T) = \frac{kT}{\lambda^3} g_{5/2}(1)$$

$$U(T) = kT^2 V g_{5/2}(1) \frac{\partial}{\partial T} \frac{1}{\lambda^3} \propto T^2 \frac{\partial}{\partial T} T^{3/2} \propto T^{5/2} \Rightarrow C_V \propto T^{3/2} \propto \frac{1}{\lambda^3}$$

Entonces para el gas de Bose la presión es menor que

la del gas ideal y converge al mismo para $T \gg T_c$

Conocido P

$$U = -\left(\frac{\partial}{\partial \beta}\ln \Xi\right) = kT^2\left[\frac{\partial}{\partial T}\left(\frac{PV}{kT}\right)\right]$$

$$\frac{C_V}{Nk} = \frac{1}{Nk}\left(\frac{\partial U}{\partial T}\right)_{N,V}$$

Si $T < T_c$

$$\frac{C_v}{Nk} = \frac{3}{2}\frac{V}{N}g_{5/2}(1)\frac{d}{dT}\left(\frac{T}{\lambda^3}\right) = \frac{15}{4}g_{5/2}(1)\frac{v}{\lambda^3}$$

$$\left[\frac{T}{\lambda^3} \propto T^{5/2} \Rightarrow \frac{\partial}{\partial T} \to \frac{5}{2}\frac{1}{\lambda^3}\right]$$

o sea que $\frac{C_v}{Nk} \propto T^{3/2}$

Si $T = T_c$

$$\frac{C_v(T_c)}{Nk} = \frac{15}{4}\frac{g_{5/2}(1)}{g_{3/2}(1)} \simeq 1.925 > 1.5$$

Si $T > T_c$

$$\frac{C_v}{Nk} = \frac{\partial}{\partial T}\left(\frac{3}{2}T\frac{g_{5/2}(z)}{g_{3/2}(z)}\right)$$

$$\frac{\partial}{\partial T}g_{3/2} = \frac{\partial}{\partial T}\frac{\lambda^3}{v} = -\frac{3}{2T}g_{3/2}$$

ademas

$$\frac{\partial}{\partial T}g_{3/2} = \frac{\partial z}{\partial T}\frac{\partial}{\partial z}g_{3/2} = \frac{\partial z}{\partial T}\frac{1}{z}g_{1/2} \Rightarrow -\frac{3}{2T}\frac{g_{3/2}}{g_{1/2}} = \frac{\partial z}{\partial T}\frac{1}{z}$$

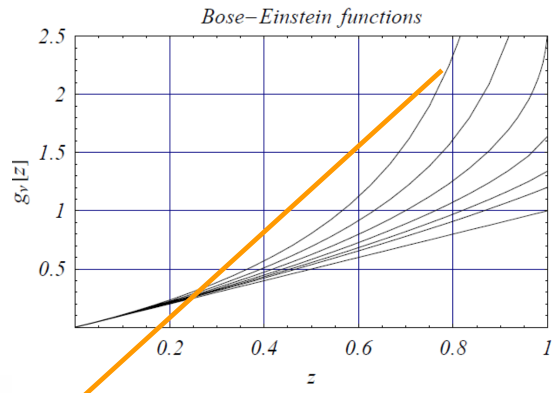
$$\frac{\partial}{\partial T}g_{5/2} = \frac{\partial z}{\partial T}\frac{\partial}{\partial z}g_{5/2} = \frac{\partial z}{\partial T}\frac{1}{z}g_{3/2} = -\frac{3}{2T}\frac{g_{3/2}^2}{g_{1/2}}$$

entonces

$$\frac{\partial}{\partial T}\left(\frac{3}{2}T\frac{g_{5/2}}{g_{3/2}}\right) = \frac{3}{2}\frac{g_{5/2}}{g_{3/2}} + \frac{3}{2}T\frac{g_{3/2}\partial_T g_{5/2} - g_{5/2}\partial_T g_{3/2}}{g_{3/2}^2}$$

$$\frac{\partial}{\partial T}\left(\frac{3}{2}T\frac{g_{5/2}}{g_{3/2}}\right) = \frac{3}{2}\frac{g_{5/2}}{g_{3/2}} - \frac{9}{4}\frac{g_{3/2}}{g_{1/2}} + \frac{9}{4}\frac{g_{5/2}}{g_{3/2}} = \frac{15}{4}\frac{g_{5/2}}{g_{3/2}} - \frac{9}{4}\frac{g_{3/2}}{g_{1/2}}$$

Si $T > T_c$



Esto resulta

$$\frac{C_v}{Nk} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)}$$

En $z = 1$ el segundo termino se anula y encontramos que C_v es continuo.

Si $T \gg T_c$

$$\frac{C_v}{Nk} = \frac{3}{2}\left(\frac{\partial}{\partial T}\left(\frac{PV}{NK}\right)\right)$$

usando en el sarrollo del virial

$$\frac{C_v}{Nk} = \frac{3}{2}\left(\frac{\partial}{\partial T}\left(T\sum_{l=1} a_l\left(\frac{\lambda^3}{v}\right)^{l-1}\right)\right)$$

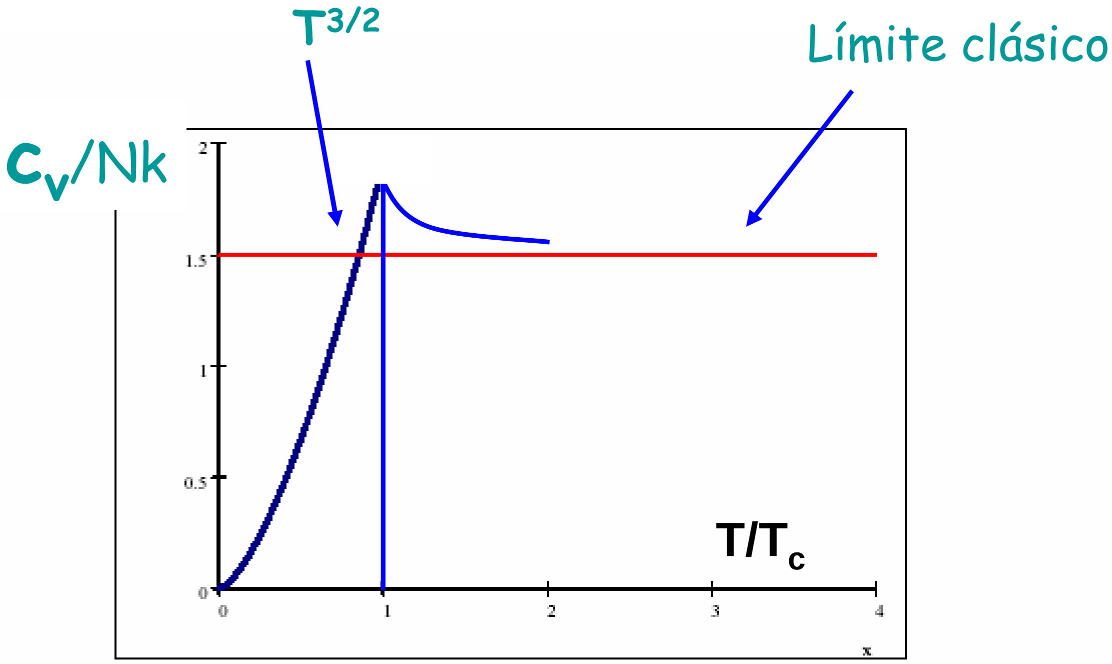
$$= \frac{3}{2}\sum_{l=1} a_l\left(\frac{\lambda^3}{v}\right)^{l-1} + (l-1)\left(\frac{-3}{2}\right)\sum_{l=1} a_l\left(\frac{\lambda^3}{v}\right)^{l-1}$$

$$\frac{C_v}{Nk} = \frac{3}{2}\sum_{l=1} \frac{5-3l}{2} a_l\left(\frac{\lambda^3}{v}\right)^{l-1}$$

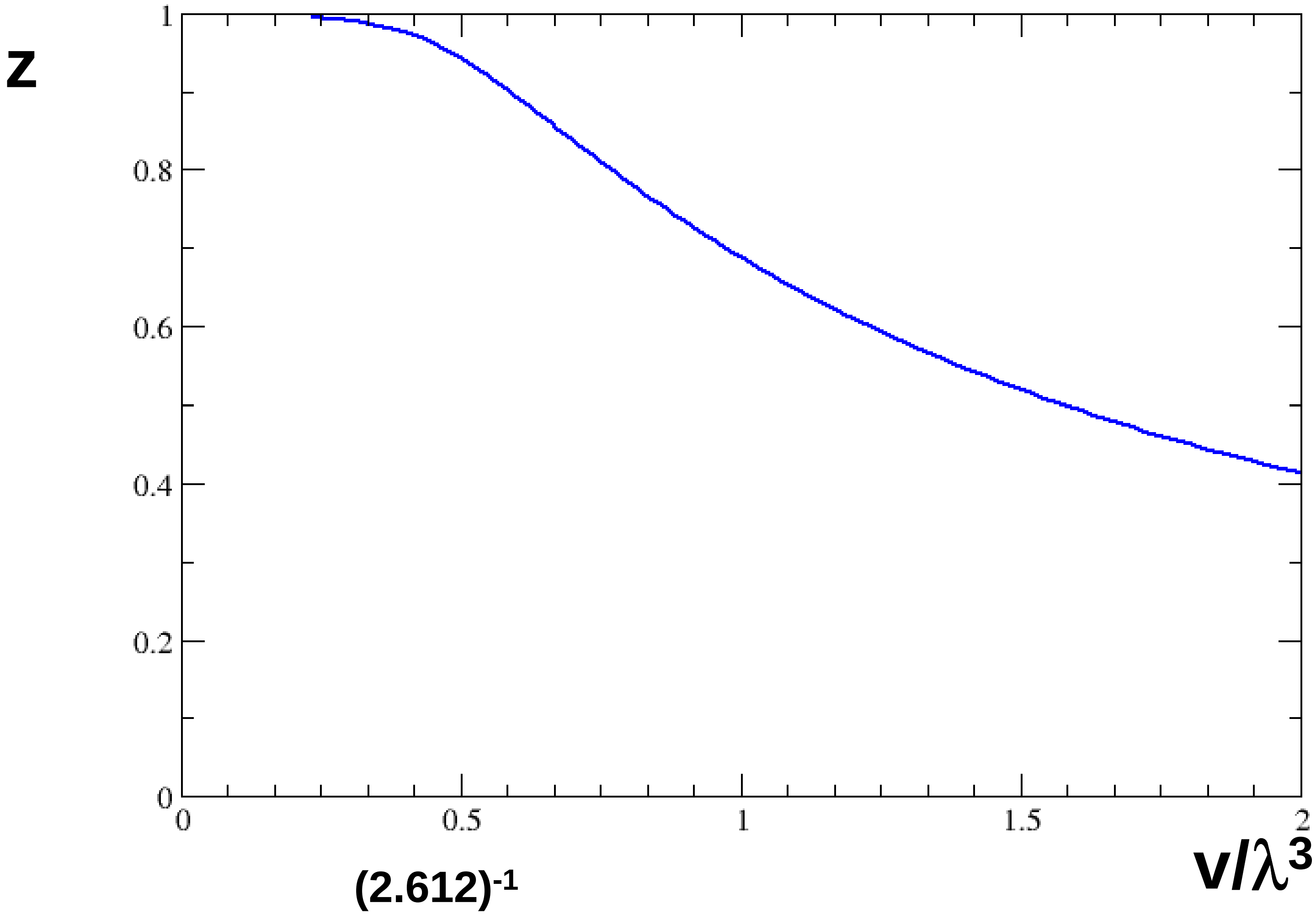
entonces

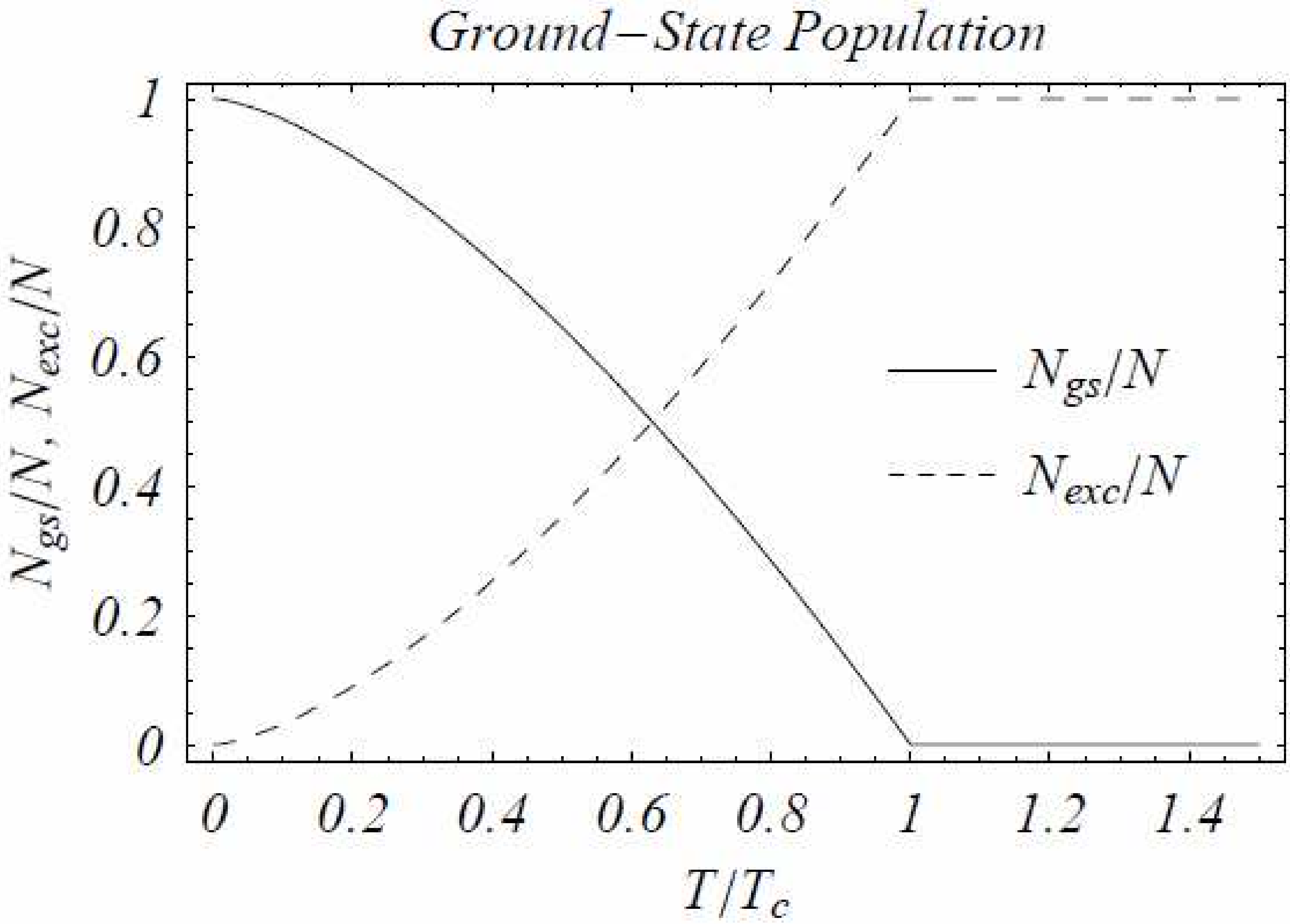
$$\boxed{\frac{C_v}{Nk} = \frac{3}{2}\left(1 + 0.0884\frac{\lambda^3}{v} + \ldots\right)}$$

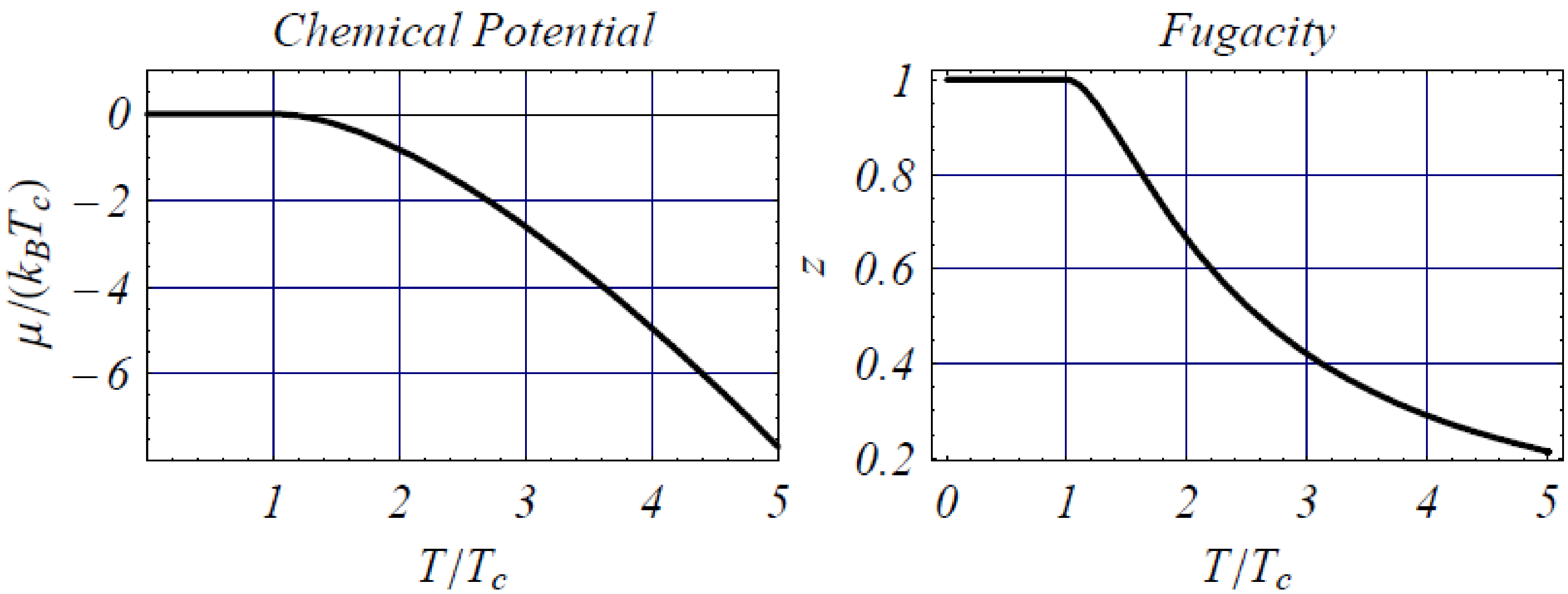
o sea que converge uniformemente al limite clasico



T3/2
Límite clásico
Cv/Nk
2
1.5
1
0.5
0
0
1
2
3
4
x
T/Tc

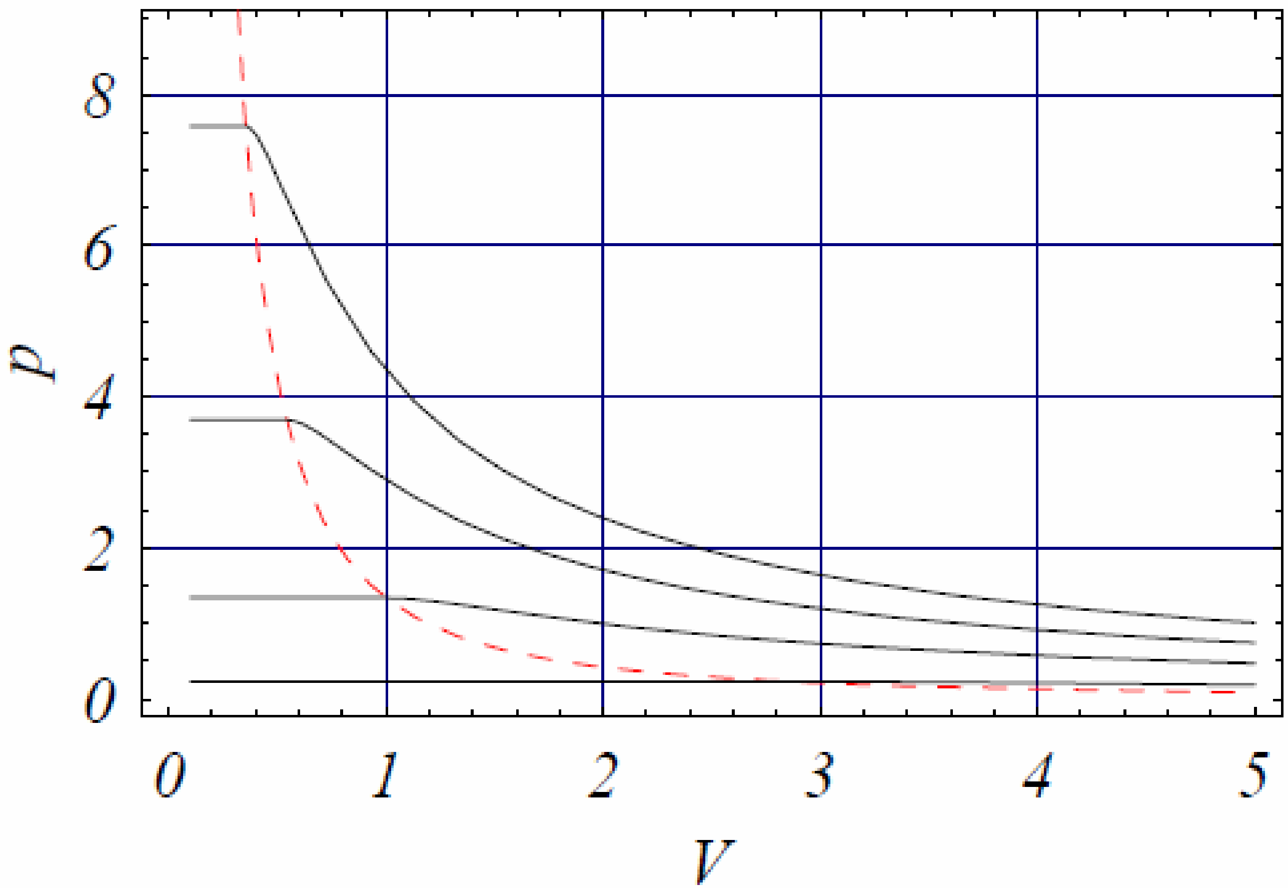

z
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
v/λ3
(2.612)-1


Ground−State Population
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
N_{gs}/N, N_{exc}/N
T/T_c
N_{gs}/N
N_{exc}/N

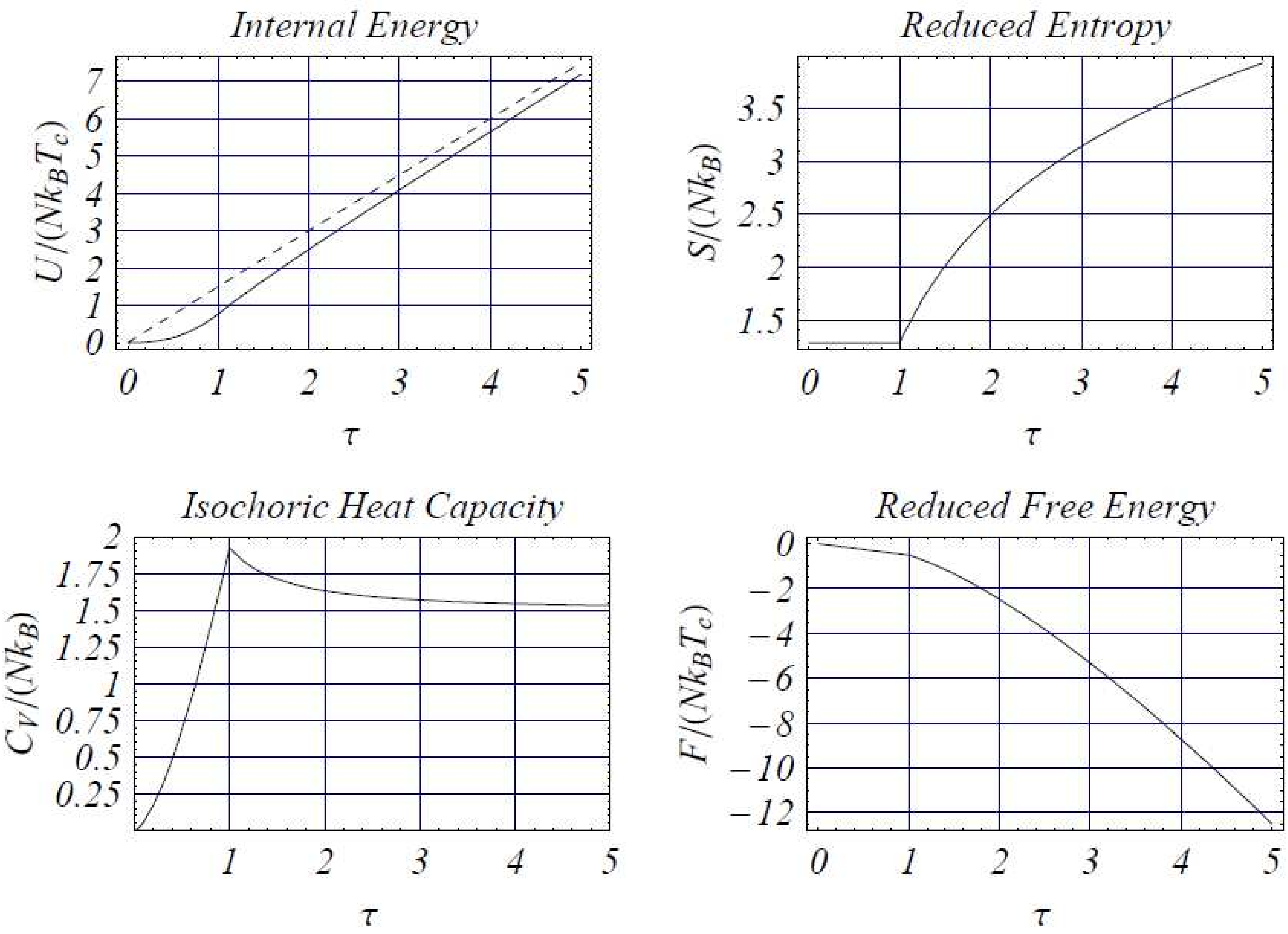


Chemical Potential
$\mu/(k_B T_c)$
0
−2
−4
−6
1
2
3
4
5
T/T_c
Fugacity
z
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
T/T_c

Isotherms for ideal Bose systems



	$T \leq T_c$	$T \geq T_c$
N_{exc}	$\left(\frac{T}{T_c}\right)^{3/2} N$	N
U	$N_{\text{exc}}\, k_B\, T\, \frac{3}{2}\, \frac{\zeta[\frac{5}{2}]}{\zeta[\frac{3}{2}]}$	$N\, k_B\, T\, \frac{3}{2}\, \frac{g_{5/2}[z]}{g_{3/2}[z]}$
C_V	$N_{\text{exc}}\, k_B\, \frac{15}{4}\, \frac{\zeta[\frac{5}{2}]}{\zeta[\frac{3}{2}]}$	$N\, k_B \left(\frac{15}{4}\, \frac{g_{5/2}[z]}{g_{3/2}[z]} - \frac{9}{4}\, \frac{g_{3/2}[z]}{g_{1/2}[z]} \right)$
S	$N_{\text{exc}}\, k_B\, \frac{5}{2}\, \frac{\zeta[\frac{5}{2}]}{\zeta[\frac{3}{2}]}$	$N\, k_B \left(\frac{5}{2}\, \frac{g_{5/2}[z]}{g_{3/2}[z]} - \ln z \right)$
F	$-N_{\text{exc}}\, k_B\, T\, \frac{\zeta[\frac{5}{2}]}{\zeta[\frac{3}{2}]}$	$-N\, k_B\, T \left(\frac{g_{5/2}[z]}{g_{3/2}[z]} + \ln z \right)$


Internal Energy
$U/(Nk_BT_c)$
0
1
2
3
4
5
6
7
0
1
2
3
4
5
τ
Reduced Entropy
$S/(Nk_B)$
1.5
2
2.5
3
3.5
0
1
2
3
4
5
τ
Isochoric Heat Capacity
$C_V/(Nk_B)$
0.25
0.5
0.75
1
1.25
1.5
1.75
2
1
2
3
4
5
τ
Reduced Free Energy
$F/(Nk_BT_c)$
0
−2
−4
−6
−8
−10
−12
0
1
2
3
4
5
τ

$$\frac{\partial}{\partial T} g_{3/2} = \frac{\partial}{\partial T} \frac{\lambda^3}{v} = -\frac{3}{2T} g_{3/2}$$

Pero también

$$\frac{\partial}{\partial T} g_{3/2} = \frac{\partial z}{\partial T} \frac{\partial}{\partial z} g_{3/2} = \frac{\partial z}{\partial T} \frac{1}{z} g_{1/2} \Rightarrow -\frac{3}{2T} \frac{g_{3/2}}{g_{1/2}} = \frac{\partial z}{\partial T} \frac{1}{z}$$

$$\frac{\partial}{\partial T} g_{5/2} = \frac{\partial z}{\partial T} \frac{\partial}{\partial z} g_{5/2} = \frac{\partial z}{\partial T} \frac{1}{z} g_{3/2} = -\frac{3}{2T} \frac{g_{3/2}^2}{g_{1/2}}$$

Entonces

$$\frac{\partial}{\partial T} \left(\frac{3}{2} T \frac{g_{5/2}}{g_{3/2}} \right) = \frac{3}{2} \frac{g_{5/2}}{g_{3/2}} + \frac{3}{2} T \frac{g_{3/2} \partial_T g_{5/2} - g_{5/2} \partial_T g_{3/2}}{g_{3/2}^2}$$

$$\frac{\partial}{\partial T} \left(\frac{3}{2} T \frac{g_{5/2}}{g_{3/2}} \right) = \frac{3}{2} \frac{g_{5/2}}{g_{3/2}} - \frac{9}{4} \frac{g_{3/2}}{g_{1/2}} + \frac{9}{4} \frac{g_{5/2}}{g_{3/2}}$$