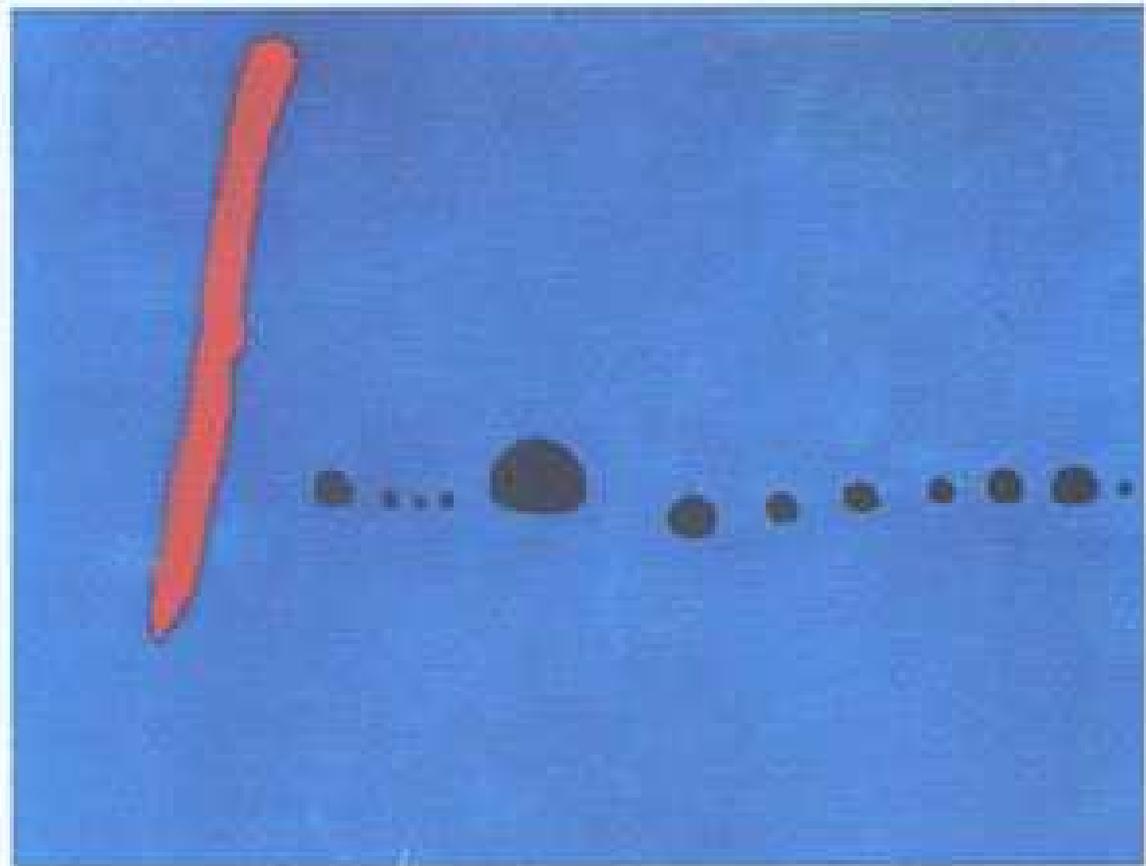


Quantal\_5



JOAN MIRÓ - BLEU II

## Es la condensacion de Bose una transicion de fase?

recordemos

La condicion  $\frac{\lambda^3}{v} > g_{\frac{3}{2}}(1)$  define una region en el diagrama de fases  $P - v - T$  en esta region podemos pensar en la coexistencia de dos fluidos ( $p = 0$  y  $p \neq 0$ )

De esta forma, si fijamos  $v$ , podemos definir una temperatura critica  $T_c \rightarrow$

como  $\lambda = \sqrt{2\pi\hbar^2/mkT} \Rightarrow$

$$\lambda_c^3 = \left[ \sqrt{2\pi\hbar^2/mkT_c} \right]^3 = v g_{\frac{3}{2}}(1)$$

$(g_{3/2}(1) \cong 2)$

luego  $\lambda_c^3$  es del orden del volumen especifico  
entonces

→  $\frac{\langle n_0 \rangle}{N} = 0$  si  $\frac{\lambda^3}{v} \leq g_{3/2}(1)$  (en esta region no hay problemas)

→  $\frac{\langle n_0 \rangle}{N} = \left[ 1 - \left( \frac{T}{T_c} \right)^{3/2} \right] = 1 - \frac{v}{v_c}$  si  $\frac{\lambda^3}{v} \geq g_{3/2}(1)$  (aparece el condensado)

En al region  $\frac{\lambda^3}{v} \leq g_{3/2}(1)$  vale

$$\frac{v_c}{v} = \frac{g_{3/2}(z)}{g_{3/2}(1)}$$

$$\left( \frac{T_c}{T} \right)^{3/2} = \frac{g_{3/2}(z)}{g_{3/2}(1)}$$

$$\frac{\lambda^3}{v_c} = g_{3/2}(1)$$

$$\frac{\lambda^3}{v} = g_{3/2}(z)$$

entonces

En al region  $\frac{\lambda^3}{v} > g_{3/2}(1)$  vale  $z = 1$

$$\frac{P}{kT} = -\frac{4\pi}{h^3} \int_0^\infty dp p^2 \log\left(1 - ze^{-\beta \frac{p^2}{2m}}\right) - \frac{1}{V} \log(1-z)$$

Si  $V \rightarrow \infty \Rightarrow \frac{\log(1-z)}{V} \rightarrow 0$  entonces

$\frac{P}{kT} = \frac{1}{\lambda^3} g_{5/2}(z)$	←	$v > v_c$
$\frac{P}{kT} = \frac{1}{\lambda^3} g_{5/2}(1)$	←	$v < v_c$

$$g_{5/2}(z) = z + \frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} + \dots$$

De donde para  $v < v_c$  la presión  $P$  no depende de  $v$

(infinitamente compresible)

En la otra región, depende de  $v$

recordemos

$$\lambda = \sqrt{2\pi\hbar^2 / m} \left( \frac{1}{kT} \right)^{1/2}$$

Entonces

$$kT = \left[ \frac{2\pi\hbar^2}{m} \right] \frac{1}{\lambda^2}$$

Solo si  $v > v_c$

$$P = \left[ \frac{2\pi\hbar^2}{m} \right]^3 \frac{1}{\lambda^5} g_{5/2}(z)$$

$$\frac{\lambda^3}{v} = g_{3/2}(z) \Rightarrow \lambda^3 = v \cdot g_{3/2}(z)$$

Reemplazando en P

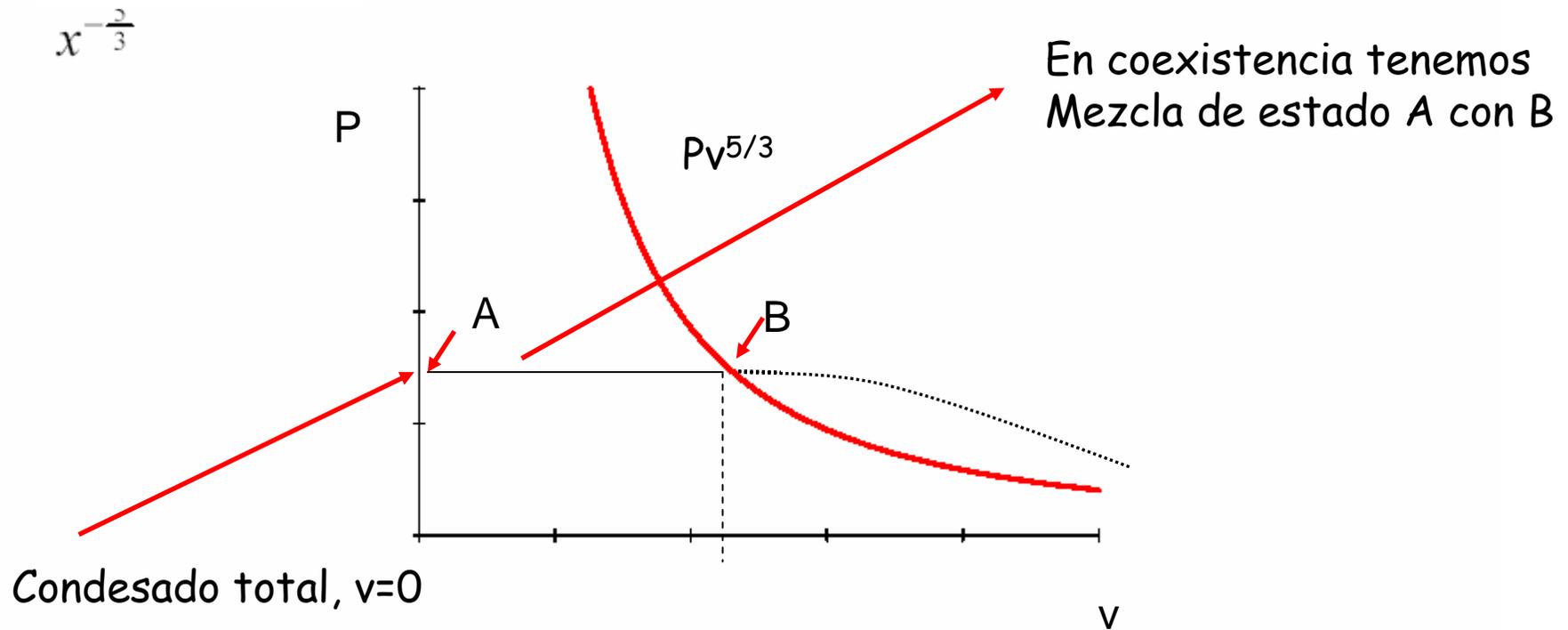
$$P = \left[ \frac{2\pi\hbar^2}{m} \right]^3 \frac{1}{v^{5/3}} \frac{g_{5/2}(z)}{(g_{3/2}(z))^{5/3}}$$

Finalmente

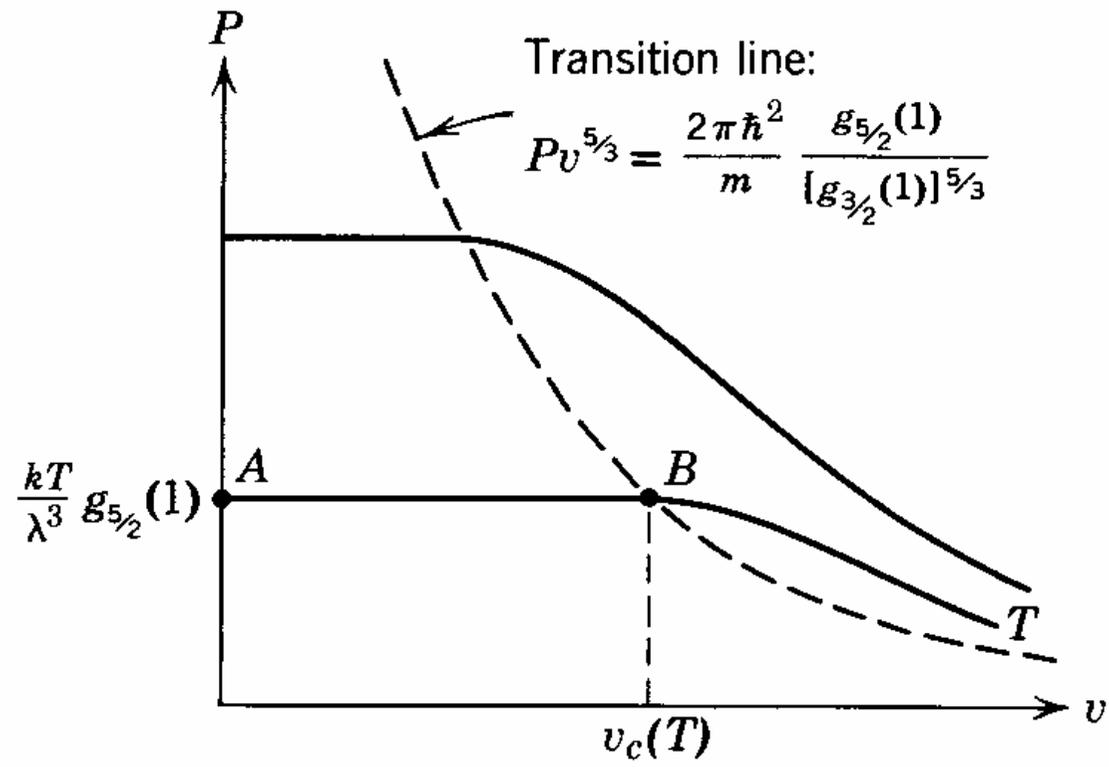
$$Pv^{5/3} = \left[ \frac{2\pi\hbar^2}{m} \right]^3 \frac{g_{5/2}(z)}{(g_{3/2}(z))^{5/3}}$$

Para la curva de transición

$$Pv^{5/3} = \left[ \frac{2\pi\hbar^2}{m} \right]^3 \frac{g_{5/2}(1)}{(g_{3/2}(1))^{5/3}}$$



Dentro de la línea de transición la presión  $P$  es cte y corresponde fases en coexistencia, en  $B$  la fase normal y en  $A$  la fase condensada.

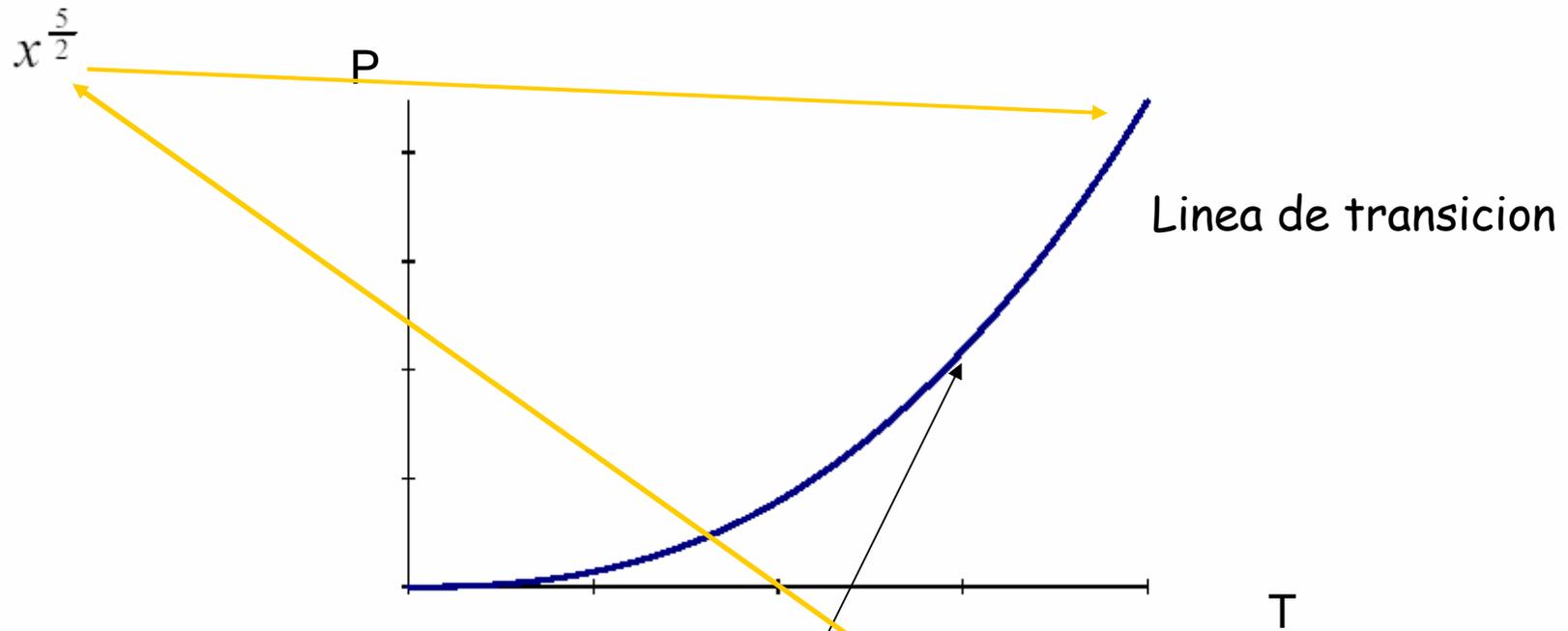


**Fig. 12.7** Isotherms of the ideal Bose gas.

# En el plano PT

La presión de vapor es

$$P_0(T) = \frac{kT}{\lambda^3} g_{5/2}(1)$$



$$P = \left[ \frac{m}{2\pi\hbar^2} \right] g_{5/2}(1) [kT]^{5/2}$$

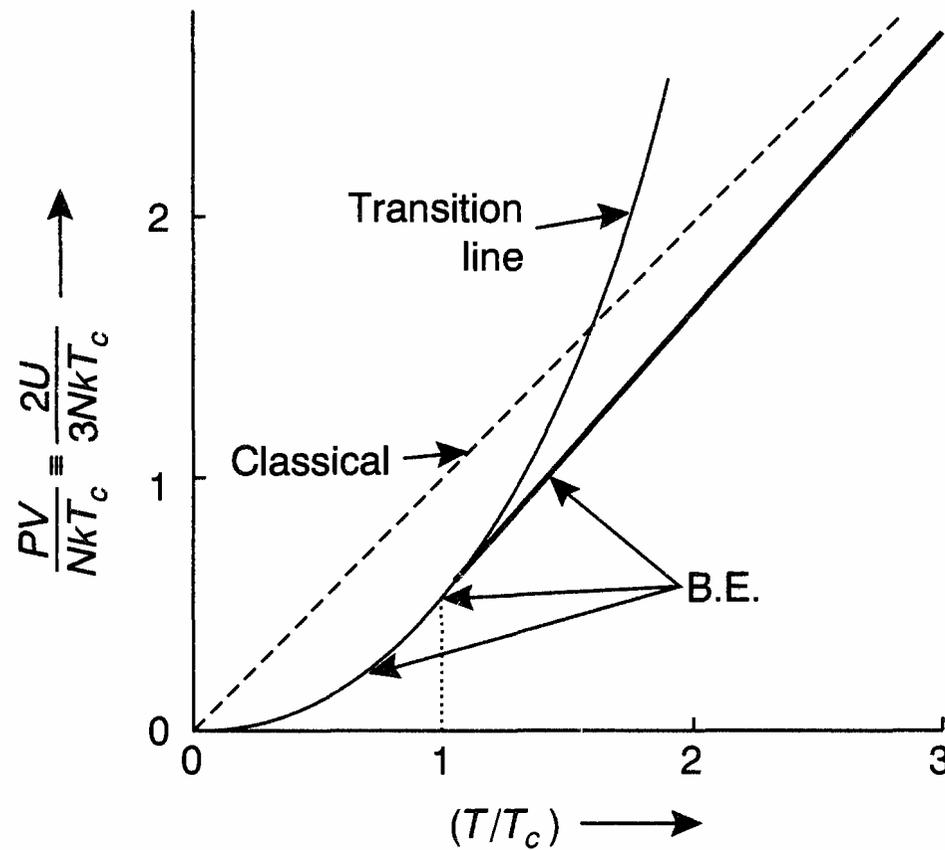


FIG. 7.3. The pressure and the internal energy of an ideal Bose gas as a function of the temperature parameter  $(T/T_c)$ .

Si derivamos esto con respecto de  $T$  con

$$\left( \lambda = \sqrt{2\pi\hbar^2/mkT} \right)$$

$$\begin{aligned} \frac{dP_0(T)}{dT} &= \frac{d}{dT} \frac{kT^{5/2}}{\left( \sqrt{2\pi\hbar^2/mk} \right)^3} g_{5/2}(1) = \frac{5}{2} \frac{k}{\lambda^3} g_{5/2}(1) \\ &= \frac{1}{Tv_c} \left[ \frac{5}{2} kT \frac{g_{5/2}(1)}{g_{3/2}(1)} \right] \end{aligned}$$

El volumen específico de la fase normal es  $v_c$  y el de la condensada es 0 de donde  $\Delta v$  es  $v_c$

**(sobre esta curva exactamente)**

Recordando la ecuación de Clayperon

$$\frac{dP}{dT} = \frac{T\Delta s}{T\Delta v} = \frac{l}{T\Delta v}$$

con  $l$  el calor latente que en este caso es

$$l = \left[ \frac{5}{2} kT \frac{g_{5/2}(1)}{g_{3/2}(1)} \right]$$

Luego es una transición de fase de primer orden

Otras funciones Termodinamicas:

$$l = T\Delta s$$

$v_c$

$$I = T\Delta S$$

$$\frac{S}{Nk} = \frac{U + PV}{NkT} - \frac{\mu}{kT} = \dots$$

$$\text{si } T > T_c \Rightarrow \left( \text{como } P = \frac{2}{3} \frac{U}{V} \right)$$

$$\frac{S}{Nk} = \frac{5}{2} \frac{PV}{NkT} - \frac{\mu}{kT} = \frac{5}{2} \frac{g_{5/2}(z)}{g_{3/2}(z)} - \log(z)$$

$$\text{si } T < T_c \Rightarrow \quad (\log(1)=0)$$

$$\frac{S}{Nk} = \frac{5}{2} \frac{PV}{NkT} - \frac{\mu}{kT} = \frac{5}{2} \frac{v}{\lambda^3} g_{5/2}(1)$$

En este ultimo caso  $\frac{S}{Nk} \propto T^{3/2} \Rightarrow$

$$S \rightarrow 0 \text{ con } T \rightarrow 0$$

## Proceso Adiabático

Tomando en cuenta lo anterior

En  $T > T_c$      $S/N$  constante  $\Rightarrow z = cte$

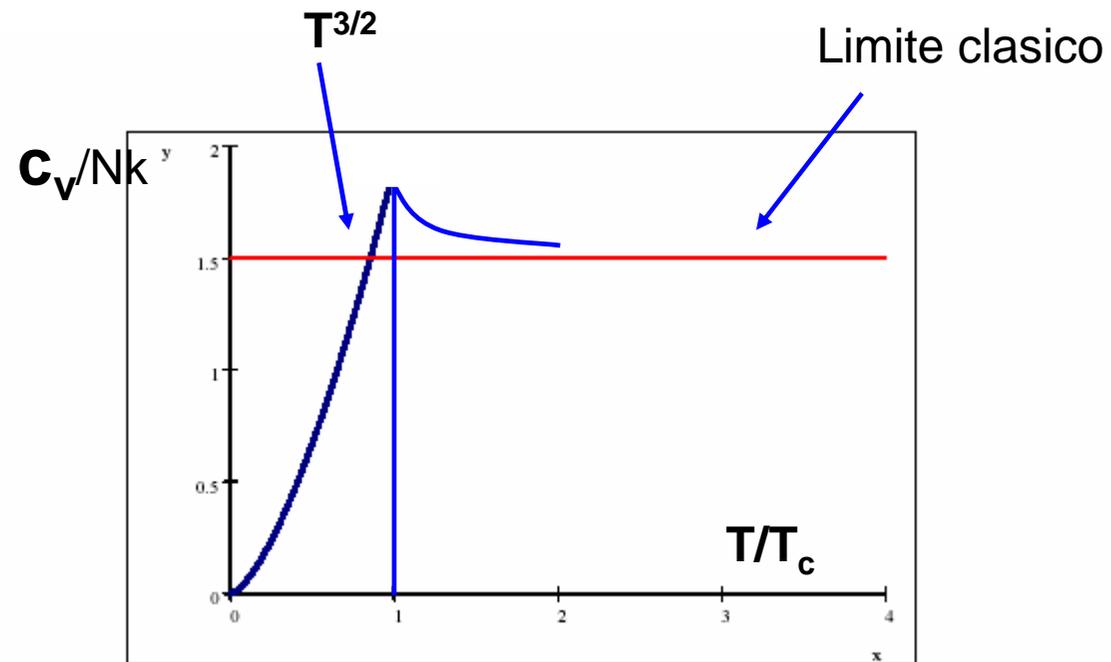
En  $T < T_c$      $S/N$  constante  $\Rightarrow v/\lambda^3$  constante

Luego para ambos casos es la misma relación, de donde

$$vT^{3/2} = cte$$

$$PT^{-5/2} = cte$$

$$Pv^{5/3} = cte$$



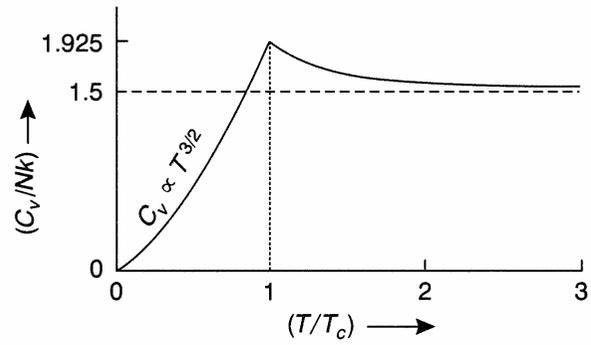


FIG. 7.4. The specific heat of an ideal Bose gas as a function of the temperature parameter  $(T/T_c)$

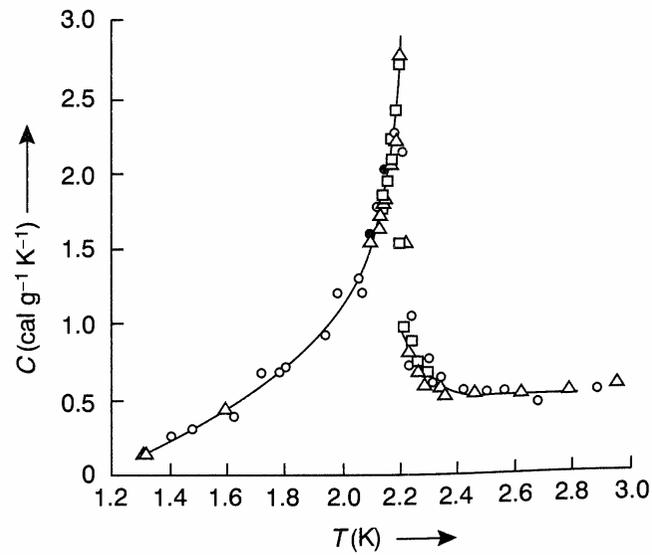


FIG. 7.5. The specific heat of liquid  $\text{He}^4$  under its own vapor pressure (after Keesom and co-workers).

## Resumen de cosas interesantes

Quantal\_5b



# Limite clasico de la Funcion de particion

Sistema :

gas ideal ,  $N$  particulas

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m}; \hat{H} = \sum_i -\frac{\hbar^2}{2m} \nabla_i^2$$

Funciones de onda (condiciones periódicas de contorno)

$$\text{Dado un volumen } V \Rightarrow \mathbf{p}_i = \frac{2\pi\hbar\mathbf{n}}{V^{1/3}}$$

$$\text{con } \mathbf{n} = (n_x, n_y, n_z); n_i = 0, \pm 1, \pm 2, \dots$$

$\Phi_P(1,2,\dots,N)$  ; donde el subindice  $P$  se refiere a las permutaciones

$$\hat{H}\Phi_P(1,2,\dots,N) = \left[ \sum_i \frac{p_i^2}{2m} \right] \Phi_P(1,2,\dots,N)$$

$$\Phi_P(1,2,\dots,N) = \frac{1}{\sqrt{N!}} \sum_P \delta_P [u_{p_1}(P_1)u_{p_2}(P_2)\dots]$$

adecuadamente "simetrizada"

la funcion de onda  $u_{p_1}()$  tendra asociada la particula  $P_1$

$$u_{p_1}(r) = \frac{1}{\sqrt{V}} \exp\left(\frac{ip_1 \cdot r}{\hbar}\right)$$

$$\delta_P = 1 \text{ Bosones}$$

$$\delta_P = -1 \text{ Fermiones con permutacion par}$$

$\delta_P = -1$  Fermiones con permutacion impar

$$\delta_P = [-1]^P$$

Sea el Canonico

$$Q_N(V, T) = \text{Tr} \rho = \sum_n \exp(-\beta E_n)$$

$$\begin{aligned} \text{Tr} \exp(-\beta \hat{H}) &= \frac{1}{N!} \sum_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots} \langle \Phi_{\mathbf{p}} | \exp(-\beta \hat{H}) | \Phi_{\mathbf{p}} \rangle \\ &= \frac{1}{N!} \sum_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots} \langle \Phi_{\mathbf{p}} | \exp(-\beta E_{\mathbf{p}}) | \Phi_{\mathbf{p}} \rangle \\ &= \frac{1}{N!} \sum_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots} \exp(-\beta E_{\mathbf{p}}) \langle \Phi_{\mathbf{p}} | \Phi_{\mathbf{p}} \rangle \end{aligned}$$

$$T_r \exp(-\beta H) = \frac{1}{N!} \sum_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots} \exp(-\beta E_{\mathbf{p}}) \int d^{3N} r |\Phi_{\mathbf{p}}|^2$$

$$\sum_{\mathbf{p}_i} \rightarrow \frac{V}{h^3} \int d^3 p$$

$$T_r \exp(-\beta \hat{H}) = T_r \exp(-\beta K) = \frac{1}{N! h^{3N}} \int d^{3N} r V^N \int d^{3N} p \exp(-\beta E_{\mathbf{p}}) |\Phi_{\mathbf{p}}|^2$$

Veamos primero el  $|\Phi_{\mathbf{p}}|^2$

permutación

Función de onda

$$|\Phi_{\mathbf{p}}|^2 = \frac{1}{\sqrt{N!}} \frac{1}{\sqrt{N!}} \sum_P \delta_P [u_{p_1}^*(P_1) u_{p_2}^*(P_2) \dots] \sum_Q \delta_Q [u_{p_1}(Q_1) u_{p_2}(Q_2) \dots]$$

$$|\Phi_{\mathbf{p}}|^2 = \frac{1}{N!} \sum_{P, Q} \delta_P \delta_Q [u_{p_1}^*(P_1) u_{p_2}^*(P_2) \dots] [u_{p_1}(Q_1) u_{p_2}(Q_2) \dots]$$

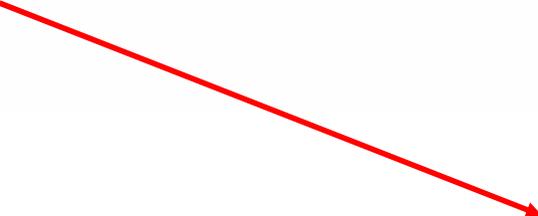
Como hay permutaciones en  $P$  y  $Q \Rightarrow N!$  están repetidas  $\Rightarrow$  multiplicamos por  $N!$  y reemplazamos  $u_{p_1}(Q_1) \leftrightarrow u_{p_1}(1)$

$$|\Phi_{\mathbf{p}}|^2 = \frac{N!}{N!} \sum_P \delta_P [u_{p_1}^*(P_1) u_{p_2}^*(P_2) \dots] [u_{p_1}(1) u_{p_2}(2) \dots]$$

$$|\Phi_{\mathbf{p}}|^2 = \frac{N!}{N!} \sum_P \delta_P [u_{p_1}^*(P_1) u_{p_1}(1)] [u_{p_2}^*(P_2) u_{p_2}(2)] \dots$$

Como

$$u_{p_1}(r) = \frac{1}{\sqrt{V}} \exp\left(\frac{i\mathbf{p}_1 \cdot \mathbf{r}}{\hbar}\right)$$

Entonces 

$$|\Phi_{\mathbf{p}}|^2 = \frac{1}{V^N} \exp[ip_1 \cdot (\mathbf{r}_1 - P\mathbf{r}_1) + ip_2 \cdot (\mathbf{r}_2 - P\mathbf{r}_2) + \dots]$$

Si fuesen 2 partículas consideraríamos  $e^{ik_1r_1}e^{ik_2r_2} - e^{ik_1r_2}e^{ik_2r_1}$

Habiendo calculado  $|\Phi_P|^2$

$$T_r \exp(-\beta \hat{H}) = \frac{1}{N!h^{3N}} \int d^{3N}r \int d^{3N}p \sum_P \delta_P \left\{ \prod_{j=1}^N e^{-\beta(\frac{p_j^2}{2m}) + ip_j \cdot (\mathbf{r}_j - P\mathbf{r}_j)/\hbar} \right\}$$

Multiplico y divido por  $\left[ \int d^3p \exp(-\beta \mathbf{p}^2/2m) \right]^N$

$$T_r \exp(-\beta \hat{H}) = \frac{1}{N!h^{3N}} \int d^{3N}r \left[ \int d^3p \exp(-\beta \mathbf{p}^2/2m) \right]^N * \\ * \sum_P \delta_P \left\{ \prod_{j=1}^N \left[ \frac{\int d^3p \exp\left(-\beta\left(\frac{p_j^2}{2m}\right) + ip_j \cdot (\mathbf{r}_j - P\mathbf{r}_j)/\hbar\right)}{\int d^3p \exp(-\beta \mathbf{p}^2/2m)} \right] \right\}$$

$$T_r \exp(-\beta \hat{H}) = \frac{1}{N! h^{3N}} \int d^{3N} r V^N \left[ \int d^3 p \exp(-\beta \mathbf{p}^2 / 2m) \right]^N * \\ * \sum_P \delta_P \left\{ \prod_{j=1}^N \left[ \frac{\int d^3 p \exp\left(-\beta \left(\frac{p_j^2}{2m}\right) + i \mathbf{p}_j \cdot (\mathbf{r}_j - P \mathbf{r}_j) / \hbar\right)}{\int d^3 p \exp(-\beta \mathbf{p}^2 / 2m)} \right] \right\}$$

La integral es de la forma

$$\frac{1}{h^3} \int d^3 p \exp\left(-\beta \left(\frac{\mathbf{p}^2}{2m}\right) - i \mathbf{p} \cdot \mathbf{r} / \hbar\right) =$$

con  $\mathbf{r} = (\mathbf{r}_j - P \mathbf{r}_j)$

ademas  $kT = \mathbf{p}_T^2 / 2m$  entonces queda

$$= \frac{1}{h^3} \int d^3 p \exp\left(-\left(\frac{\mathbf{p}^2}{\mathbf{p}_T^2}\right) - i \mathbf{p} \cdot \mathbf{r} / \hbar\right)$$

este es el numerador del termino de la derecha

$T_r \exp(-\beta \hat{H})$

Ahora lo integramos

$$= \frac{1}{h^3} \int d^3p \exp\left(-\left(\frac{p^2}{p_T^2}\right) - ip \cdot r/\hbar\right)$$

veamos que

$$\int d^3p \exp(-ap^2) \exp(-ibp)$$

$$\int dp_x \exp(-ap_x^2) \exp(-ibp_x) ; a = \frac{1}{p_T^2} ; b = \frac{r_x}{\hbar}$$

$$\int dp_x \exp(-ap_x^2) (\cos(bp_x) - i \sin(bp_x))$$

$$\int dp_x \exp(-ap_x^2) (i \sin(bp_x)) = 0$$

$$\int dp_x \exp(-ap_x^2) (\cos(bp_x)) = \frac{\sqrt{\pi}}{\sqrt{a}} \exp\left(-\frac{b^2}{4a}\right) =$$

$$T_r \exp(-\beta \hat{H}) = \frac{1}{N! h^{3N}} \int d^{3N} r V^N \left[ \int d^3 p \exp(-\beta p^2/2m) \right]^N * \\ * \sum_P \delta_P \left\{ \prod_{j=1}^N \left[ \frac{\int d^3 p \exp\left(-\beta \left(\frac{p_j^2}{2m}\right) + i p_j \cdot (r_j - P r_j)/\hbar\right)}{\int d^3 p \exp(-\beta p^2/2m)} \right] \right\}$$

(para x)

entonces resulta

$$= \frac{\sqrt{\pi}}{\sqrt{\frac{1}{p_T^2}}} \exp\left(\frac{r_x^2}{\hbar^2} \frac{p_T^2}{4}\right) = \frac{\sqrt{\pi}}{2\pi\hbar} \sqrt{2mkT} \exp\left(\frac{r_x^2}{\hbar^2} \frac{p_T^2}{4}\right) =$$

como  $\lambda = \sqrt{\frac{2\pi\hbar^2}{mkT}}$   $\Rightarrow$

$$= \exp\left(-\pi \frac{r^2}{\lambda^2}\right) \lambda^3$$

$$T_r \exp(-\beta\hat{H}) = \frac{1}{N!h^{3N}} \int d^{3N}r V^N \left[ \int d^3p \exp(-\beta\mathbf{p}^2/2m) \right]^N * \\ * \sum_P \delta_P \left\{ \prod_{j=1}^N \left[ \frac{\int d^3p \exp\left(-\beta\left(\frac{p_j^2}{2m}\right) + i\mathbf{p}_j \cdot (\mathbf{r}_j - P\mathbf{r}_j)/\hbar\right)}{\int d^3p \exp(-\beta\mathbf{p}^2/2m)} \right] \right\}$$

Este ultimo  $\lambda^3$  se simplifica con  $\int d^3p \exp(-\beta\mathbf{p}^2/2m)$  en el denominador

$$= \exp\left(-\pi \frac{r^2}{\lambda^2}\right) = f(\mathbf{r}_i - P\mathbf{r}_i)$$

Queda entonces

$$T_r \exp(-\beta \hat{H}) = \frac{1}{N! h^{3N}} \int d^{3N} r d^{3N} p \exp(-\beta \sum \frac{p_i^2}{2m}) \sum_P \delta_P \prod f(\mathbf{r}_i - P\mathbf{r})$$

$$= \left\{ \frac{1}{N! h^{3N}} \int d^{3N} p \exp(-\beta \sum \frac{p_i^2}{2m}) \right\} \left\{ \int d^{3N} r \sum_P \delta_P \prod f(\mathbf{r}_i - P\mathbf{r}_i) \right\}$$

donde

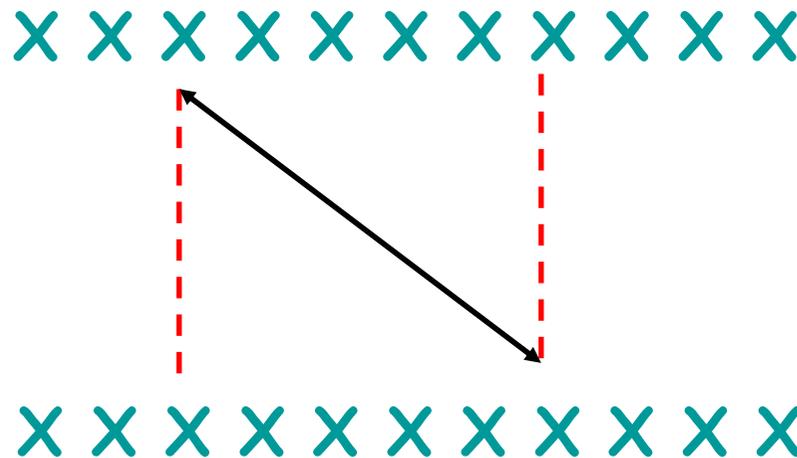
$$Q_N = \left\{ \frac{1}{N! h^{3N}} \int d^{3N} p \exp(-\beta \sum \frac{p_i^2}{2m}) \right\}$$

Estudiamos ahora  $\left\{ \int d^{3N} r \sum_P \delta_P \prod f(\mathbf{r}_i - P\mathbf{r}_i) \right\}$  (que multiplica a  $Q_N$ )

a) "No hay permutaciones"  $\Rightarrow$  dilucion " $\infty$ "  $\Rightarrow f(0) = 1$

b) "una permutacion"  $\rightarrow$  permutamos 2 indices

$$\prod [f(\mathbf{r}_i - P\mathbf{r}_i)] = [f(r_b - r_a)]^2 * \prod_{n=1}^{N-2} f(\mathbf{0}) = [f(r_b - r_a)]^2 = f_{ij}f_{ji}$$

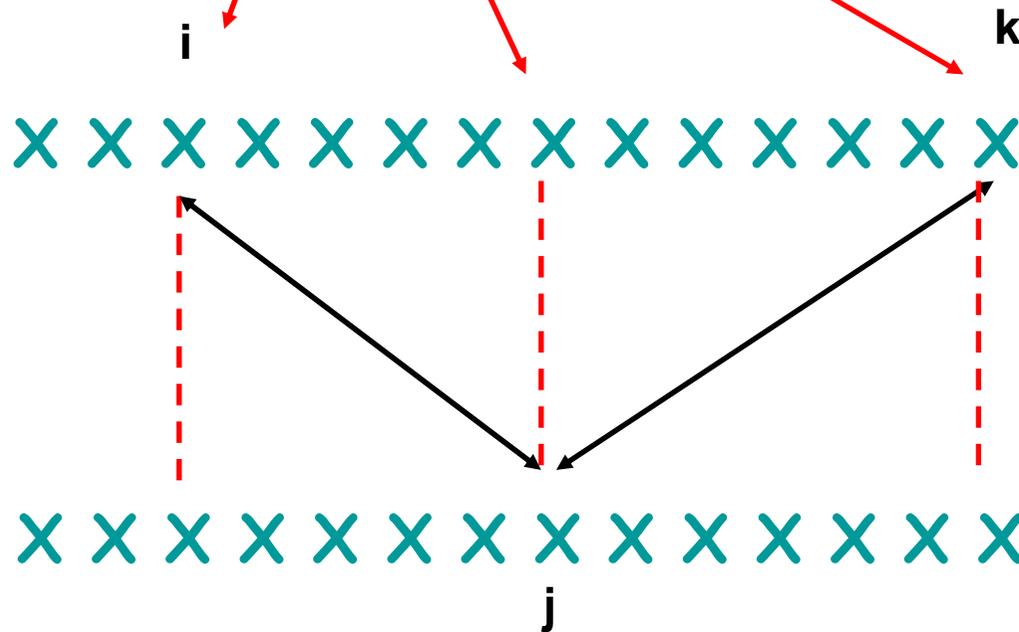


1 permutación

c) permutamos 3 indices

$$\prod f(\mathbf{r}_i - P\mathbf{r}_i) = f_{ij}f_{ik}f_{kj}$$

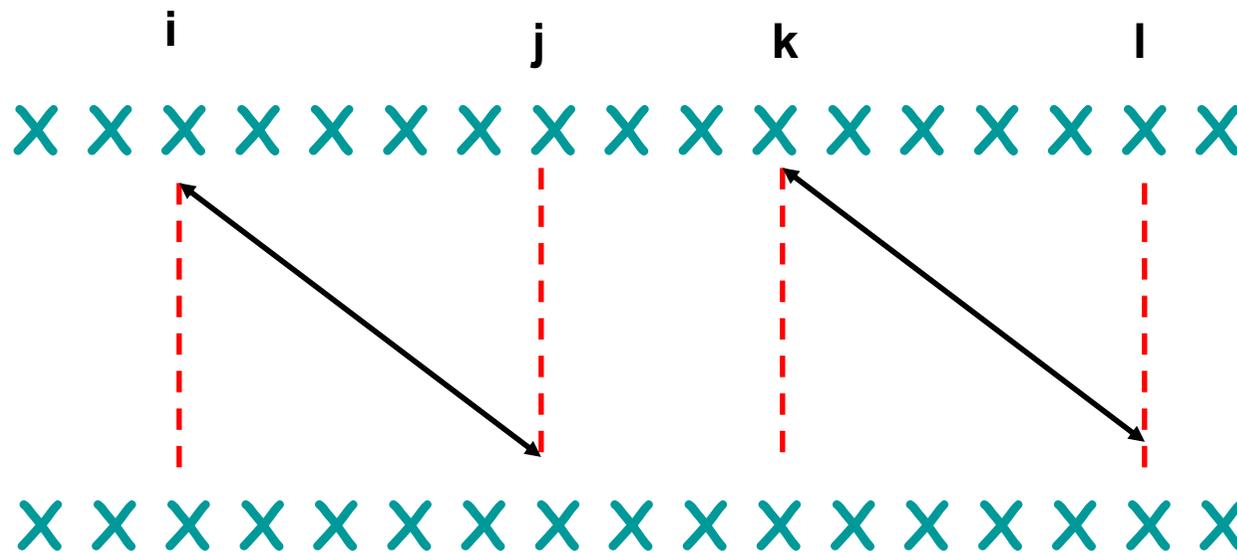
2 permutaciones



d) 4 indices "consecutivos de a pares"

$$\prod f(\mathbf{r}_i - P\mathbf{r}_i) = f_{ij}^2 f_{kl}^2$$

2 permutaciones



2 permutaciones

De esta forma

$$\prod f(\mathbf{r}_i - P\mathbf{r}_i) = 1 \pm \sum_{i < j} f_{ij}^2 + \sum_{i < j < k} f_{ij} f_{ik} f_{kj} + \sum_{i < j; k < l} f_{ij}^2 f_{kl}^2 \pm \dots$$

1 permutación      2 permutaciones

donde

$+\Rightarrow$  *bosones*

$-\Rightarrow$  *fermiones*

Como depende todo esto de la Temperatura ?

( va como  $r^2T$  )

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{mkT}} ; f = \exp(-\pi r^2/\lambda^2) \Rightarrow$$

$$T \rightarrow \infty \Rightarrow \lambda \rightarrow 0 \Rightarrow f(r) \rightarrow 0$$

$$T \rightarrow 0 \Rightarrow \lambda \rightarrow \infty \Rightarrow f(r) \rightarrow 1$$

de donde si  $T$  es alta y el sistema es diluido solo sobrevive el termino de orden mas bajo es decir 0 permutacion

$$\prod f(\mathbf{r}_i - P\mathbf{r}_i) = 1$$

entonces a orden 0 recuperamos lo clasico

$$T_r \exp(-\beta\hat{H}) = Q_N$$

Si la dilucion no es tan extrema

$$\prod f(\mathbf{r}_i - P\mathbf{r}_i) = 1 \pm \sum_{i < j} f_{ij}^2 \approx \prod_{i < j} [1 \pm f_{ij}^2] \approx \exp(-\beta \sum_{i < j} v_{ij}^{\pm})$$

con  $v_{ij}^{\pm}$  "interaccion efectiva"

$$v_{ij}^{\pm} = -kT \log[1 \pm f_{ij}^2] = -kT \log \left[ 1 \pm \exp\left(\frac{-2\pi|r_i - r_j|^2}{\lambda^2}\right) \right]$$

Si tomamos  $|r_i - r_j| \sim 0$

Para bosones

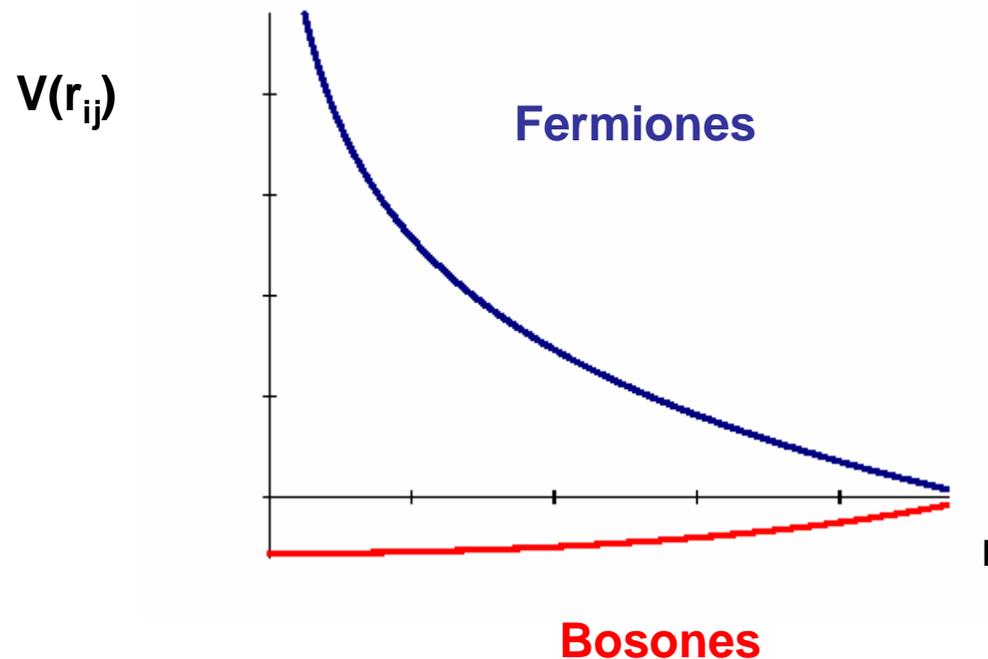
$$v_{ij}^B = -kT \log \left[ 1 + \left\{ 1 - \frac{2\pi|r_i-r_j|^2}{\lambda^2} \right\} \dots \right] \xrightarrow{|r_i-r_j| \rightarrow 0} kT \log(2)$$

---

Para fermiones

$$v_{ij}^F = -kT \log \left[ \frac{2\pi|r_i-r_j|^2}{\lambda^2} \dots \right] \xrightarrow{|r_i-r_j| \rightarrow 0} -kT \log(0)$$

---



Luego en este limite

Fermiones se comportan como si hubiese un potencial de interaccion repulsivo

Bosones se comportan como si hubiese un potencial de interaccion atractivo

Pero las particulas son no interactuantes

