FÍSICA TEÓRICA 3

Liliana Arrachea

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GASES CUÁNTICOS DE FERMIONES

REPASO

N partículas no interactuantes con spin semientero

$$\hat{H} = \sum_{l=1}^{N} \hat{H}_l$$

identifica partícula

Cada partícula está descripta por el mismo Hamiltoniano de "partícula independiente" (PI):

$$\hat{H}^{(PI)}|\alpha_j\rangle = \epsilon_{\alpha_j}|\alpha_j\rangle$$

identifica estado de PI

Las N partículas comparten los números cuánticos α que identifican a los estados de "partícula independiente".

Estados de N partículas en la representación nro de ocupación

$$|\ldots, n_{\alpha_{j-1}}, n_{\alpha_j}, n_{\alpha_{j+1}}, \ldots\rangle$$

ocupación de estado
j de PI

Antisimetría de la función de onda de N partículas ➡ Ppio de exclusión de Pauli ➡ $n_{\alpha_j} = 0, 1$

$$a^\dagger_{\alpha_j}|\ldots, 0, \ldots\rangle = |\ldots, 1, \ldots\rangle$$
$$a_{\alpha_j}|\ldots, 1, \ldots\rangle = |\ldots, 0, \ldots\rangle$$

En la representación de 2da cuantización

$$\hat{H} = \sum_j \epsilon_{\alpha_j} a^\dagger_{\alpha_j} a_{\alpha_j} \equiv \sum_\alpha \epsilon_\alpha a^\dagger_\alpha a_\alpha \qquad \hat{N} = \sum_\alpha a^\dagger_\alpha a_\alpha$$

Antisimetría de la función de onda de las N partículas garantizada por:

$$\{a_\alpha, a^\dagger_{\alpha'}\} = \delta_{\alpha,\alpha'}, \quad \{a_\alpha, a_{\alpha'}\} = \{a^\dagger_\alpha, a^\dagger_{\alpha'}\} = 0$$

El conjunto gran canónico es el más cómodo para estudiar estos sistemas.

$$\hat{\rho} = \frac{e^{-\beta(\hat{H}-\mu\hat{N})}}{Z} \qquad Z = \mathrm{Tr}[e^{\beta(\hat{H}-\mu\hat{N})}]$$

Satisface

$$\max\{S = -k_B \mathrm{Tr}[\hat{\rho}\ln\hat{\rho}]\} \qquad U = \mathrm{Tr}[\hat{\rho}\hat{H}], \quad N = \mathrm{Tr}[\hat{\rho}\hat{N}]$$

Calculando Tr[..] respecto de los estados de N partículas:

$$Z = \prod_{\alpha} (1 + z e^{-\beta \epsilon_\alpha}), \qquad z = e^{\beta \mu}$$

$$\Omega(T, \mu, V) = -k_B T \sum_{\alpha} \ln(1 + z e^{-\beta \epsilon_\alpha})$$

$$U = -\frac{\partial \ln Z}{\partial \beta} = \sum_{\alpha} \epsilon_\alpha n_F(\epsilon_\alpha) = \int_{-\infty}^{\infty} d\epsilon \mathcal{D}(\epsilon)\, \epsilon\, n_F(\epsilon_\alpha)$$

$$N = z \frac{\partial \ln Z}{\partial z} = \sum_{\alpha} n_F(\epsilon_\alpha) = \int_{-\infty}^{\infty} d\epsilon \mathcal{D}(\epsilon) n_F(\epsilon)$$

Función de Fermi-Dirac

$$n_F(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} + 1}$$

Densidad de estados

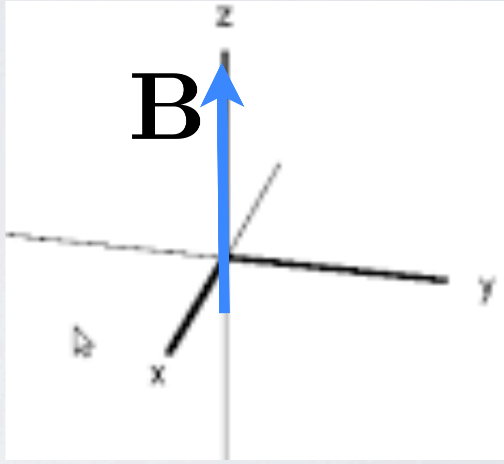
$$\mathcal{D}(\epsilon) = \sum_{\alpha} \delta(\epsilon - \epsilon_\alpha)$$

GAS DE ELECTRONES EN CAMPO MAGNÉTICO EXTERNO

PARAMAGNETISMO DE PAULI

Hamiltoniano de PI, volúmen V=LxLxL

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} - \mu_B \hat{\sigma} \cdot \mathbf{B} = \frac{\hat{\mathbf{p}}^2}{2m} - \mu_B \hat{\sigma_z} B$$



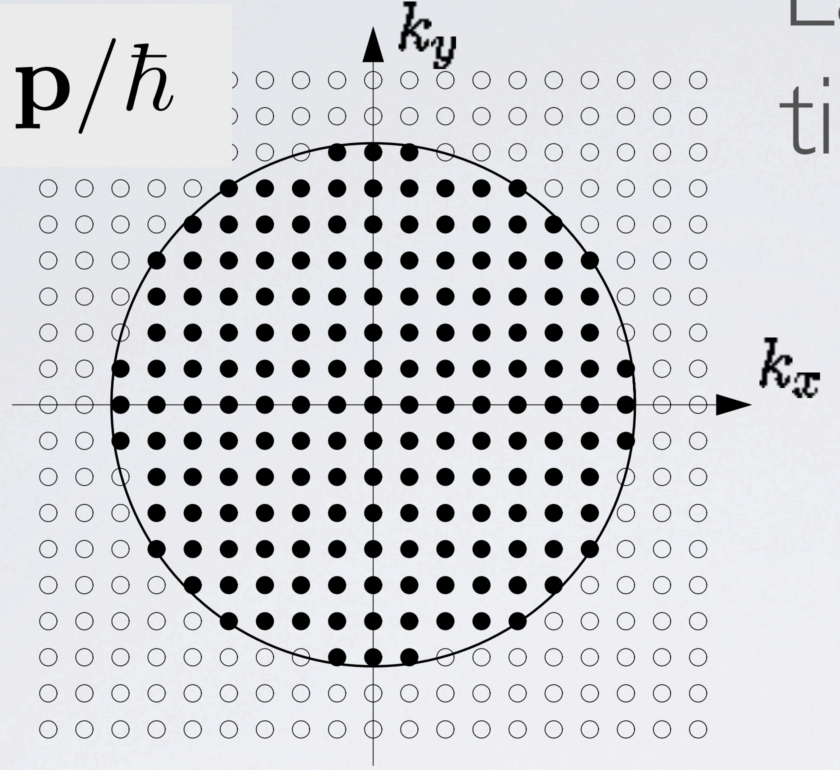
$$\hat{H}|\mathbf{p}, \sigma\rangle = \epsilon_{\mathbf{p},\sigma}|\mathbf{p}, \sigma\rangle, \quad \sigma = \uparrow, \downarrow$$

$$|\mathbf{p}, \sigma\rangle, \quad \to \psi_{\mathbf{p},\sigma}(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}} \chi_\sigma$$

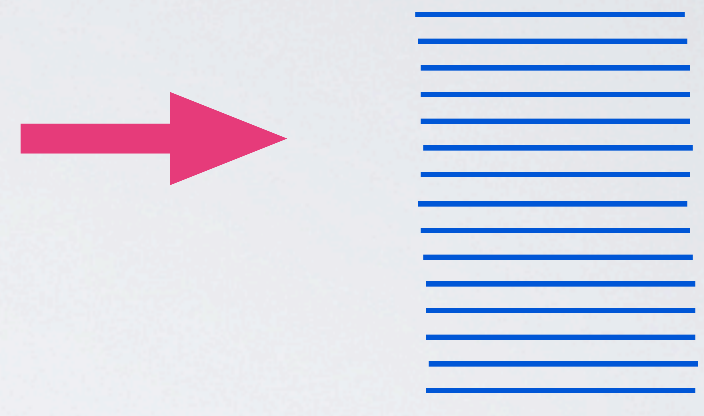
$$\epsilon_{\mathbf{p},\sigma} = \frac{p^2}{2m} - s_\sigma \mu_B B, \qquad s_{\uparrow,\downarrow} = \pm, \qquad \mathbf{p} = \frac{2\pi\hbar}{L}(n_x, n_y, n_z)$$

B=0

$$\mathbf{k} = \mathbf{p}/\hbar$$



Las dos especies de spin tienen = relación dispersión



$$\epsilon_{\mathbf{p}} = \frac{p^2}{2m}$$

B≠0

$\sigma = \downarrow$ $\sigma = \uparrow$

$$\epsilon_{\mathbf{p}\downarrow} = \frac{p^2}{2m} + \mu_B B$$

$$\epsilon_{\mathbf{p}\uparrow} = \frac{p^2}{2m} - \mu_B B$$

$$N = N_\uparrow + N_\downarrow$$ nro total de partículas

$$M = \mu_B(N_\uparrow - N_\downarrow)$$ momento magnético total

$$N_\sigma = \sum_{\mathbf{p}} n_F(\epsilon_{\mathbf{p},\sigma}) = \sum_{\mathbf{p}} n_F(\epsilon_{\mathbf{p}} + s_\sigma \mu_B B) = \sum_{\mathbf{p}} \frac{1}{e^{\beta(\epsilon_{\mathbf{p}} + s_\sigma \mu_B B - \mu)} + 1}$$

$$= \sum_{\mathbf{p}} \frac{1}{e^{\beta(\epsilon_{\mathbf{p}} - \mu_\sigma)} + 1}, \qquad \mu_\sigma = \mu - s_\sigma \mu_B B$$

En términos de la densidad de estados:

$$N_\sigma = \int d\epsilon \mathcal{D}(\epsilon) \frac{1}{e^{\beta(\epsilon - \mu_\sigma)} + 1},$$

$$\mathcal{D}(\epsilon) = \sum_{\mathbf{p}} \delta(\epsilon - \epsilon_{\mathbf{p}}) \sim \frac{V}{h^3} \int d^3\mathbf{p}\, \delta(\epsilon - \epsilon_{\mathbf{p}}) = \frac{V}{h^3} 2\pi (2m)^{2/3} \sqrt{\epsilon}$$

Reemplazando:

$$N_\sigma = \frac{V}{\lambda^3} f_{3/2}(z_\sigma), \qquad z_\sigma = e^{\beta\mu_\sigma} = e^{\beta(\mu - s_\sigma \mu_B B)}$$

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}}{z^{-1}e^x + 1}, \qquad \lambda = \sqrt{\frac{2\pi\hbar^2}{mk_B T}}$$

*Altas temperaturas $z_\sigma \ll 1$ $\qquad f_\nu(z_\sigma) \sim z_\sigma$

$$n = \frac{N}{V} = n_\uparrow + n_\downarrow \sim \frac{e^{\beta\mu}}{\lambda^3}(e^{\beta\mu_B B} + e^{-\beta\mu_B B}) \sim \frac{e^{\beta\mu}}{\lambda^3} 2$$

$$m = \frac{M}{V} = n_\uparrow - n_\downarrow \sim \frac{e^{\beta\mu}}{\lambda^3}(e^{\beta\mu_B B} - e^{-\beta\mu_B B}) \sim \frac{e^{\beta\mu}}{\lambda^3} 2\beta\mu_B^2 B$$

$$\chi(T) = \frac{\partial M}{\partial B}|_{B=0} = \frac{2e^{\beta\mu}}{\lambda^3}\beta\mu_B^2 \sim N\frac{\mu_B^2}{k_B T}$$ Ley de Curie

*Bajas temperaturas: $z_\sigma \gg 1 \longrightarrow$ Sommerfeld

$$N_\sigma \sim \int_0^{\mu_\sigma} d\epsilon \mathcal{D}(\epsilon) + \frac{\pi^2}{6}(k_B T)^2 \mathcal{D}'(\mu_\sigma)$$

$$n_\sigma \sim \frac{4\pi}{3}(\frac{2m}{h^2})^{3/2}(k_B T \ln z_\sigma)^{3/2}\{1 + \frac{\pi^2}{8}(\ln z_\sigma^{-2}\}$$

$$\mu_\sigma = \mu - s_\sigma \mu_B B = k_B T \ln z_\sigma \sim \epsilon_{F,\sigma}[1 - \frac{\pi^2}{12}(\frac{k_B T}{\epsilon_{F\sigma}})^2]$$

$$\epsilon_{F\sigma} = \frac{h^2}{2m}(\frac{3}{4\pi})^{2/3} n_\sigma^{2/3}$$

Energía de Fermi de un sistema con densidad de fermiones n_σ

Resultado $\mu_B B \ll \epsilon_F$ (Energía de Fermi a B=0)

$$m(B,T) \sim \frac{3n\mu_B B}{2\epsilon_F}[1 - \frac{\pi^2}{12}(\frac{k_B T}{\epsilon_F})^2]$$

$$\chi(T) \frac{\partial M}{\partial B}|_{B=0} =\sim \frac{3N\mu_B}{2\epsilon_F}[1 - \frac{\pi^2}{12}(\frac{k_B T}{\epsilon_F})^2]$$

FERMIONES RELATIVISTAS

Tratamiento "sui generis"

$$\epsilon_{\mathbf{p}} = mc^2[\sqrt{1 + (\frac{p}{mc})^2} - 1]$$

$$\begin{aligned}\ln Z &= g_s \frac{4\pi V}{h^3} \int_0^\infty dp p^2 \ln(1 + z e^{-\beta\epsilon}) \\ &= g_s \frac{4\pi V \beta}{3h^3} \int_0^\infty dp p^3 \frac{d\epsilon}{dp} \frac{1}{z^{-1} e^{\beta\epsilon} + 1}\end{aligned}$$

por partes

T=0

$$P = k_B T \ln Z = g_s \frac{4\pi V}{3h^3} \int_0^{p_F} dp p^3 \frac{d\epsilon}{dp},$$

$$N = g_s \frac{4\pi V}{3h^3} \int_0^{p_F} dp p^2 = g_s \frac{4\pi V p_F^3}{3h^3},$$

$$p_F = (\frac{3}{4\pi} \frac{N h^3}{V g_s})^{1/3}$$

Momento de Fermi = gas no relativista

$$U = g_s \frac{4\pi V}{3h^3} \int_0^{p_F} dp p^2 \epsilon_p$$

Límites para T=0

$$P = \frac{g_s \pi m^4 c^5}{6h^3} A(y_f), \qquad U = \frac{g_s \pi m^4 c^5}{6h^3} B(y_f)$$

$$p_f = mc \, \sinh x_f, \qquad y_f = \frac{p_f}{mc}$$

* No relativista: $y_f \ll 1$

$$A(y) \sim \frac{8}{5} y^5, \qquad B(y) \sim \frac{12}{5} y^5$$

*Ultrarelativista $y_f \gg 1$

$$A(y) \sim 2y^4, \qquad B(y) \sim 6y^4$$

Resultados

* No relativista:

$$P = \frac{2}{3}\frac{U}{V}$$

$$U = \frac{2\pi g_s V}{5h^3 m} p_F^5$$

* Ultrarelativista:

$$P = \frac{1}{3}\frac{U}{V}$$

$$U = \frac{\pi g_s V}{h^3 m} p_F^4$$

Enanas blancas

Modelo: esfera de He ionizado

$$M \sim 10^{30} kg \qquad \rho \sim 10^{10} kg m^{-3} \qquad T \sim 10^{7} K$$

Densidad de electrones

$$n = 3 \times 10^{36} \frac{\text{electrones}}{m^3}$$

$$p_f \sim 0.9 \frac{MeV}{c}$$ Relativista

$$\epsilon_F \sim 0.5 MeV, \qquad k_B T \sim 1 keV \ll \epsilon_F$$ T aprox 0 :)