1. (a) $\log Z_{GC} = - \sum_{i} \log\left(1 - z e^{-\beta \epsilon_i}\right)$

$\uparrow$ estados monopart.

$$= - \sum_{s=-1}^{1} \int \frac{d^3q \, d^3p}{h^3} \log\left[1 - z e^{-\beta\left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 - \mu_m B s\right)}\right]$$

$$= - \sum_{s=-1}^{1} \int \frac{d^3q \, d^3p}{h^3} \log\left[1 - z_s e^{-\beta\left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2\right)}\right]$$

$\uparrow$ $z_s \equiv z e^{\beta \mu_m B s}$

$$= - \left(\frac{2}{\cancel{m}\omega^2}\right)^{3/2} (2\cancel{m})^{3/2} \sum_{s=-1}^{1} \int d^3Q \, d^3P \log\left[1 - z_s e^{-\beta\left(P^2 + Q^2\right)}\right]$$

$\uparrow$

$\vec{Q} \equiv \sqrt{\frac{m\omega^2}{2}} \, \vec{q}$

$\vec{P} \equiv \frac{\vec{p}}{\sqrt{2m}}$

$$= - \left(\frac{2}{h\omega}\right)^3 \pi^3 \sum_{s=-1}^{1} \int_0^\infty dr \, r^5 \log\left(1 - z_s e^{-\beta r^2}\right)$$

$\uparrow$ esféricas

$\uparrow$ área de la esfera de radio 1

$$= - \left(\frac{1}{\hbar\omega}\right)^3 \frac{1}{\beta^3} \frac{1}{2} \sum_{s=-1}^{1} \int_0^\infty dx \, x^2 \log\left(1 - z_s e^{-x}\right)$$

$\uparrow$

$\beta r^2 = x$

$\Longrightarrow 2r\,dr = \frac{dx}{\beta}$

$$= - \left(\frac{kT}{\hbar\omega}\right)^3 \frac{1}{2} \sum_{s=-1}^{1} \left[\cancel{\frac{x^3}{3} \log\left(1 - z_s e^{-x}\right) \Big|_0^\infty} - \frac{1}{3} \int_0^\infty dx \, \frac{x^3}{1 - z_s e^{-x}} z_s e^{-x} \right]$$

$\uparrow$ Por partes

$$\Longrightarrow \quad \log Z_{GC} = \left(\frac{kT}{\hbar\omega}\right)^3 \sum_{s=-1}^{1} \underbrace{\frac{1}{6}\int_0^\infty dx\, \frac{x^3}{z_s^{-1}e^x - 1}}_{= g_4(z_s)}$$

Estado fundamental $\epsilon_0 = -\mu_m B$. Agregando la contribución de este estado,

$$\boxed{\log Z_{GC} = -\log\left(1 - z e^{\beta\mu_m B}\right) + \left(\frac{kT}{\hbar\omega}\right)^3 \sum_{s=-1}^{1} g_4\left(z e^{\beta\mu_m B s}\right)}$$

$$N = z\frac{\partial}{\partial z}\log Z_{GC} = \frac{1}{z^{-1}e^{-\beta\mu_m B} - 1} + \left(\frac{kT}{\hbar\omega}\right)^3 \sum_{s=-1}^{1} z\, g_4'\left(z e^{\beta\mu_m B s}\right) e^{\beta\mu_m B s}$$

$$= \boxed{\frac{1}{z^{-1}e^{-\beta\mu_m B} - 1} + \left(\frac{kT}{\hbar\omega}\right)^3 \sum_{s=-1}^{1} g_3\left(z e^{\beta\mu_m B s}\right) = N}$$

(b) $\mu_{max} = \epsilon_0 = -\mu_m B \Longrightarrow z_{max} = e^{-\beta\mu_m B}$

$T = T_c \Longleftrightarrow N = N_{exc}$ y $z = z_{max} = e^{-\beta_c\mu_m B}$

$$\Longrightarrow N = \left(\frac{kT_c}{\hbar\omega}\right)^3 \left[g_3(1) + g_3\left(e^{-\beta_c\mu_m B}\right) + g_3\left(e^{-2\beta_c\mu_m B}\right) \right]$$

$$B \to 0 \implies N = 3\,\zeta(3)\left(\frac{kT_c}{\hbar\omega}\right)^3$$

$$\implies \boxed{T_c = \frac{\hbar\omega}{k}\left[\frac{N}{3\,\zeta(3)}\right]^{1/3}}$$

$$B \to \infty \implies N = \left(\frac{kT_c}{\hbar\omega}\right)^3 \zeta(3)$$

$$\implies \boxed{T_c = \frac{\hbar\omega}{k}\left[\frac{N}{\zeta(3)}\right]^{1/3}}$$

(c) $$M = \frac{1}{N\beta}\frac{\partial}{\partial B}\log Z_{GC} = \frac{1}{N\beta}\left\{\beta\mu_m \frac{1}{z^{-1}e^{-\beta\mu_m B}-1}\right.$$

$$\left. + \left(\frac{kT}{\hbar\omega}\right)^3 \sum_{s=-1}^{1} g_4'\left(ze^{\beta\mu_m Bs}\right)\beta\mu_m s\, ze^{\beta\mu_m Bs}\right\}$$

$$= \frac{1}{N\cancel{\beta}}\cancel{\beta}\mu_m\left\{\underset{\substack{\uparrow\\ \text{Núm part.}\\ \text{en el fund.}}}{N_0} + \left(\frac{kT}{\hbar\omega}\right)^3 \sum_{s=-1}^{1} g_3\left(ze^{\beta\mu_m Bs}\right)s\right\}$$

$$\implies M = \mu_m\left\{\underset{\substack{\uparrow\\ \text{fracción de}\\ \text{part. en el fund.}}}{f_0} + \frac{1}{N}\left(\frac{kT}{\hbar\omega}\right)^3\left[g_3\left(ze^{\beta\mu_m B}\right) - g_3\left(ze^{-\beta\mu_m B}\right)\right]\right\}$$

$$T < T_c \Rightarrow z = z_{max} = e^{-\beta\mu_m B}$$

$$N = N_0 + \left(\frac{kT}{\hbar\omega}\right)^3 \left[g_3(1) + g_3(e^{-\beta\mu_m B}) + g_3(e^{-2\beta\mu_m B}) \right]$$

$$\Rightarrow 1 = f_0 + \frac{1}{N}\left(\frac{kT}{\hbar\omega}\right)^3 \left[g_3(1) + g_3(e^{-\beta\mu_m B}) + g_3(e^{-2\beta\mu_m B}) \right]$$

$$\Rightarrow f_0 = 1 - \frac{1}{N}\left(\frac{kT}{\hbar\omega}\right)^3 \left[g_3(1) + g_3(e^{-\beta\mu_m B}) + g_3(e^{-2\beta\mu_m B}) \right]$$

$$M = \mu_m \left\{ f_0 + \frac{1}{N}\left(\frac{kT}{\hbar\omega}\right)^3 \left[g_3(1) - g_3(e^{-2\beta\mu_m B}) \right] \right\}$$

$$= \boxed{\mu_m \left\{ 1 - \frac{1}{N}\left(\frac{kT}{\hbar\omega}\right)^3 \left[g_3(e^{-\beta\mu_m B}) + 2 g_3(e^{-2\beta\mu_m B}) \right] \right\} = M}$$

S: $B \to \infty$, $M = \mu_m$

S: $B \to 0$, $M = \mu_m \left\{ 1 - \frac{1}{N}\left(\frac{kT}{\hbar\omega}\right)^3 3\zeta(3) \right\}$

$$= \mu_m \left[1 - \left(\frac{T}{T_c}\right)^3 \right]$$

2. (a)

$$\delta F = \int d^3r \left\{ \frac{\hbar^2}{2m} \left(\vec{\nabla}\psi^* \cdot \vec{\nabla}\delta\psi + \vec{\nabla}\psi \cdot \vec{\nabla}\delta\psi^* \right) \right.$$

$$+ \frac{a(T-T_c)}{2} \left(\psi^* \delta\psi + \psi\,\delta\psi^* \right)$$

$$\left. + \frac{b}{4}\, 2|\psi|^2 \left(\psi^* \delta\psi + \psi\,\delta\psi^* \right) \right\}$$

$$= \int d^3r \left\{ \left[-\frac{\hbar^2}{2m}\Delta\psi^* + \frac{a(T-T_c)}{2}\psi^* + \frac{b}{2}|\psi|^2\psi^* \right] \delta\psi \right.$$

$$\left. + \left[-\frac{\hbar^2}{2m}\Delta\psi + \frac{a(T-T_c)}{2}\psi + \frac{b}{2}|\psi|^2\psi \right] \delta\psi^* \right\}$$

$$= \int d^3r \; \mathrm{Re} \left\{ \left[-\frac{\hbar^2}{m}\Delta\psi + a(T-T_c)\psi + b|\psi|^2\psi \right] \delta\psi^* \right\}$$

$$= \int d^3r \left\{ \mathrm{Re}[\;\;]\, \mathrm{Re}\,\delta\psi^* - \mathrm{Im}[\;\;]\, \mathrm{Im}\,\delta\psi^* \right\} = 0$$

$$xy = (\mathrm{Re}\,x + i\,\mathrm{Im}\,x)(\mathrm{Re}\,y + i\,\mathrm{Im}\,y)$$

$$= (\mathrm{Re}\,x)(\mathrm{Re}\,y) - (\mathrm{Im}\,x)(\mathrm{Im}\,y)$$

$$+ i\left((\mathrm{Im}\,x)(\mathrm{Re}\,y) + (\mathrm{Re}\,x)(\mathrm{Im}\,y) \right)$$

$$\Rightarrow \mathrm{Re}(xy) = (\mathrm{Re}\,x)(\mathrm{Re}\,y) - (\mathrm{Im}\,x)(\mathrm{Im}\,y)$$

$\Rightarrow \quad \mathrm{Re}\,[\ \ ] = \mathrm{Im}\,[\ \ ] = 0 \quad \Rightarrow \quad [\ \ ] = 0$

$$\Rightarrow \boxed{-\frac{\hbar^2}{m}\Delta\psi + a(T-T_c)\,\psi + b\,|\psi|^2\psi = 0}$$

(b) Densidad uniforme n, velocidad constante $\vec{v}$

$$\Rightarrow \quad \psi = \sqrt{n}\, e^{i\vec{k}\cdot\vec{r}} \qquad \vec{p} = \hbar\vec{k} = m\vec{v} \quad \Rightarrow \quad \vec{k} = \frac{m\vec{v}}{\hbar}$$

$$\Rightarrow \boxed{\psi = \sqrt{n}\, e^{i\frac{m\vec{v}}{\hbar}\cdot\vec{r}}}$$

Reemplazado en la ec. diferencial obtenemos

$$\underbrace{\frac{\hbar^2}{m}\left(\frac{m\vec{v}}{\hbar}\right)^2}_{= \, m v^2}\sqrt{n} + a(T-T_c)\sqrt{n} + b\,n^{3/2} = 0$$

$$\Rightarrow \sqrt{n}\left[\, m v^2 + a(T-T_c) + b n \,\right] = 0$$

$$n \neq 0 \quad \Rightarrow \quad \boxed{n = \frac{1}{b}\left[\, a(T_c - T) - m v^2 \,\right]}$$

$$n > 0 \implies mv^2 < a(T_c - T)$$

$$\boxed{v < \sqrt{\frac{a(T_c - T)}{m}}}$$

(c) $F(T) = F\left[\sqrt{n}\, e^{i\frac{m\vec{v}}{\hbar}\cdot\vec{r}} ; T\right)$

$$= \int d^3r \left\{ \frac{\hbar^2}{2m}\left(\frac{mv}{\hbar}\right)^2 n + \frac{a(T-T_c)}{2} n + \frac{b}{4} n^2 \right\}$$

$$= V \frac{1}{2} n \left\{ mv^2 + a(T-T_c) + \frac{b}{2} n \right\}$$

Volumen

$$= V \frac{1}{2} n \left\{ mv^2 + a(T-T_c) - \frac{1}{2}\left[mv^2 + a(T-T_c)\right] \right\}$$

$$= \frac{V}{4} n \left[mv^2 + a(T-T_c)\right] = -\frac{V}{4b}\left[mv^2 + a(T-T_c)\right]^2$$

$$\implies S = -\frac{dF}{dT} = \frac{V}{2b}\left[mv^2 + a(T-T_c)\right] a$$

$$\implies C_V = T\frac{dS}{dT} = \frac{V}{2b} a^2 T$$

Calor específico $c_V = \frac{C_V}{N} = \frac{C_V}{nV} = \frac{C_V}{\frac{1}{b}\left[a(T_c - T) - mv^2\right] V}$

$$\implies \boxed{c_V = \frac{a^2 T}{2\left[a(T_c - T) - mv^2\right]}}$$

3. (a) $$P(x,t) = \int_{-\infty}^{\infty} dk \; P_k(t) \, e^{ikx}$$

$$\Rightarrow \frac{\partial P}{\partial t} = \int_{-\infty}^{\infty} dk \; \dot{P}_k(t) \, e^{ikx}$$

$$\frac{\partial^2 P}{\partial x^2} = - \int_{-\infty}^{\infty} dk \; P_k(t) \, k^2 \, e^{ikx}$$

$$\Rightarrow \int_{-\infty}^{\infty} dk \; \dot{P}_k \, e^{ikx} = - \frac{g}{2\gamma^2} \int_{-\infty}^{\infty} dk \; k^2 P_k \, e^{ikx}$$

Las funciones $\{ e^{ikx}, \; k \in \mathbb{R} \}$ forman base

$$\Rightarrow \dot{P}_k = - \frac{g}{2\gamma^2} k^2 P_k \;\longrightarrow\; P_k(t) = A_k \, e^{-\frac{gk^2}{2\gamma^2} t}$$

$$\Rightarrow \boxed{P(x,t) = \int_{-\infty}^{\infty} dk \; A_k \, e^{-\frac{gk^2}{2\gamma^2} t + ikx}}$$

(b) $$P(x,0) = \int_{-\infty}^{\infty} dk \; A_k \, e^{ikx} = \delta(x)$$

$$\Rightarrow A_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \; \delta(x) \, e^{-ikx} = \frac{1}{2\pi}$$

$$\Rightarrow P(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{-\frac{gt}{2\gamma^2} k^2 + ikx} =$$

$$= \frac{1}{2\pi} \sqrt{\frac{\pi}{gt/2\gamma^2}} \; e^{-\frac{\gamma^2}{2gt} x^2}$$

$$\Rightarrow \boxed{P(x,t) = \sqrt{\frac{\gamma^2}{2\pi g t}}\, e^{-\frac{\gamma^2}{2gt}x^2}}$$

Comprobamos normalización:

$$\int dx\, P(x,t) = \sqrt{\frac{\gamma^2}{2\pi g t}} \sqrt{\frac{2\pi g t}{\gamma^2}} = 1 \quad ✓$$

(c) $$\langle x(t) \rangle = \int_{-\infty}^{\infty} dx \underset{\substack{\uparrow \\ \text{impar}}}{x} \underset{\substack{\uparrow \\ \text{par}}}{P(x,t)} = \boxed{0 = \langle x(t) \rangle}$$

$$\langle x^2(t) \rangle = \int_{-\infty}^{\infty} dx\, x^2 P(x,t) = \sqrt{\frac{\gamma^2}{2\pi g t}} \underbrace{\int_{-\infty}^{\infty} dx\, x^2 e^{-\overbrace{\frac{\gamma^2}{2gt}}^{\alpha} x^2}}$$

$$= -\frac{d}{d\alpha} \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}$$

$$= -\frac{d}{d\alpha} \sqrt{\frac{\pi}{\alpha}} =$$

$$= \frac{1}{2} \frac{\sqrt{\pi}}{\alpha^{3/2}}$$

$$\Rightarrow \langle x^2(t) \rangle = \cancel{\sqrt{\frac{\gamma^2}{2\pi g t}}}\ \frac{1}{\cancel{2}} \cancel{\sqrt{\pi}} \cancel{\sqrt{\frac{2gt}{\gamma^2}}}\ \frac{\cancel{2}gt}{\gamma^2}$$

$$= \boxed{\frac{g}{\gamma^2} t = \langle x^2(t) \rangle}$$

$$S(t) = -\int_{-\infty}^{\infty} dx \; P(x,t) \log P(x,t)$$

$$= -\int_{-\infty}^{\infty} dx \; P(x,t) \left[-\frac{\gamma^2}{2gt} x^2 - \frac{1}{2} \log\left(\frac{2\pi g}{\gamma^2} t \right) \right]$$

$$= \frac{\gamma^2}{2gt} \langle x^2(t) \rangle + \frac{1}{2} \log\left(\frac{2\pi g}{\gamma^2} t \right)$$

$$= \boxed{\frac{1}{2} + \frac{1}{2} \log\left(\frac{2\pi g}{\gamma^2} t \right) = S(t)}$$

La entropía de Shannon aumenta con el tiempo, lo cual es consistente con la idea de que la partícula alcanza el equilibrio.

A tiempos pequeños $S(t) < 0$ porque una densidad de probabilidad no tiene por qué ser < 1.