1. (a) $$Q = \frac{1}{N!} Q_1^N$$

$$Q_1 = \sum_{n=0}^{\infty} \int \frac{d^3q\, d^3p}{h^3}\, e^{-\beta\left[\frac{p^2}{2m} + \hbar\omega\left(n+\frac{1}{2}\right)\right]}$$

$$= \sum_{n=0}^{\infty} e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)} \frac{V}{\lambda^3}$$

$$= \frac{V}{\lambda^3} e^{-\beta\frac{\hbar\omega}{2}} \underbrace{\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}}_{= \frac{1}{1-e^{-\beta\hbar\omega}}}$$

$$= \frac{V}{\lambda^3} \frac{1}{2\,\mathrm{senh}\,\beta\frac{\hbar\omega}{2}} = Q_1$$

Energía por partícula $u = -\frac{\partial}{\partial\beta} \log Q_1$

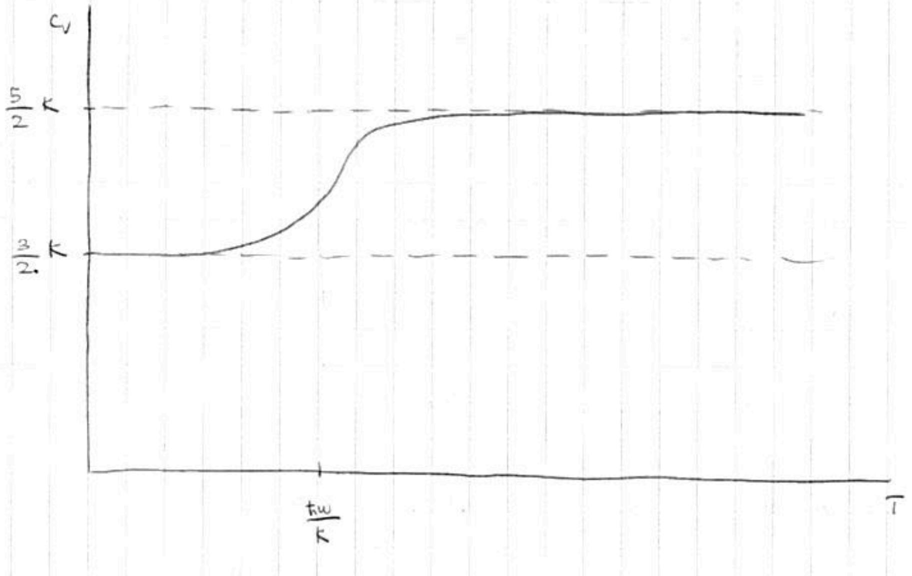
$$u = \frac{\partial}{\partial\beta} \log \lambda^3 + \frac{\partial}{\partial\beta} \log\left(\mathrm{senh}\,\frac{\beta\hbar\omega}{2}\right)$$

$$= \frac{3}{2} KT + \left(\coth\frac{\beta\hbar\omega}{2}\right)\frac{\hbar\omega}{2}$$

$$\Rightarrow \quad c_V = \frac{\partial U}{\partial T} = \frac{3}{2} k + \frac{1}{\sinh^2 \frac{\beta \hbar \omega}{2}} \left(\frac{\hbar \omega}{2} \right)^2 \frac{1}{k T^2}$$

$$\coth' x = - \frac{1}{\sinh^2 x}$$

$$\Rightarrow \quad \boxed{c_V = \left\{ \frac{3}{2} + \left[\frac{\frac{\beta \hbar \omega}{2}}{\sinh\left(\frac{\beta \hbar \omega}{2}\right)} \right]^2 \right\} k}$$



A temperaturas altas el grado de libertad de vibración se comporta clásicamente y obtenemos equipartición. A temperaturas bajas el grado de libertad de vibración no se excita y es como si no estuviera

(b) $$Q_1 = \frac{V}{\lambda^3} \frac{1}{2 \operatorname{senh} \frac{\beta \hbar\omega}{2}}$$

$$\Rightarrow \quad Q = \frac{1}{N!} \left(\frac{V}{\lambda^3} \frac{1}{2 \operatorname{senh} \frac{\beta \hbar\omega}{2}} \right)^N$$

$$F = -kT \log Q = -kT \left[N \log \left(\frac{V}{\lambda^3} \frac{1}{2 \operatorname{senh} \frac{\beta \hbar\omega}{2}} \right) - N \log N + N \right]$$

$$= -NkT \left[\log \left(\frac{v}{\lambda^3} \frac{1}{2 \operatorname{senh} \frac{\beta \hbar\omega}{2}} \right) + 1 \right]$$

$$= U - TS$$

$$\Rightarrow \quad S = \frac{U-F}{T} \underset{\substack{\uparrow \\ \text{item} \\ \text{anterior}}}{=} \frac{1}{T} \left\{ \frac{3}{2} NkT + N \frac{\hbar\omega}{2} \coth \frac{\beta \hbar\omega}{2} + NkT \left[\log \left(\frac{v}{\lambda^3} \frac{1}{2 \operatorname{senh} \frac{\beta \hbar\omega}{2}} \right) + 1 \right] \right\}$$

$$= Nk \left\{ \frac{3}{2} + \frac{\beta \hbar\omega}{2} \coth \frac{\beta \hbar\omega}{2} + \log \left(\frac{v}{\lambda^3} \frac{1}{2 \operatorname{senh} \frac{\beta \hbar\omega}{2}} \right) + 1 \right\}$$

$$\Rightarrow \quad \boxed{S = Nk \left[\frac{\beta \hbar\omega}{2} \coth \frac{\beta \hbar\omega}{2} + \log \left(\frac{v}{\lambda^3} \frac{1}{2 \operatorname{senh} \frac{\beta \hbar\omega}{2}} \right) + \frac{5}{2} \right]}$$

(c) $\beta\hbar\omega << 1 \Longrightarrow \coth\frac{\beta\hbar\omega}{2} \simeq \frac{1}{\beta\hbar\omega/2}$

$$\sinh\frac{\beta\hbar\omega}{2} \simeq \frac{\beta\hbar\omega}{2}$$

$$\Longrightarrow S = NK\left[\log\left(\frac{v}{\lambda^3}\,\frac{KT}{\hbar\omega}\right) + \frac{7}{2}\right]$$

Adiabática y cuasiestática $\Longrightarrow \Delta S = 0$

$$\Longrightarrow \log\frac{V'T'^{5/2}}{V\,T^{5/2}} = 0 \Longrightarrow V'T'^{5/2} = V\,T^{5/2}$$

$$T'^{5/2} = \frac{V}{V'}\,T^{5/2}$$

$$\boxed{T' = \left(\frac{V}{V'}\right)^{2/5} T}$$

2. (a) Proponemos

$$f(r,p) = A\, e^{-\beta\left(\frac{p^2}{2m} + \phi(r)\right)} = A\left(\frac{r}{R}\right)^{\beta\mathcal{K}} e^{-\beta\frac{p^2}{2m}}$$

Veamos que satisface Boltzmann:

1) $\dfrac{\partial f}{\partial t} = 0$

2) $f(r,p) = g(H(r,p)) \Rightarrow$ Se conserva $\Rightarrow \{f, H\} = 0.$

3) $f(r,p) = g(r)\, \underset{\substack{\uparrow \\ \text{Maxwell - Boltzmann}}}{f_{MB}}(p)$

$$\Rightarrow \left(\frac{\partial f}{\partial t}\right)_{col}(r,p) = g(r)\underbrace{\left(\frac{\partial f_{MB}}{\partial t}\right)_{col}(p)}_{=0} = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} + \{f, H\} = \left(\frac{\partial f}{\partial t}\right)_{col}$$

Determinamos A

$$n(r) = \int d^3p \; f(r,p) = A \left(\frac{r}{R}\right)^{\beta \mathcal{K}} \underbrace{\int d^3p \; e^{-\beta p^2/2m}}_{= (2\pi m k T)^{3/2}}$$

$$\Longrightarrow \; n_R = A \; (2\pi m k T)^{3/2} \;\Longrightarrow\; A = \frac{n_R}{(2\pi m k T)^{3/2}}$$

$$\Longrightarrow \; \boxed{f(r,p) = \frac{n_R}{(2\pi m k T)^{3/2}} \left(\frac{r}{R}\right)^{\beta \mathcal{K}} e^{-\beta \frac{p^2}{2m}}}$$

(b) Presión $\quad P = J_{\vec{p}\cdot\hat{n}} = n \langle \vec{p}\cdot\hat{n} \; \vec{v}\cdot\hat{n} \rangle = \frac{n}{m} \langle (\vec{p}\cdot\hat{n})^2 \rangle$

$$= \frac{n}{m} \langle p_z^2 \rangle = \frac{n}{m} \frac{\int d^3p \; p_z^2 \; e^{-\beta p^2/2m}}{\int d^3p \; e^{-\beta p^2/2m}}$$

elijo $\hat{n} = \hat{z}$

$$= \frac{n}{m} \underbrace{\frac{\int dp_z \; p_z^2 \; e^{-\beta p_z^2/2m}}{\int dp_z \; e^{-\beta p_z^2/2m}}}_{= \sigma^2 = mkT} = nkT = P$$

$$\Rightarrow \quad F = \int P\, dS = 2\pi \int_0^R dr\, r\, P(r) = 2\pi\, n_R kT \int_0^R dr\, r \left(\frac{r}{R}\right)^{\beta\mathcal{K}}$$

$$= 2\pi\, n_R kT\, \frac{1}{R^{\beta\mathcal{K}}} \underbrace{\int_0^R dr\, r^{\beta\mathcal{K}+1}}_{= \frac{1}{\beta\mathcal{K}+2} R^{\beta\mathcal{K}+2}}$$

$$= 2\pi\, n_R kT\, \frac{R^2}{\beta\mathcal{K}+2} = \boxed{\pi R^2 \frac{n_R kT}{1 + \beta\mathcal{K}/2} = F}$$

(c) $$J = n \langle \vec{v}\cdot\hat{n} \rangle \underset{\hat{n} = -\hat{z}}{=} - n \langle v_z \rangle = - \frac{n}{m} \langle p_z \rangle$$

$$= - \frac{1}{m} \frac{n_R}{(2\pi m kT)^{3/2}} \left(\frac{r}{R}\right)^{\beta\mathcal{K}} \int_{p_z<0} d^3p\, p_z\, e^{-\beta p^2/2m}$$

Agujero $\longrightarrow$ No hay partículas con $p_z > 0$.

$$= - \frac{1}{m} \frac{n_R}{(2\pi m kT)^{3/2}} \left(\frac{r}{R}\right)^{\beta\mathcal{K}} 2\pi m kT \int_{-\infty}^{0} dp_z\, p_z\, e^{-\beta p_z^2/2m}$$

$$\underset{x \equiv -p_z}{=} + \frac{1}{m} \frac{n_R}{\sqrt{2\pi m kT}} \left(\frac{r}{R}\right)^{\beta\mathcal{K}} \underbrace{\int_0^\infty dx\, x\, e^{-\beta x^2/2m}}_{= -mkT\, e^{-\beta x^2/2m}\Big|_0^\infty = mkT}$$

$\Rightarrow \quad J = n_R \left(\frac{r}{R}\right)^{\beta\mathcal{K}} \sqrt{\frac{kT}{2\pi m}}$

$\Rightarrow$ Núm. de partículas que escapan por unidad de tiempo

$$\boxed{-\dot{N} = n_R \left(\frac{r}{R}\right)^{\beta\mathcal{K}} \sqrt{\frac{kT}{2\pi m}} \; a}$$

Si $r = 0$ no escapan partículas. Eso pasa pq $\phi(r=0) = \infty$

$\Rightarrow$ Las partículas no tienen energía para llegar al agujero.

3. (a) $N = N_0 + N_\epsilon$

$E = N_\epsilon \, \epsilon \quad \Rightarrow \quad N_\epsilon = \frac{E}{\epsilon}, \quad N_0 = N - \frac{E}{\epsilon}$

$$\Omega(E, N) = \Omega(N_0, N_\epsilon) = \binom{N}{N_\epsilon} \; g^{N_\epsilon}$$

Núm. de formas en que puedo elegir qué partículas tienen energía ϵ

Núm. de formas en que puedo distribuir estas partículas entre los g estados

$$\Longrightarrow \quad S = k \log \Omega = k \left\{ N \log N - N_\epsilon \log N_\epsilon - N_0 \log N_0 + N_\epsilon \log g \right\}$$

$$\boxed{= k \left\{ N \log N - \frac{E}{\epsilon} \log \frac{E}{\epsilon} - \left(N - \frac{E}{\epsilon}\right) \log \left(N - \frac{E}{\epsilon}\right) + \frac{E}{\epsilon} \log g \right\} = S}$$

$$S(E = 0) = 0$$

$$S(E = N\epsilon) = . N \log g$$

$$\frac{1}{T} = \frac{\partial S}{\partial E} = k \left\{ - \frac{1}{\epsilon} \log \frac{E}{\epsilon} - \cancel{\frac{1}{\epsilon}} + \frac{1}{\epsilon} \log \left(N - \frac{E}{\epsilon}\right) + \cancel{\frac{1}{\epsilon}} + \frac{1}{\epsilon} \log g \right\}$$

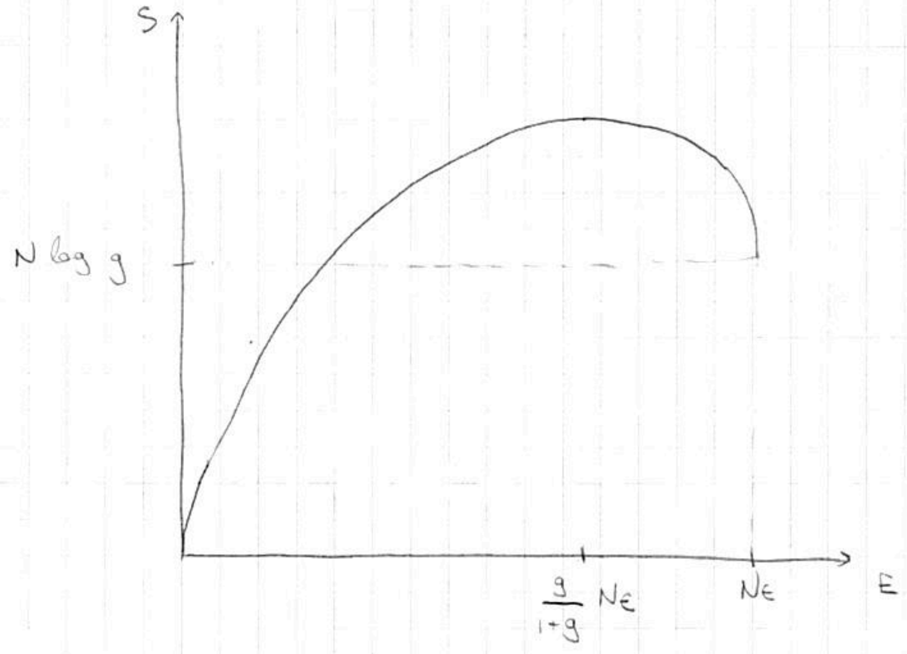
$$= \frac{k}{\epsilon} \log \frac{g\,(N - E/\epsilon)}{E/\epsilon}$$

En $E = 0$, $\frac{1}{T} = \infty$

En $E = N\epsilon$, $\frac{1}{T} = -\infty$

Extremos $\frac{1}{T} = 0 \iff g \dfrac{N - E/\epsilon}{E/\epsilon} = 1 \Rightarrow \frac{E}{\epsilon} = g\left(N - \frac{E}{\epsilon}\right)$

$\Rightarrow \frac{E}{\epsilon}\left(1 + g\right) = g N \Rightarrow \frac{E}{\epsilon} = \frac{g}{1+g} N \Rightarrow E = \frac{g}{1+g} N\epsilon$



Para $E > \frac{g}{1+g} N\epsilon$, S decrece con E $\Rightarrow$ Temperatura negativa.

(b) $$\frac{p}{T} = \frac{\partial S}{\partial V} = k \frac{E}{\epsilon} \frac{1}{V}$$

$$\frac{1}{T} = \frac{k}{\epsilon} \log \frac{g\,(N - E/\epsilon)}{E/\epsilon}$$

$$\beta\epsilon = \log \frac{g\,(N - E/\epsilon)}{E/\epsilon}$$

$$e^{\beta\epsilon} = g\,\frac{N - E/\epsilon}{E/\epsilon}$$

$$\frac{E}{\epsilon}\, e^{\beta\epsilon} = g\left(N - E/\epsilon\right)$$

$$\frac{E}{\epsilon}\left(e^{\beta\epsilon} + g\right) = g\,N$$

$$\Rightarrow \frac{E}{\epsilon} = \frac{g\,N}{e^{\beta\epsilon} + g} = \frac{\alpha V\, N}{e^{\beta\epsilon} + \alpha V} \underset{\substack{\uparrow \\ \text{Lim} \\ \text{termodinámico}}}{\simeq} N$$

$$\Rightarrow \frac{p}{T} = \frac{Nk}{V} \qquad \boxed{p = \frac{NkT}{V}}$$

(c) $$\Omega(E, N) = \Omega(N_0, N_\epsilon) = \binom{N_\epsilon + g - 1}{N_\epsilon}$$

Núm. de formas en que puedo distribuir N_ϵ parti. indistinguibles entre g estados (cajas)

$$\Rightarrow S = k \left\{ (N_\epsilon + g - 1) \log (N_\epsilon + g - 1) - N_\epsilon \log N_\epsilon - (g-1) \log (g-1) \right\}$$

$$\boxed{S = k \left\{ \left(\frac{E}{\epsilon} + g - 1\right) \log \left(\frac{E}{\epsilon} + g - 1\right) - \frac{E}{\epsilon} \log \frac{E}{\epsilon} - (g-1) \log (g-1) \right\}}$$

En $E = 0$, $S = 0$

$E = N\epsilon$ $$S = k \left\{ (N + g - 1) \log (N + g - 1) - N \log N - (g-1) \log (g-1) \right\}$$

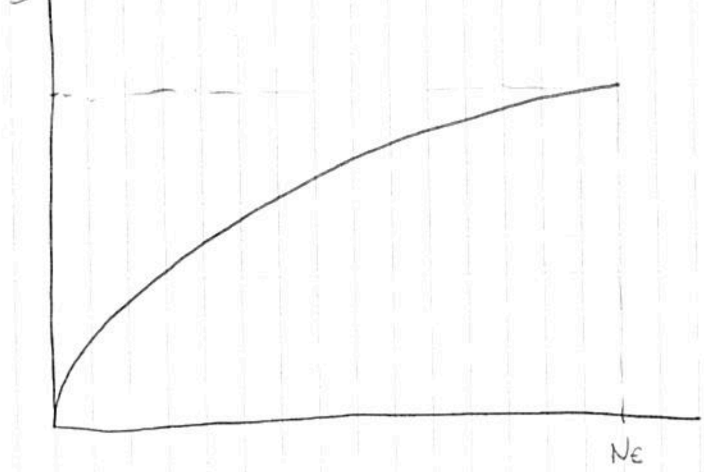
$$\frac{1}{T} = \frac{\partial S}{\partial E} = k \left\{ \frac{1}{\epsilon} \log\left(\frac{E}{\epsilon} + g - 1\right) - \frac{1}{\epsilon} \log \frac{E}{\epsilon} \right\}$$

$$= \frac{k}{\epsilon} \log \frac{\frac{E}{\epsilon} + g - 1}{E/\epsilon}$$

En $E = 0$, $\frac{1}{T} = \infty$

$E = N\epsilon$, $\frac{1}{T} = \frac{k}{\epsilon} \log \frac{N + g - 1}{N}$

$\frac{1}{T}$ nunca es $0 \Rightarrow \nexists$ extremos



T siempre positiva.