1. (a) $N_0 = g_s \frac{V}{2\lambda^3} f_{3/2}(z)$

$$N_\Delta = g_s \int \frac{d^3q\, d^3p}{h^3} \frac{1}{z^{-1} e^{\beta\left(\frac{p^2}{2m}+\Delta\right)}+1} = g_s \frac{V}{2\lambda^3} f_{3/2}(z e^{-\beta\Delta}) .$$

(potencial químico es el mismo para ambos estados pq están en equilibrio)

$$\Rightarrow N = g_s \frac{V}{2\lambda^3} \left[f_{3/2}(z) + f_{3/2}(z e^{-\beta\Delta}) \right]$$

T bajas $\longrightarrow$ Sommerfeld:

$$f_{3/2}(z) \simeq \frac{(\log z)^{3/2}}{\Gamma(5/2)} = \frac{4}{3\sqrt{\pi}} (\log z)^{3/2}$$

$\Rightarrow$ A $T=0$,

$$N = \underbrace{g_s \frac{V}{2} \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{4}{3\sqrt{\pi}}}_{\left(\frac{2m}{4\pi \hbar^2}\right)^{3/2} \frac{4}{3\sqrt{\pi}} = \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{6\pi^2}} \left[\epsilon_F^{3/2} + (\epsilon_F - \Delta)^{3/2} \right]$$

$$\Rightarrow N = g_s \frac{V}{2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{6\pi^2} \left[\epsilon_F^{3/2} + (\epsilon_F - \Delta)^{3/2} \right] .$$

$$\Rightarrow \epsilon_F^{3/2} + (\epsilon_F - \Delta)^{3/2} = \underbrace{2 \left(\frac{\hbar^2}{2m}\right)^{3/2} \frac{6\pi^2}{g_s} \frac{N}{V}}_{(\epsilon_F^0)^{3/2}}$$

$\llcorner$ Energía de Fermi si $\Delta = 0$.

Δ chico $\Rightarrow$ Calculo ϵ_F hasta 1er orden en Δ.

$$(\epsilon_F - \Delta)^{3/2} = \epsilon_F^{3/2}\left(1 - \frac{\Delta}{\epsilon_F}\right)^{3/2} = \epsilon_F^{3/2}\left(1 - \frac{3}{2}\frac{\Delta}{\epsilon_F} + O(\Delta^2)\right)$$

$$= \epsilon_F^{3/2}\left(1 - \frac{3}{2}\frac{\Delta}{\epsilon_F^0} + O(\Delta^2)\right)$$

$$\uparrow$$

$$\epsilon_F = \epsilon_F^0 + O(\Delta)$$

$$\Rightarrow \epsilon_F^{3/2}\left(2 - \frac{3}{2}\frac{\Delta}{\epsilon_F^0} + O(\Delta^2)\right) = 2(\epsilon_F^0)^{3/2}$$

$$\Rightarrow \epsilon_F^{3/2} = \frac{(\epsilon_F^0)^{3/2}}{1 - \frac{3}{4}\frac{\Delta}{\epsilon_F^0} + O(\Delta^2)} = (\epsilon_F^0)^{3/2}\left[1 + \frac{3}{4}\frac{\Delta}{\epsilon_F^0} + O(\Delta^2)\right]$$

$$\Rightarrow \epsilon_F = \epsilon_F^0\left(1 + \frac{1}{2}\frac{\Delta}{\epsilon_F^0} + O(\Delta^2)\right)$$

$$\Rightarrow \boxed{\epsilon_F \simeq \epsilon_F^0 + \frac{1}{2}\Delta}$$

b) $\underline{T \text{ altas}} \Rightarrow z \ll 1 \Rightarrow f_\nu(z) \simeq z$

$$\Rightarrow \quad N_0 = g_s \frac{V}{2\lambda^3} z$$

$$N_\Delta = g_s \frac{V}{2\lambda^3} z e^{-\beta\Delta}$$

$$N = g_s \frac{V}{2\lambda^3} z \left(1 + e^{-\beta\Delta}\right)$$

$$\Rightarrow \boxed{\frac{N_0}{N} = \frac{1}{1+e^{-\beta\Delta}} \qquad \frac{N_\Delta}{N} = \frac{e^{-\beta\Delta}}{1+e^{-\beta\Delta}}}$$

$\underline{T=0}$

$$N_0 = g_s \frac{V}{2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{6\pi^2} \epsilon_F^{3/2}$$

$$N_\Delta = g_s \frac{V}{2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{6\pi^2} (\epsilon_F - \Delta)^{3/2}$$

$$N = g_s \frac{V}{2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{6\pi^2} \left[\epsilon_F^{3/2} + (\epsilon_F - \Delta)^{3/2}\right]$$

$$\Rightarrow \frac{N_0}{N} = \frac{\epsilon_F^{3/2}}{\epsilon_F^{3/2} + (\epsilon_F - \Delta)^{3/2}} = \frac{(\epsilon_F^0 + \Delta/2)^{3/2}}{(\epsilon_F^0 + \Delta/2)^{3/2} + (\epsilon_F^0 - \Delta/2)^{3/2}}$$

$$= \frac{(1 + \Delta/2\epsilon_F^0)^{3/2}}{(1 + \Delta/2\epsilon_F^0)^{3/2} + (1 - \Delta/2\epsilon_F^0)^{3/2}} \underset{\uparrow}{\simeq} \boxed{\frac{1}{2}\left[1 + \frac{3}{4}\frac{\Delta}{\epsilon_F^0}\right] = \frac{N_0}{N}}$$

1^er orden en Δ

$$\Rightarrow \boxed{\frac{N_\Delta}{N} = 1 - \frac{N_0}{N} = \frac{1}{2}\left[1 - \frac{3}{4}\frac{\Delta}{e_F^0}\right]}$$

Si valiera la estadística de Boltzmann a $T=0$ tendríamos $N_0 = N$, $N_\Delta = 0$. El principio de exclusión de Pauli impide eso.

(c)
$$\log \mathcal{Z}_0 = g_s \frac{V}{2\lambda^3} f_{5/2}(z)$$
$$\log \mathcal{Z}_\Delta = g_s \frac{V}{2\lambda^3} f_{5/2}(z e^{-\beta\Delta})$$

$$U_0 = -\left(\frac{\partial}{\partial\beta}\log \mathcal{Z}_0\right)_{V,z} = \frac{3}{2}(kT)\, g_s \frac{V}{2\lambda^3} f_{5/2}(z)$$

$$U_\Delta = -\left(\frac{\partial}{\partial\beta}\log \mathcal{Z}_\Delta\right)_{V,z} = \frac{3}{2} kT\, g_s \frac{V}{2\lambda^3} f_{5/2}(z e^{-\beta\Delta}) + N_\Delta \Delta$$

<u>T altos</u>: $f_{5/2}(z) \simeq z \Rightarrow g_s \frac{V}{2\lambda^3} f_{5/2}(z) \simeq g_s \frac{V}{2\lambda^3} z \simeq N_0$

$$g_s \frac{V}{2\lambda^3} f_{5/2}(z e^{-\beta\Delta}) \simeq g_s \frac{V}{2\lambda^3} z e^{-\beta\Delta} \simeq N_\Delta$$

$$\Rightarrow U = U_0 + U_\Delta = \frac{3}{2} kT (N_0 + N_\Delta) + N_\Delta \Delta$$

$$= \boxed{\frac{3}{2} NkT + N \frac{e^{-\beta\Delta}}{1+e^{-\beta\Delta}}\Delta = U}$$

T bajos $\Rightarrow$ Sommerfeld: $f_{5/2}(z) \simeq \dfrac{(\log z)^{5/2}}{\Gamma(7/2)} = \dfrac{(\log z)^{5/2}}{5/2\ \Gamma(5/2)} = \dfrac{2}{5}\dfrac{(\log z)^{5/2}}{3\sqrt{\pi}/4}$

$\Rightarrow$ $\underline{T=0}$.

$$U_0 = \frac{3}{2}\, g_s\, \frac{V}{2}\, \underbrace{\left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{4}{3\sqrt{\pi}}}_{=\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{1}{6\pi^2}}\, \frac{2}{5}\, \epsilon_F^{5/2}$$

$$U_\Delta = \frac{3}{2}\, g_s\, \frac{V}{2}\, \overbrace{\left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{4}{3\sqrt{\pi}}}\, \frac{2}{5} (\epsilon_F - \Delta)^{5/2} + N_\Delta\, \Delta$$

$$\Rightarrow \quad U = \frac{3}{5}\, g_s\, \frac{V}{2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{6\pi^2} \left[\epsilon_F^{5/2} + (\epsilon_F - \Delta)^{5/2} \right] + N_\Delta\, \Delta$$

$$= \frac{3}{5}\, g_s\, \frac{V}{2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{6\pi^2} \left[\left(\epsilon_F^0 + \frac{\Delta}{2}\right)^{5/2} + \left(\epsilon_F^0 - \frac{\Delta}{2}\right)^{5/2} \right] + N_\Delta\, \Delta$$

$$\simeq \frac{3}{5}\, g_s\, \frac{V}{\cancel{2}} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{6\pi^2}\, \cancel{2} (\epsilon_F^0)^{5/2} + \frac{1}{2} N \Delta$$

$1^{er.}$ orden en Δ

$$N_\Delta = \frac{1}{2} N + O(\Delta)$$

$$\Rightarrow \quad \boxed{U = \frac{3}{5}\, N\, \epsilon_F^0 + \frac{1}{2} N \Delta}$$

2. (a) $$\log \mathcal{Z} = - \sum_{n_1,\dots,n_d} \log \left[1 - z e^{-\beta\hbar\omega(n_1+\dots+n_d)} \right]$$

$$\underset{\substack{\uparrow \\ m = n_1+\dots+n_d}}{=} - \sum_{m=0}^{\infty} g(m) \log\left(1 - z e^{-\beta\hbar\omega m}\right)$$

$g(m)$: núm. de formas de distribuir m cuantos de energía en d cajas

$$\Rightarrow g(m) = \binom{m+d-1}{m}$$

$$\Rightarrow \log \mathcal{Z} = - \sum_{m=0}^{\infty} \binom{m+d-1}{m} \log\left(1 - z e^{-\beta\hbar\omega m}\right)$$

$$\underset{\substack{\uparrow \\ \beta\hbar\omega \ll 1}}{\simeq} - \frac{1}{(d-1)!} \sum_{m=0}^{\infty} m^{d-1} \log\left(1 - z e^{-\beta\hbar\omega m}\right)$$

$\Rightarrow$ suma dominada por m grande. Uso ayuda

$$\underset{\substack{\uparrow \\ \beta\hbar\omega \ll 1}}{\simeq} - \frac{1}{(d-1)!} \int_0^{\infty} dm \; m^{d-1} \log\left(1 - z e^{-\beta\hbar\omega m}\right)$$

$\Rightarrow$ función que sumo decae lentamente

$\Rightarrow$ puedo aproximar $\sum$ por $\int$

$$\underset{\substack{\uparrow \\ x \equiv \beta\hbar\omega m}}{=} - \left(\frac{kT}{\hbar\omega}\right)^{d} \frac{1}{(d-1)!} \int_0^{\infty} dx \; x^{d-1} \log\left(1 - z e^{-x}\right)$$

$$\Rightarrow \log \mathcal{Z} \underset{\substack{\uparrow \\ \text{partes}}}{=} -\left(\frac{kT}{\hbar\omega}\right)^d \frac{1}{(d-1)!}\left[\cancel{\frac{x^d}{d}\log(1-ze^{-x})\Big|_0^\infty}^{\,0} - \int_0^\infty dx\, \frac{x^d}{d}\frac{ze^{-x}}{1-ze^{-x}}\right]$$

$$= +\underbrace{\left(\frac{kT}{\hbar\omega}\right)^d \underbrace{\frac{1}{d!}}_{\Gamma(d+1)} \int_0^\infty dx\, \frac{x^d}{z^{-1}e^{x}-1}}_{g_{d+1}(z)}$$

$$\Rightarrow \boxed{\log \mathcal{Z} = \left(\frac{kT}{\hbar\omega}\right)^d g_{d+1}(z)}$$

(b) $$N = z\frac{\partial}{\partial z}\log \mathcal{Z} = \left(\frac{kT}{\hbar\omega}\right)^d g_d(z)$$

Para que haya condensación, $g_d(1)$ finito $\Rightarrow \boxed{d>1.}$

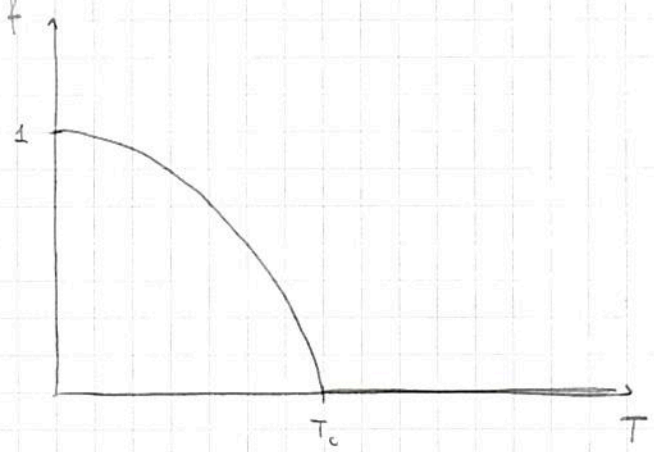
$$N = \left(\frac{kT_c}{\hbar\omega}\right)^d \underbrace{g_d(1)}_{\zeta(d)} \Rightarrow \boxed{T_c = \frac{\hbar\omega}{k}\left[\frac{N}{\zeta(d)}\right]^{1/d}}$$

(c) $T > T_c \Rightarrow f = 0.$

$$T < T_c \Rightarrow N = N_0 + \left(\frac{kT}{\hbar\omega}\right)^d \zeta(d)$$

$$\Rightarrow 1 = f + \underbrace{\frac{1}{N}\left(\frac{kT}{\hbar\omega}\right)^d \zeta(d)}_{= \left(\frac{T}{T_c}\right)^d}$$

$$\Rightarrow \boxed{f = 1 - \left(\frac{T}{T_c}\right)^d}$$



(d) $$U = -\left(\frac{\partial}{\partial \beta} \log Z\right)_z = d\,\frac{(kT)^{d+1}}{(\hbar\omega)^d}\, g_{d+1}(z)$$

$$* \quad T \leq T_c \Rightarrow U = d\,\frac{(kT)^{d+1}}{(\hbar\omega)^d}\,\zeta(d+1)$$

$$\Rightarrow c_v = d(d+1)\,k\,\underbrace{\frac{1}{N}\left(\frac{kT}{\hbar\omega}\right)^d}_{= \frac{1}{\zeta(d)}\left(\frac{T}{T_c}\right)^d} \zeta(d+1)$$

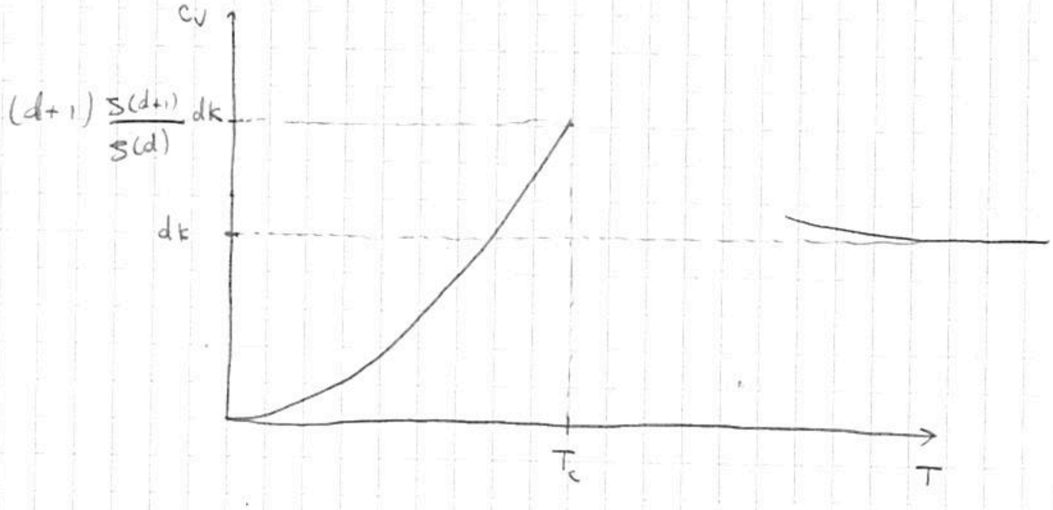
$$\Rightarrow \boxed{c_v = d(d+1)\,\frac{\zeta(d+1)}{\zeta(d)}\left(\frac{T}{T_c}\right)^d k} \qquad T \leq T_c$$

$$T \gg T_c \Rightarrow z \ll 1 \Rightarrow g_\nu(z) \simeq z$$

$$\left.\begin{aligned} \Rightarrow U &\simeq d\,\frac{(kT)^{d+1}}{(\hbar\omega)^d}\,z \\ N &\simeq \left(\frac{kT}{\hbar\omega}\right)^d z \end{aligned}\right\} \Rightarrow U \simeq d\,N\,kT$$

$$\Rightarrow \boxed{c_v \simeq d k} \qquad T \gg T_c$$

$\Rightarrow$ A $T \gg T_c$, recuperamos la predicción de equipartición.



3. (a) $H = -J \sum_{i=1}^{N/2} \left(\delta_{s_{2i-1} s_{2i}} + \delta_{s_{2i} s_{2i+1}} \right)$

$$Q = \sum_{\{s_{2i-1}\}} \underbrace{\sum_{\{s_{2i}\}} e^{K \sum_{i=1}^{N/2} \left(\delta_{s_{2i-1} s_{2i}} + \delta_{s_{2i} s_{2i+1}} \right)}}$$

$$\|$$

$$\prod_{i=1}^{N/2} \underbrace{\sum_{s} e^{K(\delta_{s_{2i-1} s} + \delta_{s_{2i} s_{2i+1}})}}$$

$$\|$$

$$\sum_{s} \left[1 + \delta_{s_{2i-1} s} (e^{K} - 1) \right] \left[1 + \delta_{s\, s_{2i+1}} (e^{K} - 1) \right]$$

$$\|$$

$$\sum_{s} \Big[1 + \delta_{s_{2i-1} s} (e^{K} - 1) + \delta_{s\, s_{2i+1}} (e^{K} - 1)$$

$$+ \delta_{s_{2i-1} s} \delta_{s\, s_{2i+1}} (e^{K} - 1)^2 \Big]$$

$$\|$$

$$q + 2(e^{K} - 1) + \delta_{s_{2i-1} s_{2i+1}} (e^{K} - 1)^2$$

$$\Rightarrow Q = \sum_{\{s_{2i-1}\}} \prod_{i=1}^{N/2} \left[q + 2(e^{K} - 1) + \delta_{s_{2i-1} s_{2i+1}} (e^{K} - 1)^2 \right]$$

$$\underset{s'_i = s_{2i-1}}{=} \sum_{\{s'_i\}} \prod_{i=1}^{N/2} \left[q + 2(e^{K} - 1) + \delta_{s'_i s'_{i+1}} (e^{K} - 1)^2 \right]$$

$$= \sum_{\{s'_i\}} e^{\sum_{i=1}^{N/2} \left(K'_0 + K' \delta_{s'_i s'_{i+1}} \right)}$$

$$\Rightarrow e^{K'_0 + K' \delta_{s'_i s'_{i+1}}} = e^{K'_0} \left[1 + \delta_{s'_i s'_{i+1}} (e^{K'} - 1) \right]$$

$$= q + 2(e^{K} - 1) + \delta_{s'_i s'_{i+1}} (e^{K} - 1)^2$$

$$\Rightarrow \quad e^{K_0'} = q + 2(e^{K}-1)$$

$$e^{K_0'}(e^{K'}-1) = e^{K}-1$$

$$\boxed{\begin{aligned} \Rightarrow \quad & e^{K_0'} = q + 2(e^{K}-1) \\ & e^{K'}-1 = \frac{(e^{K}-1)^2}{q+2(e^{K}-1)} \end{aligned}}$$

$$(b) \quad e^{K'} = \frac{(e^{K}-1)^2}{q+2(e^{K}-1)} + 1$$

$$= \frac{e^{2K} - \cancel{2e^{K}} + 1 + \cancel{2e^{K}} + q - 2}{2e^{K} + q - 2}$$

$$= \frac{e^{2K} + q - 1}{2e^{K} + q - 2}$$

$$\Rightarrow \quad x' = \frac{2e^{K} + q - 2}{e^{2K} + q - 1} = \frac{\frac{2}{e^{-K}} + q - 2}{\frac{1}{e^{-2K}} + q - 1} = \frac{\frac{2}{x} + q - 2}{\frac{1}{x^2} + q - 1}$$

$$= \boxed{\underbrace{\frac{2x + (q-2)x^2}{1 + (q-1)x^2}}_{R(x)} = x'}$$

Puntos fijos

$$x = \frac{2x + (q-2)x^2}{1 + (q-1)x^2}$$

$x = 0$ punto fijo.

$$x \neq 0 \Rightarrow 1 = \frac{2 + (q-2)x}{1 + (q-1)x^2}$$

$$\Rightarrow 1 + (q-1)x^2 = 2 + (q-2)x$$

$$(q-1)x^2 - (q-2)x - 1 = 0$$

$$x^2 - \frac{q-2}{q-1}x - \frac{1}{q-1} = 0$$

$$x = \frac{q-2}{2(q-1)} \pm \sqrt{\left[\frac{q-2}{2(q-1)}\right]^2 + \frac{1}{q-1}}$$

$(x \geq 0)$

$$= \frac{q-2}{2(q-1)} + \sqrt{\frac{(q-2)^2 + 4(q-1)}{[2(q-1)]^2}} = \frac{q-2}{2(q-1)} + \sqrt{\left[\frac{q}{2(q-1)}\right]^2}$$

$$= \frac{q-2}{2(q-1)} + \frac{q}{2(q-1)} = \frac{2q-2}{2(q-1)} = 1$$

Estabilidad : $R(x) = \dfrac{2x + (q-2)x^2}{1 + (q-1)x^2}$

$$R'(x) = \frac{2 + 2(q-2)x}{1+(q-1)x^2} - \frac{2x + (q-2)x^2}{\left[1+(q-1)x^2\right]^2}\, 2(q-1)x$$

$\Rightarrow \; R'(0) = 2$

$$R'(1) = \frac{2 + 2(q-2)}{1+(q-1)} - \frac{2 + (q-2)}{\left[1+(q-1)\right]^2}\, 2(q-1)$$

$$= \frac{2(q-1)}{q} - \frac{2\cancel{q}(q-1)}{q^2} = 0$$

$\Rightarrow \; |R'(0)| > 1$

$\Rightarrow$ $\boxed{x = 0 \text{ inestable}, \quad x = 1 \text{ estable}}$

$|R'(1)| < 1$