Quantal_4

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Recordemos

Condiciones periodicas de contorno

$$\boxed{\Psi(x,y,z) = \Psi(x+a,y,z) = \Psi(x,y+b,z) = \Psi(x,y,z+c)}$$

Resulta

$$\Psi_{lmn}(\mathbf{r}) = \exp[i(\mathbf{k}\cdot\mathbf{r})]$$

con

$$k = \left(\frac{l}{a}, \frac{m}{b}, \frac{n}{c}\right)$$

con $l, m, n = 0, \pm 1, \pm 2, \pm 3, \ldots$

Si uno calcula la densidad de estados para estos casos (calcular el numero de puntos del lattice)

$$g(k) = \sum_{l,m,n}{}' f^*(l,m,n)$$

Con $f^*(l,m,n) = 1$ si l,m,n se dan en la region y satisfacen

$$\left(\frac{l^2}{a^2},\frac{m^2}{b^2},\frac{n^2}{c^2}\right) \leq \frac{k^2}{\pi^2}\left(\frac{1}{4}\right)_{pbc}$$

En este caso

$$g(k) \propto V$$

Gas ideal en el GC

en el GC tenemos

$$\boxed{\Xi(z,V,T) = \sum_{N=0} z^N Q_N(V,T)}$$

$$\langle O \rangle = \frac{1}{\Xi} \sum_{N=0} z^N \langle O \rangle_N$$

o tambien

$$\boxed{\Xi(z,V,T) = Tr\exp\left[-\beta\left(\hat{H} - \mu N\right)\right]}$$

Entonces planteamos el Q_N

$$\boxed{Q_N = \sum_{\{n_\mathbf{p}\}} g\{n_\mathbf{p}\} e^{-\beta E\{n_\mathbf{p}\}}}$$

Donde $g\{n_{\mathbf{p}}\}$ es la "degeneracion" del nivel $\{n_{\mathbf{p}}\}$ ("combinatorial")

Donde $\{n_{\mathbf{p}}\}$ es el conjunto de numeros de ocupacion de los niveles $\mathbf{p}$ con

$$\boxed{E\{n_{\mathbf{p}}\} = \sum_{\mathbf{p}} \varepsilon_{\mathbf{p}} n_{\mathbf{p}}} \text{ y } \boxed{N = \sum_{\mathbf{p}} n_{\mathbf{p}}}$$

Que forma adoptan estas expresiones cuando particularizamos para los ≠ "tipos" de particulas que hemos definido?

$$\begin{array}{ll} Bose & n_{\mathbf{p}} = 0,1,2,3,.. \\ Boltzmann & n_{\mathbf{p}} = 0,1,2,3,\ldots \\ Fermi & n_{\mathbf{p}} = 0,1 \end{array}$$

Para los $g\{n_{\mathbf{p}}\}$

Para los $g\{n_{\mathbf{p}}\}$

$$
\boxed{\begin{array}{ll}
Bose & 1 \\
Boltzmann & \frac{1}{N!}\left[\frac{N!}{\prod n_{\mathbf{p}}!}\right] \\
Fermi & 1
\end{array}}
$$

Donde para el caso de Boltzmann primero distribuimos en los niveles , importando en que celda van, pero no las permutaciones dentro de cada celda y luego "indistinguibilizamos"

Bose y Fermi

(lo resolvemos en el GC directamente para obviar el problema de la condicion sobre N)

$$\Xi(z,V,T) = \sum_{N=0} z^N Q_N(V,T) = \sum_{N=0} z^N \sum_{\substack{\{n_{\mathbf{p}}\} \\ N=\Sigma n_{\mathbf{p}}}} e^{-\beta \sum_{\mathbf{p}} \varepsilon_{\mathbf{p}} n_{\mathbf{p}}}$$

$$= \sum_{N=0} \sum_{\substack{\{n_{\mathbf{p}}\} \\ N=\Sigma n_{\mathbf{p}}}} \prod (ze^{-\beta\varepsilon_{\mathbf{p}}})^{n_{\mathbf{p}}}$$

Se ve que:

$$= \sum \sum \ldots [(ze^{-\beta\varepsilon_{\mathbf{0}}})^{n_0} (ze^{-\beta\varepsilon_{\mathbf{1}}})^{n_1} \ldots] = \sum (ze^{-\beta\varepsilon_{\mathbf{0}}})^{n_0} \sum (ze^{-\beta\varepsilon_{\mathbf{1}}})^{n_1} \sum$$

$$= \prod_{\mathbf{p}} \sum_n (ze^{-\beta\varepsilon_{\mathbf{p}}})^n$$

Para Bose tenemos $n = 0, 1, 2, 3 \ldots$, luego geometrica

Bose :

$$\Xi(z,V,T) = \prod_{\mathbf{p}} \frac{1}{1 - ze^{-\beta\varepsilon_{\mathbf{p}}}}$$

Para Fermi $n = 0, 1$

$$\Xi(z,V,T) = \prod_{\mathbf{p}} (1 + ze^{-\beta\varepsilon_{\mathbf{p}}})$$

De donde es inmediato calcular la EOS es

$$\frac{PV}{kT} = \log \Xi(z,V,T)$$

Para Bose

$$\frac{PV}{kT} = -\sum \log(1 - ze^{-\beta\varepsilon_{\mathbf{p}}})$$

Para Fermi

$$\frac{PV}{kT} = \sum \log(1 + ze^{-\beta\varepsilon_{\mathbf{p}}})$$

Necesitamos otra ecuacion para z

$$N = z\frac{\partial}{\partial z}\log \Xi(z, V, T)$$

Para Bose

$$N = \sum_{\mathbf{p}} \frac{ze^{-\beta\varepsilon_{\mathbf{p}}}}{1 - ze^{-\beta\varepsilon_{\mathbf{p}}}}$$

Para Bosones

$$N = z\frac{\partial}{\partial z}\log\prod_p \frac{1}{1 - ze^{-\beta\varepsilon_p}}$$

$$= -z\frac{\partial}{\partial z}\sum_p \log\left(1 - ze^{-\beta\varepsilon_p}\right)$$

$$= -z\sum_p \frac{\partial}{\partial z}\log\left(1 - ze^{-\beta\varepsilon_p}\right)$$

$$= -z\sum_p \frac{-e^{-\beta\varepsilon_p}}{\left(1 - ze^{-\beta\varepsilon_p}\right)} = \sum_p \frac{ze^{-\beta\varepsilon_p}}{\left(1 - ze^{-\beta\varepsilon_p}\right)}$$

Para Fermi

$$\frac{PV}{kT} = \sum \log(1 + ze^{-\beta\varepsilon_{\mathbf{p}}})$$

Necesitamos otra ecuacion para z

$$N = z\frac{\partial}{\partial z}\log \Xi(z,V,T)$$

Para Bose

$$N = \sum_{\mathbf{p}} \frac{ze^{-\beta\varepsilon_{\mathbf{p}}}}{1 - ze^{-\beta\varepsilon_{\mathbf{p}}}}$$

Para Fermi

$$N = \sum_{\mathbf{p}} \frac{ze^{-\beta\varepsilon_{\mathbf{p}}}}{1 + ze^{-\beta\varepsilon_{\mathbf{p}}}}$$

Si ahora calculamos la ocupacion media por nivel

La ocupacion media por nivel :

$$\langle n_{\mathbf{p}} \rangle = \frac{1}{\Xi} \sum_{N=0}^{\infty} z^N \sum_{n_{\mathbf{p}};\, \Sigma n_{\mathbf{p}}=N} n_{\mathbf{p}} \exp(-\beta \Sigma_{\mathbf{p}} \varepsilon_{\mathbf{p}} n_{\mathbf{p}}) = \frac{-1}{\beta} \frac{\partial}{\partial \varepsilon_{\mathbf{p}}} \log \Xi$$

$\boxed{\text{Para Bose}}$

$$< n_p >= \frac{-1}{\beta} \frac{\partial}{\partial \varepsilon_p} \log \Xi = \frac{-1}{\beta} \frac{\partial}{\partial \varepsilon_p} \log \prod_p \frac{1}{1 - z e^{-\beta \varepsilon_p}}$$

$$< n_p >= \frac{-1}{\beta} \frac{\partial}{\partial \varepsilon_p} \sum \log \frac{1}{1 - z e^{-\beta \varepsilon_p}}$$

$$< n_p >= \frac{1}{\beta} \frac{\partial}{\partial \varepsilon_p} \sum \log(1 - z e^{-\beta \varepsilon_p})$$

$$< n_p >= \frac{1}{\beta} \frac{\beta z e^{-\beta \varepsilon_p}}{(1 - z e^{-\beta \varepsilon_p})} = \frac{z e^{-\beta \varepsilon_p}}{(1 - z e^{-\beta \varepsilon_p})}$$

Gas de Bose

1)

$$\boxed{\langle n_{\mathbf{p}} \rangle = \frac{ze^{-\beta\varepsilon_{\mathbf{p}}}}{1 - ze^{-\beta\varepsilon_{\mathbf{p}}}}}$$

$$0 \leq \frac{ze^{-\beta\varepsilon_{\mathbf{p}}}}{1 - ze^{-\beta\varepsilon_{\mathbf{p}}}} = \frac{1}{e^{\beta(\varepsilon_{\mathbf{p}}-\mu)} - 1}$$

$$0 \leq e^{\beta(\varepsilon_{\mathbf{p}}-\mu)} - 1 \Rightarrow 1 \leq e^{\beta(\varepsilon_{\mathbf{p}}-\mu)} \Rightarrow 0 \leq (\varepsilon_{\mathbf{p}} - \mu) \Rightarrow \varepsilon_{\mathbf{p}} \geq \mu \Rightarrow$$

para el fundamental $\boxed{\varepsilon_{\mathbf{p}} = 0 \Rightarrow 0 \geq \mu}$ de otra forma el fundamental tendria poblacion negativa

$$\boxed{\frac{PV}{kT} = -\sum \log(1 - ze^{-\beta\varepsilon_{\mathbf{p}}}) \qquad N = \sum_{\mathbf{p}} \frac{ze^{-\beta\varepsilon_{\mathbf{p}}}}{1 - ze^{-\beta\varepsilon_{\mathbf{p}}}}}$$

Separamos explícitamente el termino de población del fundamental

2)

$$\frac{P}{kT} = -\frac{4\pi}{h^3}\int_0^\infty dp p^2 \log\left(1 - z e^{-\beta\frac{p^2}{2m}}\right) - \frac{1}{V}\log(1-z)$$

ademas

$$\frac{1}{v} = \frac{4\pi}{h^3}\int_0^\infty dp p^2 \frac{1}{-1 + z^{-1} e^{\beta\frac{p^2}{2m}}} + \frac{1}{V}\frac{z}{1-z}$$

Lo cual puede ser reescrito siguiendo metodo usado para el gas de Fermi

$$\boxed{\frac{P}{kT} = \frac{1}{\lambda^3} g_{\frac{5}{2}}(z) - \frac{1}{V}\log(1-z)}$$

Tenemos $\langle n_p \rangle = \dfrac{1}{e^{\beta(e_p-\mu)}-1} \Rightarrow \boxed{\langle n_0 \rangle = \dfrac{1}{e^{\beta(-\mu)}-1} = \dfrac{1}{\frac{1}{z}-1} = \dfrac{z}{1-z}}$

con $$\frac{P}{kT} = -\frac{4\pi}{h^3}\int_0^\infty dpp^2 \log\left(1 - ze^{-\beta\frac{p^2}{2m}}\right) - \frac{1}{V}\log(1-z)$$

pero $N_0 = \dfrac{z}{1-z} \Rightarrow \boxed{z = \dfrac{N_0}{N_0+1} \Rightarrow 1-z = 1 - \dfrac{N_0}{N_0+1} = \dfrac{1}{N_0+1}}$

$\Rightarrow$

$$-\frac{1}{V}\log(1-z) \to \frac{1}{V}\log(N_0+1) \approx \frac{1}{N_0}\log(N_0+1)$$

Resulta entonces que este termino se va a 0 con $N_0 \to \infty$
o sea que no es necesario... (gracias al log)

$$\boxed{\frac{1}{v} = \frac{4\pi}{h^3}\int_0^\infty dpp^2 \frac{1}{-1+z^{-1}e^{\beta\frac{p^2}{2m}}} + \frac{1}{V}\frac{z}{1-z}}$$

Este, no
Sigue siendo
importante

2)

$$\frac{P}{kT} = -\frac{4\pi}{h^3}\int_0^\infty dp\,p^2 \log\left(1 - z e^{-\beta\frac{p^2}{2m}}\right) - \frac{1}{V}\log(1-z)$$

ademas

$$\frac{1}{v} = \frac{4\pi}{h^3}\int_0^\infty dp\,p^2 \frac{1}{-1 + z^{-1}e^{\beta\frac{p^2}{2m}}} + \frac{1}{V}\frac{z}{1-z}$$

Lo cual puede ser reescrito siguiendo metodo usado para el gas de Fermi

$$\boxed{\frac{P}{kT} = \frac{1}{\lambda^3} g_{\frac{5}{2}}(z) - \frac{1}{V}\log(1-z)}$$

$$\boxed{\frac{1}{v} = \frac{1}{\lambda^3} g_{\frac{3}{2}}(z) + \frac{1}{V}\frac{z}{1-z}}$$

con

$$\boxed{g_{\frac{5}{2}}(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^{\frac{5}{2}}}}$$

$$\boxed{g_{\frac{3}{2}}(z) = z\frac{\partial}{\partial z} g_{\frac{5}{2}}(z)}$$

Entonces

$\boxed{\langle n_0 \rangle = \frac{z}{1-z}}$; $\boxed{\frac{1}{v} = \frac{1}{\lambda^3} g_{\frac{3}{2}}(z) + \frac{\langle n_0 \rangle}{V}}$ este ultimo termino tendra relevancia en el limite V$\rightarrow \infty$ si…

Para la energía

$$U(z,T) = \frac{-\partial}{\partial\beta}\log\Xi = \frac{-\partial}{\partial\beta}\frac{PV}{kT} = \frac{-\partial}{\partial\beta}\frac{g_{\frac{5}{2}}(z)}{\lambda^3}V \Rightarrow$$

$$\boxed{U(z,T)\frac{1}{V} = \frac{3kT}{2}\frac{g_{\frac{5}{2}}(z)}{\lambda^3}}$$

Propiedades de la $g_{\frac{3}{2}}(z)$

$$g_{\frac{3}{2}}(z) = \sum_{l-1}\frac{z^l}{l^{\frac{3}{2}}}$$

Con $\begin{cases} z = e^{\beta\mu} \\ \mu \le 0 \Rightarrow 0 \le z \le 1 \end{cases}$

Para Bose $0 \leq z \leq 1$ **($\mu \leq 0$)**

(recordemos que para Fermi se cumple $0 \leq z \leq \infty$)

Si z es pequeňo

$$g_{\frac{3}{2}}(z) = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \ldots$$

Para $z = 1$

$g_{\frac{3}{2}}(1) = \zeta(1) = 2.612$ (y la derivada diverge)

$$g_{3/2}(z) = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} +$$

$$g'_{3/2}(z) = 1 + \frac{z}{2^{1/2}} + \frac{z^2}{3^{1/2}} +$$

$$g'_{3/2}(1) = 1 + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \ldots = \sum_{1}^{\infty} \frac{1}{n^{1/2}}$$

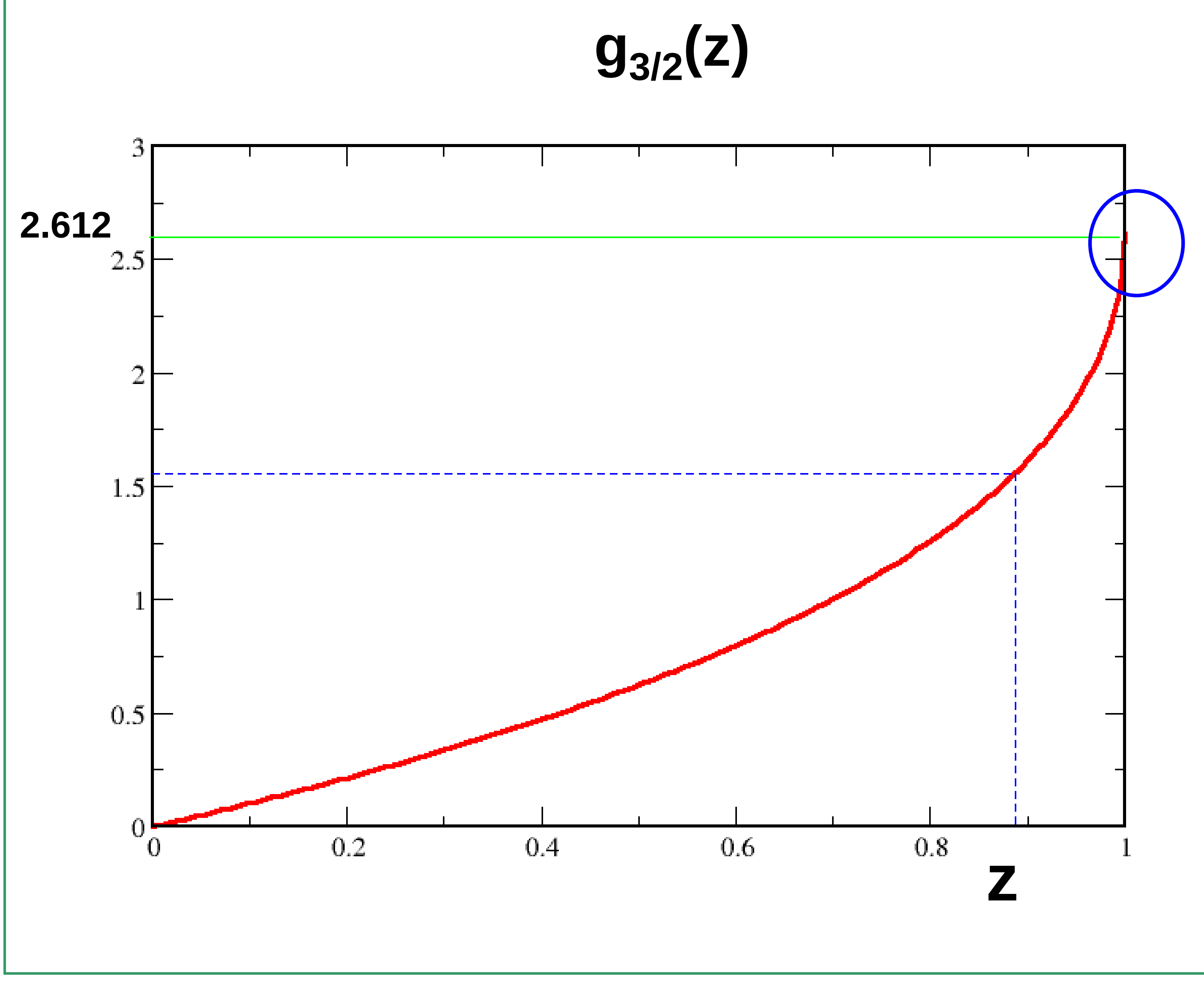
pero

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

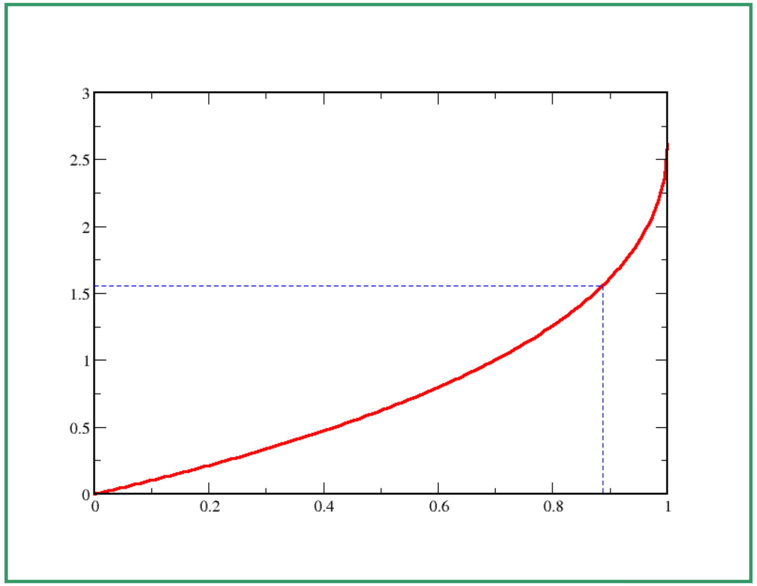
como

$$\frac{1}{n} \leq \frac{1}{n^{1/2}}$$

diverge



g3/2(z)
2.612
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
z



Si z es pequeňo

$$g_{\frac{3}{2}}(z) = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \ldots$$

Para $z = 1$

$g_{\frac{3}{2}}(1) = \zeta(1) = 2.612$ (y la derivada diverge)

De donde es claro que si erroneamente hubiesemos asumido que $\frac{\lambda^3}{v} = g_{\frac{3}{2}}(z)$ no podriamos superar $\frac{\lambda^3}{v} = 2.612$!!!!

Podemos escribir

Tomando en cuenta el termino de población del fundamental

$$\boxed{\frac{\lambda^3}{v} = g_{\frac{3}{2}}(z) + \frac{\lambda^3}{V}\langle n_0 \rangle}$$

de donde

$$\lambda^3 \frac{\langle n_0 \rangle}{V} = \frac{\lambda^3}{v} - g_{\frac{3}{2}}(z)$$

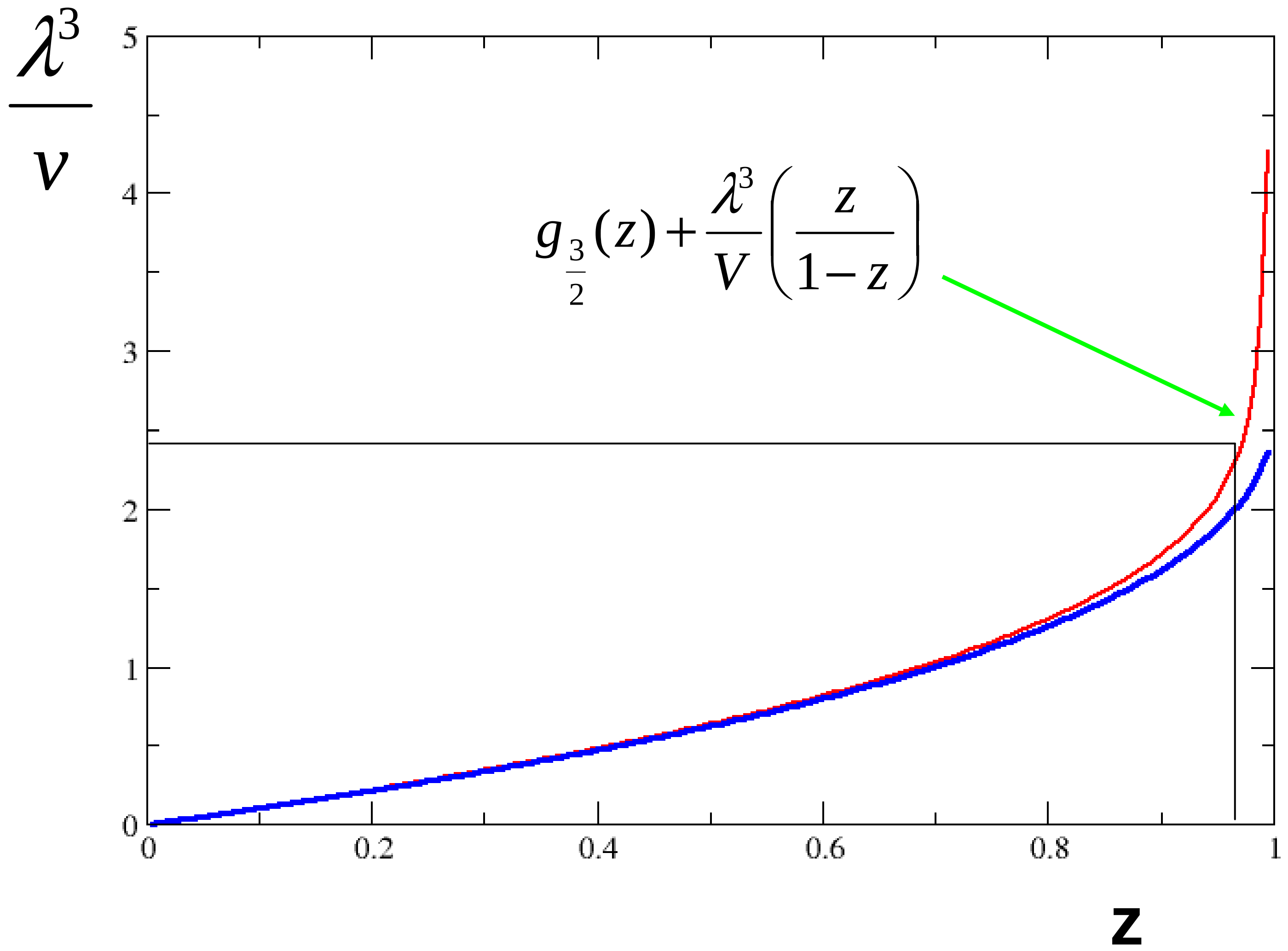
Entonces cuando $\frac{\lambda^3}{v} > g_{\frac{3}{2}}(1)$ $\rightarrow$ se debe cumplir

$$\boxed{\frac{\langle n_0 \rangle}{V} > 0}$$

Esto es lo que nos salva, pues de otra forma….

como $V \longrightarrow \infty \Longrightarrow \langle n_0 \rangle \longrightarrow \infty \Longrightarrow$ una fraccion macroscopica ocupa el nivel con $\mathbf{p} = 0$, esto se llama *condensacion de Bose-Einstein*

Tomar en cuenta que estamos considerando la población de un único nivel !!!!!



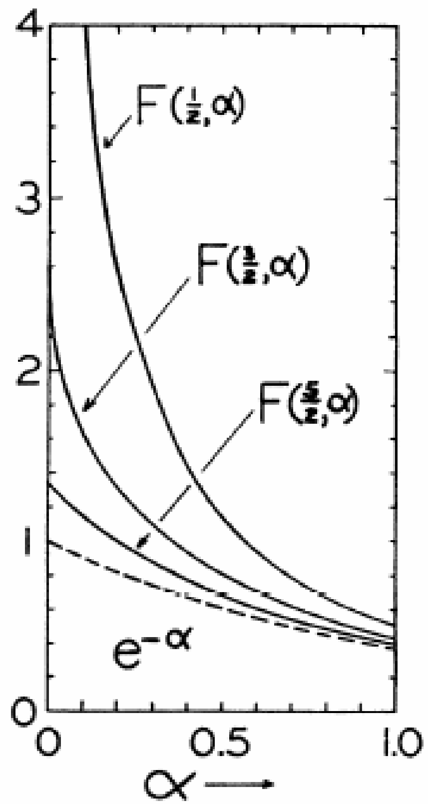
$\frac{\lambda^3}{v}$
$g_{\frac{3}{2}}(z)+\frac{\lambda^3}{V}\left(\frac{z}{1-z}\right)$
0
1
2
3
4
5
0
0.2
0.4
0.6
0.8
1
z

Cuidado con la notación!

$\exp(\alpha)=1/z \Rightarrow \alpha=\log(1/z)$
$0 \leq z \leq 1 \Rightarrow$

$$\boxed{0 \leq \alpha \leq \infty}$$

FIG. 2. The Bose-Einstein integral functions $F(\frac{1}{2},\alpha)$, $F(\frac{3}{2},\alpha)$, and $F(5/2,\alpha)$ for the range $0 \leqq \alpha \leqq 1$.



$$F(\tfrac{1}{2},\alpha) = 1.77\alpha^{-\frac{1}{2}} - 1.46 + 0.208\alpha - 0.0128\alpha^2,$$
$$F(\tfrac{3}{2},\alpha) = -3.54\alpha^{\frac{1}{2}} + 2.61 + 1.46\alpha - 0.104\alpha^2 + 0.00425\alpha^3,$$
$$F(5/2,\alpha) = 2.36\alpha^{\frac{3}{2}} + 1.34 - 2.61\alpha - 0.730\alpha^2 + 0.0347\alpha^3.$$

Note on the Bose-Einstein Integral Functions*

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(Received June 18, 1951)

Vemos que pasa con los niveles de energia con $\mathbf{p} \neq 0$

Por ejemplo:

$$\frac{<n_1>}{V} = \frac{1}{V}\frac{1}{\frac{1}{z}\exp(\beta\epsilon_1)-1} \leq \frac{1}{V}\frac{1}{\exp(\beta\epsilon_1)-1}$$

[$0 \leq z \leq 1$]

Dejamos de lado el fundamental $\Rightarrow$

como $2m\epsilon_i = \left(2\pi\hbar\right)^2 \frac{l_i}{V^{2/3}}$, con l_i el la suma de los cuadrados de eneteros no todos 0 , pues $\mathbf{p} = \frac{2\pi\hbar\,\mathbf{n}}{V^{1/3}}$

Entonces desarrollando en serie con $l_1 = 1$

$$\boxed{\frac{<n_1>}{V} \leq \frac{1}{V}\frac{2mV^{2/3}}{\left(2\pi\hbar\right)^2\beta} = \frac{2mV^{-1/3}}{\left(2\pi\hbar\right)^2\beta} \longrightarrow 0 \text{ si } V \longrightarrow \infty}$$

$$\exp(x) \approx 1 + x + \frac{1}{2}x^2 + \ldots$$

Las poblaciones parciales de los niveles distintos del fund. son muy diluidas
Luego es apropiado hablar del condensado por un lado y "el resto" por otro

La condicion $\frac{\lambda^3}{v} > g_{\frac{3}{2}}(1)$ define una region en el diagrama de fases $P - v - T$,en esta region podemos pensar en la coexistencia de dos fluidos ($\mathbf{p} = 0$ y $\mathbf{p} \neq 0$)

De esta forma, si fijamos v , podemos definir una temperatura critica $T_c \longrightarrow \frac{\lambda_c^3}{v} = g_{3/2}(1)$

como $\lambda = \sqrt{2\pi\hbar^2/mkT} \Rightarrow$

$$\boxed{\lambda_c^3 = \left[\sqrt{2\pi\hbar^2/mkT_c}\right]^3 = v g_{\frac{3}{2}}(1)}$$

luego λ_c^3 es del orden del volumen especifico ($g_{3/2}(1) \cong 2$)

entonces

$$kT_c = \frac{2\pi\hbar^2}{m(vg_{\frac{3}{2}}(1))^{\frac{2}{3}}}$$

$$(kT_c)^{3/2} = \frac{h^3}{(2\pi m)^{3/2}} \frac{1}{vg_{3/2}(1)} \Rightarrow$$

$$\frac{(2\pi m)^{3/2}}{h^3} vg_{3/2}(1) = \frac{1}{(kT_c)^{3/2}}$$

del mismo modo para una dada T existe un $v_c = \frac{\lambda^3}{g_{\frac{3}{2}}(1)}$

$\Rightarrow$ el fenomeno de aparicion de la fase macroscopica en **p** $= 0$ ocurre cuando $T < T_c$ o $v < v_c$

Podemos expresar la coexistencia de fases como:

i) una fase normal consistente en ($\frac{\lambda^3}{v} = g_{\frac{3}{2}}(1)$)

$$N_e = V\frac{(2\pi mkT)^{3/2}}{h^3} g_{\frac{3}{2}}(1) = N(\frac{T}{T_c})^{3/2}$$

(o sea que $g_{3/2}$ describe la parte normal)

ii) una fase condensada compuesta por

$N_0 = N - N_e = N(1 - (\frac{T}{T_c})^{3/2})$, acumuladas en $\mathbf{p} = 0$

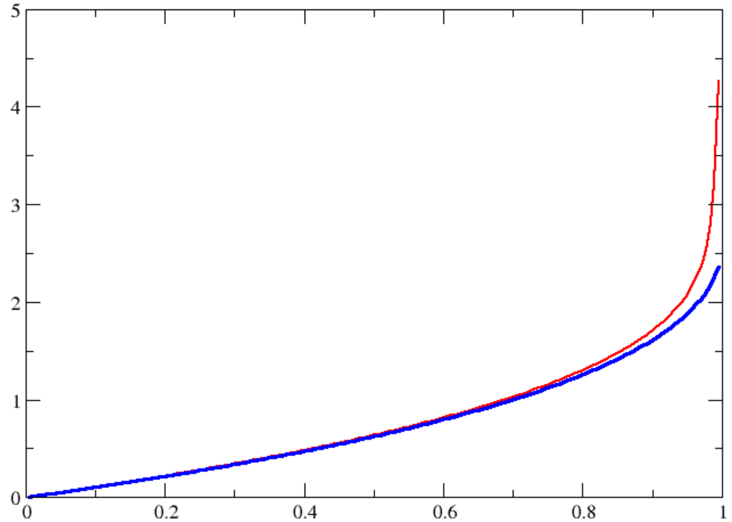
$N_0 = \underline{N - N_e} = N(1-(\frac{T}{T_c})^{3/2})$, acumuladas en $\mathbf{p} = 0$

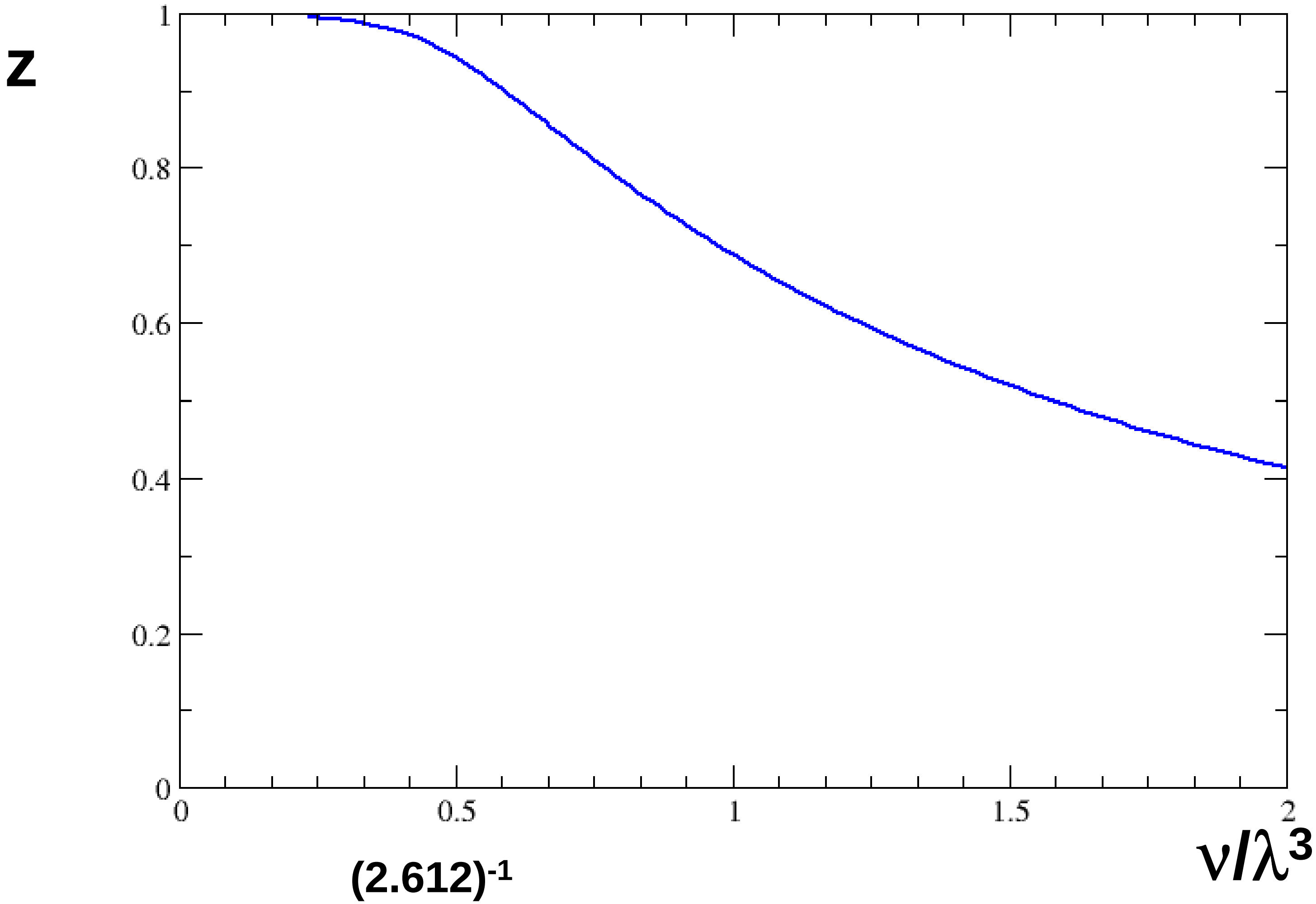
(expresado como diferencia)

Para encontrar z en funcion de T y v resolvemos numericamente

$$\frac{1}{v} = \frac{1}{\lambda^3} g_{3/2}(z) + \frac{1}{V}\frac{z}{1-z}$$

Si V es grande pero finito podemos usar el grafico





z
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
ν/λ3
(2.612)-1

En el limite $V \to \infty$ resulta

$$\boxed{z = 1 \longleftarrow \frac{\lambda^3}{v} \geq g_{\frac{3}{2}}(1)}$$

$$\boxed{\text{raiz de } \frac{\lambda^3}{v} = g_{\frac{3}{2}}(z) \leftarrow \frac{\lambda^3}{v} < g_{\frac{3}{2}}(1)}$$

Diagrama P-T del gas de Bose

Estudiamos la variacion de P con T a v cte.

$$\frac{P}{kT} = \frac{1}{\lambda^3} g_{5/2}(z)$$

-Si $T < T_c$ [$\Rightarrow$ z = 1]

$$P(T) = \frac{kT}{\lambda^3} g_{5/2}(1)$$

como $\lambda^{-3} \propto T^{3/2} \Rightarrow P(T) \propto T^{5/2}$ y ademas independiente de v

$\Rightarrow$ P(T) tiende a 0 con T $\rightarrow$0

-Si $T = T_c$ (punto de transicion)

$$P(T_c) = \left(\frac{2\pi m}{h^2}\right)^{3/2} (kT_c)^{\frac{5}{2}} g_{5/2}(1)$$

reemplazando $T_c = \frac{h^2}{2\pi mk}\left(\frac{N}{V g_{3/2}(1)}\right)^{2/3}$

$$\boxed{P(T_c) = \left(\frac{N}{V}kT_c\right)\frac{g_{5/2}(1)}{g_{3/2}(1)} \simeq 0.51\left(\frac{N}{V}kT_c\right)}$$

pues $g_{5/2}(1) \simeq 1.34$

$$P(T_c) = \left(\frac{2\pi m}{h^2}\right)^{3/2} (kT_c)^{5/2} g_{5/2}(1) = kT_c \left(\frac{2\pi m}{h^2}\right)^{3/2} (kT_c)^{3/2} g_{5/2}(1)$$

$$T_c = \frac{h^2}{2\pi m k}\left(\frac{N}{V g_{3/2}(1)}\right)^{2/3}$$

$$P(T_c) = kT_c \left(\frac{2\pi m}{h^2}\right)^{3/2} \left(k \frac{h^2}{2\pi m k}\left(\frac{N}{V g_{3/2}(1)}\right)^{2/3}\right)^{3/2} g_{5/2}(1)$$

$$\boxed{P(T_c) = \left(\frac{N}{V} kT_c\right)\left(\frac{g_{5/2}(1)}{g_{3/2}(1)}\right)}$$

-Si $T < T_c$ [$\Rightarrow$ z = 1]

$$\boxed{P(T) = \frac{kT}{\lambda^3} g_{5/2}(1)}$$

como $\lambda^3 \propto T^{3/2} \Rightarrow P(T) \propto T^{5/2}$ y ademas independiente de v

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reemplazando $T_c = \frac{h^2}{2\pi mk}\left(\frac{N}{Vg_{3/2}(1)}\right)^{2/3}$

$$\boxed{P(T_c) = \left(\frac{N}{V}kT_c\right)\frac{g_{5/2}(1)}{g_{3/2}(1)} \simeq 0.51\left(\frac{N}{V}kT_c\right)}$$

pues $g_{5/2}(1) \simeq 1.34$

resulta entonces que a la temperatura critica la presion del gas de Bose es aprox. la mitad que el de Boltzmann.

-**Si** $T > T_c$

$$\boxed{P(T) = \frac{N}{V}kT\frac{g_{5/2}(z)}{g_{3/2}(z)}}$$

y esto es lo mas simple que podemos escribir

pero si $T \gg T_c$

z sera pequeňo

N_0 despreciable frente a $N \Rightarrow$

proponemos una expansion del virial

$$\boxed{\frac{PV}{NkT} = \sum a_l \left(\frac{\lambda^3}{v}\right)^{l-1}}$$

Como sabemos tambien que

$$\boxed{\frac{P}{kT} = \frac{1}{\lambda^3} g_{5/2}(z)}$$

y

$$\boxed{\frac{N}{V} = \frac{N - N_0}{V} = \frac{1}{\lambda^3} g_{3/2}(z)}$$

Resulta de las dos ultimas que

$$\frac{PV}{NkT} = \sum a_l \left(\frac{\lambda^3}{v}\right)^{l-1} = \frac{g_{5/2}(z)}{g_{3/2}(z)}$$

de donde usando las expansiones de $g_\tau(z)$

Obtenemos

$$z+\frac{z^2}{2^{5/2}}+\frac{z^3}{3^{5/2}}+\ldots=\left[z+\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}+\ldots\right]*$$

$$*\left[\sum a_l\left(z+\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}+\ldots\right)^{l-1}\right]$$

$$z+\frac{z^2}{2^{5/2}}+\frac{z^3}{3^{5/2}}+\ldots=\left[z+\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}+\ldots\right]*$$

$$*\left[a_1+a_2\left(z+\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}+\ldots\right)+a_3\left(z+\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}+\ldots\right)^2+\right]$$

$$z=za_1$$

$$\frac{z^2}{2^{5/2}}=a_2z^2+a_1\frac{z^2}{2^{3/2}}$$

$$\frac{z^3}{3^{5/2}}=a_3z^3+2a_2\frac{z^3}{2^{3/2}}+a_1\frac{z^3}{3^{3/2}}$$

$$z = za_1 \Rightarrow \boxed{a_1 = 1}$$

$$\frac{z^2}{2^{5/2}} = a_2 z^2 + a_1 \frac{z^2}{2^{3/2}} \Rightarrow$$

$$\frac{1}{2^{5/2}} = a_2 + \frac{1}{2^{3/2}} \Rightarrow \boxed{a_2 = \frac{1}{2^{5/2}} - \frac{1}{2^{3/2}}}$$

$$\frac{z^3}{3^{5/2}} = a_3 z^3 + a_2 \frac{z^3}{2^{3/2}} + a_1 \frac{z^3}{3^{3/2}} \Rightarrow$$

$$\frac{1}{3^{5/2}} = a_3 + 2\left(\frac{1}{2^{5/2}} - \frac{1}{2^{3/2}}\right)\frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} \Rightarrow$$

$$\boxed{\frac{1}{3^{5/2}} - \left(\frac{1}{2^{5/2}} - \frac{1}{2^{3/2}}\right)\frac{2}{2^{3/2}} - \frac{1}{3^{3/2}} = a_3}$$

Los valores resultantes son (comparar con Fermi)

$$\boxed{\begin{aligned} a_1 &= 1 \\ a_2 &= -0.17678 \\ a_3 &= -0.00330 \end{aligned}}$$

Luego cuando T se hace muy grande se converge al limite clasico.

$$\frac{PV}{NkT} = 1 - 0.17678\frac{\lambda^3}{v} - 0.0033\frac{\lambda^6}{v^2}\ldots$$

$$\frac{PV}{NkT} = 1 - 0.17678\frac{1}{vT^{3/2}}\left[\frac{2\pi\hbar^2}{mk}\right]^{3/2} - 0.0033\frac{1}{v^2T^3}\left[\frac{2\pi\hbar^2}{mk}\right]^3\ldots$$

$$\boxed{PV = NkT - 0.17678\frac{1}{T^{1/2}}\frac{Nk}{v}\left[\frac{2\pi\hbar^2}{mk}\right]^{3/2} - 0.0033\frac{1}{T^2}[]\ldots}$$

Entonces para el gas de Bose la presión es menor que la del gas ideal y converge al mismo para $T \gg T_c$

Conocido P

$$U = -\left(\frac{\partial}{\partial\beta}\ln\Xi\right) = kT^2\left[\frac{\partial}{\partial T}\left(\frac{PV}{kT}\right)\right]$$

$$\frac{C_V}{Nk} = \frac{1}{Nk}\left(\frac{\partial U}{\partial T}\right)_{N,V}$$

Si $T < T_c$

$$\frac{C_v}{Nk} = \frac{3}{2}\frac{V}{N}g_{5/2}(1)\frac{d}{dT}\left(\frac{T}{\lambda^3}\right) = \frac{15}{4}g_{5/2}(1)\frac{v}{\lambda^3}$$

$$\left[\frac{T}{\lambda^3} \propto T^{5/2} \Rightarrow \frac{\partial}{\partial T} \to \frac{5}{2}\frac{1}{\lambda^3}\right]$$

o sea que $\frac{C_v}{Nk} \propto T^{3/2}$

Si $T = T_c$

$$\frac{C_v(T_c)}{Nk} = \frac{15}{4}\frac{g_{5/2}(1)}{g_{3/2}(1)} \simeq 1.925 > 1.5$$

Recordemos que :

$$T < T_c$$

$$P(T) = \frac{kT}{\lambda^3} g_{5/2}(1)$$

$$U(T) = kT^2 V g_{5/2}(1) \frac{\partial}{\partial T}\frac{1}{\lambda^3} \propto T^2 \frac{\partial}{\partial T} T^{3/2} \propto T^{5/2} \Rightarrow C_V \propto T^{3/2} \propto \frac{1}{\lambda^3}$$

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la del gas ideal y converge al mismo para $T \gg T_c$

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$\boxed{\text{Si } T > T_c}$

$$\frac{C_v}{Nk} = \frac{\partial}{\partial T}\left(\frac{3}{2} T \frac{g_{5/2}(z)}{g_{3/2}(z)} \right)$$

$$\frac{\partial}{\partial T} g_{3/2} = \frac{\partial}{\partial T} \frac{\lambda^3}{v} = -\frac{3}{2T} g_{3/2}$$

ademas

$$\frac{\partial}{\partial T} g_{3/2} = \frac{\partial z}{\partial T} \frac{\partial}{\partial z} g_{3/2} = \frac{\partial z}{\partial T} \frac{1}{z} g_{1/2} \Rightarrow -\frac{3}{2T} \frac{g_{3/2}}{g_{1/2}} = \frac{\partial z}{\partial T} \frac{1}{z}$$

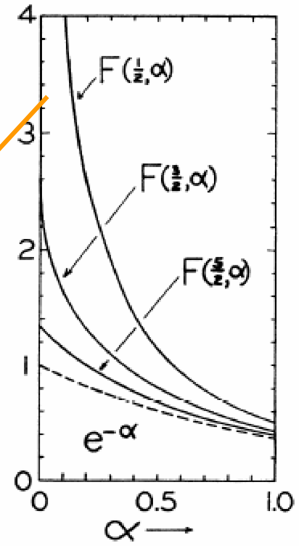
$$\frac{\partial}{\partial T} g_{5/2} = \frac{\partial z}{\partial T} \frac{\partial}{\partial z} g_{5/2} = \frac{\partial z}{\partial T} \frac{1}{z} g_{3/2} = -\frac{3}{2T} \frac{g_{3/2}^2}{g_{1/2}}$$

entonces

$$\frac{\partial}{\partial T}\left(\frac{3}{2} T \frac{g_{5/2}}{g_{3/2}} \right) = \frac{3}{2} \frac{g_{5/2}}{g_{3/2}} + \frac{3}{2} T \frac{g_{3/2} \partial_T g_{5/2} - g_{5/2} \partial_T g_{3/2}}{g_{3/2}^2}$$

$$\boxed{\frac{\partial}{\partial T}\left(\frac{3}{2} T \frac{g_{5/2}}{g_{3/2}} \right) = \frac{3}{2} \frac{g_{5/2}}{g_{3/2}} - \frac{9}{4} \frac{g_{3/2}}{g_{1/2}} + \frac{9}{4} \frac{g_{5/2}}{g_{3/2}} = \frac{15}{4} \frac{g_{5/2}}{g_{3/2}} - \frac{9}{4} \frac{g_{3/2}}{g_{1/2}}}$$

Si $T > T_c$



Esto resulta

$$\boxed{\frac{C_v}{Nk} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)}}$$

En $z = 1$ el segundo termino se anula y encontramos que C_v es continuo.

Si $T \gg T_c$

$$\boxed{\frac{C_v}{Nk} = \frac{3}{2}\left(\frac{\partial}{\partial T}\left(\frac{PV}{NK}\right)\right)}$$

usando el desarrollo del virial

$$\frac{C_v}{Nk} = \frac{3}{2}\left(\frac{\partial}{\partial T}\left(T\sum_{l=1} a_l\left(\frac{\lambda^3}{v}\right)^{l-1}\right)\right)$$

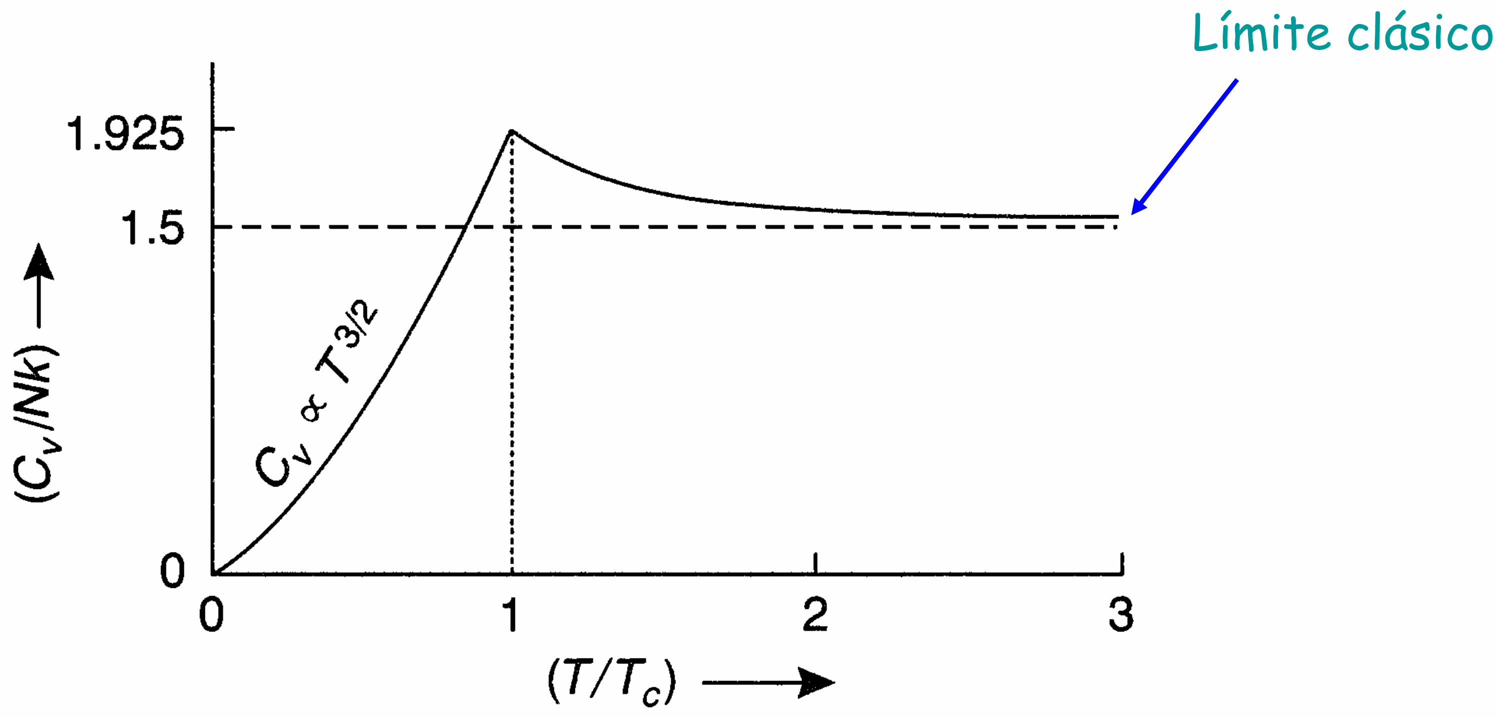
$$= \frac{3}{2}\sum_{l=1} a_l\left(\frac{\lambda^3}{v}\right)^{l-1} + (l-1)\left(\frac{-3}{2}\right)\sum_{l=1} a_l\left(\frac{\lambda^3}{v}\right)^{l-1}$$

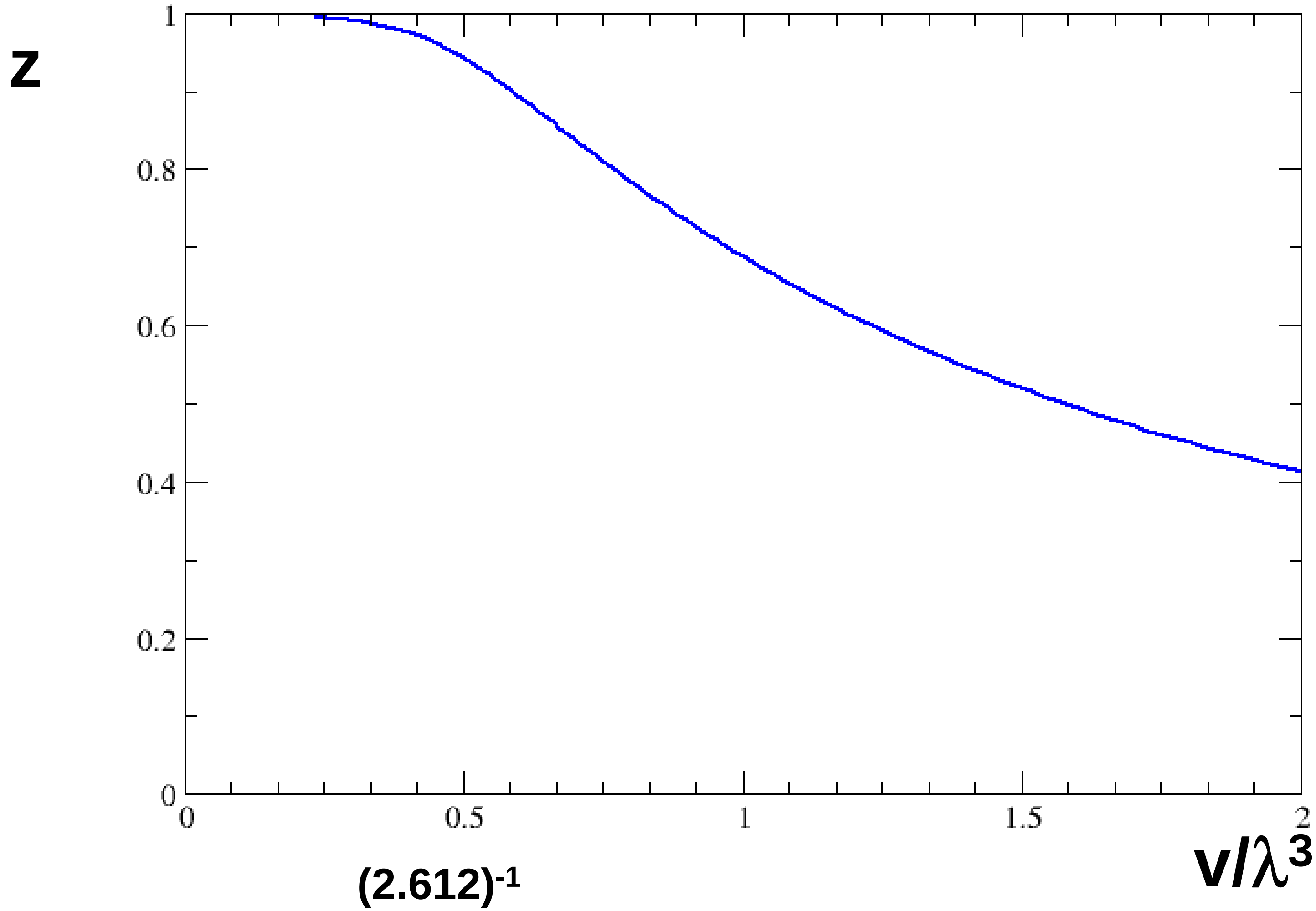
$$\frac{C_v}{Nk} = \frac{3}{2}\sum_{l=1}\frac{5-3l}{2}a_l\left(\frac{\lambda^3}{v}\right)^{l-1}$$

entonces

$$\boxed{\frac{C_v}{Nk} = \frac{3}{2}\left(1 + 0.0884\frac{\lambda^3}{v} + \ldots\right)}$$

o sea que converge uniformemente al limite clasico


Límite clásico
1.925
1.5
0
(C_V/Nk)
$C_V \propto T^{3/2}$
0
1
2
3
(T/T_c)



z
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
v/λ3
(2.612)-1

$$\frac{\partial}{\partial T} g_{3/2} = \frac{\partial}{\partial T} \frac{\lambda^3}{v} = -\frac{3}{2T} g_{3/2}$$

Pero también

$$\frac{\partial}{\partial T} g_{3/2} = \frac{\partial z}{\partial T} \frac{\partial}{\partial z} g_{3/2} = \frac{\partial z}{\partial T} \frac{1}{z} g_{1/2} \Rightarrow -\frac{3}{2T} \frac{g_{3/2}}{g_{1/2}} = \frac{\partial z}{\partial T} \frac{1}{z}$$

$$\frac{\partial}{\partial T} g_{5/2} = \frac{\partial z}{\partial T} \frac{\partial}{\partial z} g_{5/2} = \frac{\partial z}{\partial T} \frac{1}{z} g_{3/2} = -\frac{3}{2T} \frac{g_{3/2}^2}{g_{1/2}}$$

Entonces

$$\frac{\partial}{\partial T}\left(\frac{3}{2} T \frac{g_{5/2}}{g_{3/2}} \right) = \frac{3}{2} \frac{g_{5/2}}{g_{3/2}} + \frac{3}{2} T \frac{g_{3/2} \partial_T g_{5/2} - g_{5/2} \partial_T g_{3/2}}{g_{3/2}^2}$$

$$\frac{\partial}{\partial T}\left(\frac{3}{2} T \frac{g_{5/2}}{g_{3/2}} \right) = \frac{3}{2} \frac{g_{5/2}}{g_{3/2}} - \frac{9}{4} \frac{g_{3/2}}{g_{1/2}} + \frac{9}{4} \frac{g_{5/2}}{g_{3/2}}$$