Guía 4, clase 2

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Ec. Boltzmann

$$\frac{\partial f}{\partial t} + \underbrace{\{f, H\}}_{= \frac{\vec{p}}{m}\cdot\vec{\nabla}_r f + \vec{F}\cdot\vec{\nabla}_p f} = \left(\frac{\partial f}{\partial t}\right)_{col}$$

$$\left(\frac{\partial f}{\partial t}\right)_{col}(\vec{r}, \vec{p}_1, t) = \int d^3p_2\, d^3p_1'\, d^3p_2'\, \underbrace{\omega(\vec{p}_1 \vec{p}_2 | \vec{p}_1{}' \vec{p}_2{}')}_{\text{función de scattering}} (\underbrace{f_1'}_{= f(\vec{r}, \vec{p}_1{}', t)} f_2' - f_1 f_2)$$

Si tenga 2 parts en d^3r con momentos alrededor de $\vec{p}_1$ y $\vec{p}_2$, la probabilidad de que colisionen en dt y adquieran momentos alrededor de $\vec{p}_1{}'$ y $\vec{p}_2{}'$ es

$$w(\vec{p}_1{}' \, \vec{p}_2{}' \,|\, \vec{p}_1 \, \vec{p}_2) \, \frac{d^3p_1' \, d^3p_2' \, dt}{d^3r}$$

Propiedades de la función de scattering:

* $w(\vec{p}_1{}' \, \vec{p}_2{}' \,|\, \vec{p}_1 \, \vec{p}_2)$ sólo es $\neq 0$ si
$$\begin{cases} \vec{p}_1{}' + \vec{p}_2{}' = \vec{p}_1 + \vec{p}_2 \\ \dfrac{p_1'^2}{2m} + \dfrac{p_2'^2}{2m} = \dfrac{p_1^2}{2m} + \dfrac{p_2^2}{2m} \end{cases}$$

* Invariancia bajo paridad e inversión temporal

$$\longrightarrow \; w(\vec{p}_1{}' \, \vec{p}_2{}' \,|\, \vec{p}_1 \, \vec{p}_2) = w(\vec{p}_1 \, \vec{p}_2 \,|\, \vec{p}_1{}' \, \vec{p}_2{}')$$

Teorema $\cancel{H}$ de Boltzmann (~ 2ª ley):

η

Si no hay flujos a través de las paredes del recipiente, la función $\eta(t) \equiv \int d^3r\, d^3p\, f \log f$ cumple $\dot{\eta} \leq 0$, y

Paradoja de Loschmidt

la igualdad $\dot{\eta} = 0$ se da si y sólo si

$$\underbrace{f_1'}_{= f(\bar{r}, \bar{p}_1', t)} f_2' - f_1 \underbrace{f_2}_{= f(\bar{r}, \bar{p}_2, t)} = 0$$

(Balance detallado)

3. Una solución f de la ecuación de Boltzmann se dice de equilibrio si satisface $\partial f/\partial t = 0$ y, además, para toda colisión binaria $\boldsymbol{p}_1, \boldsymbol{p}_2 \to \boldsymbol{p}_1', \boldsymbol{p}_2'$ se tiene

$$f(\boldsymbol{r}, \boldsymbol{p}_1')f(\boldsymbol{r}, \boldsymbol{p}_2') = f(\boldsymbol{r}, \boldsymbol{p}_1)f(\boldsymbol{r}, \boldsymbol{p}_2).$$

Esta última condición se conoce como condición de balance detallado. Pruebe que, en ausencia de fuerzas externas, la distribución de equilibrio más general es la de Maxwell-Boltzmann

$$f(\boldsymbol{p}) = \frac{n}{(2\pi mkT)^{3/2}} \exp\left[-\frac{(p - m\boldsymbol{u})^2}{2mkT}\right],$$

donde n, $\boldsymbol{u}$ y T son constantes. Muestre que estas constantes son respectivamente la densidad, la velocidad media y la temperatura del gas. Calcule $\boldsymbol{q}$ y P_{ij} para esta distribución.

Equilibrio: $\frac{\partial f}{\partial t} = 0 \longrightarrow \dot{\eta} = 0 \longrightarrow$ Balance detallado

$\nexists$ flujos a través de las paredes del recipiente $\longrightarrow$ Aplica el teorema η

$$\Longrightarrow \left(\frac{\partial f}{\partial t}\right)_{col} = 0 \quad \underset{\text{Ec Boltzmann}}{\overset{}{\Longrightarrow}} \quad \{f, H\} = 0$$

$\Longrightarrow$ f constante del movimiento

$\nexists$ fuerzas externas: $H = \dfrac{p^2}{2m}$ $\Longrightarrow$ $\boxed{f(\vec{r}, \vec{p}) = f(\vec{p})}$

Balance detallado:

$$f(\vec r,\vec p_1')\, f(\vec r,\vec p_2') = f(\vec r,\vec p_1)\, f(\vec r,\vec p_2)$$

$$\Updownarrow$$

$$\log f(\vec r,\vec p_1') + \log f(\vec r,\vec p_2') = \log f(\vec r,\vec p_1) + \log f(\vec r,\vec p_2)$$

$\Longleftrightarrow$ $\log f$ se conserva en colisiones

$$\Rightarrow \log f = a(\vec r)\frac{p^2}{2m} + \vec b(\vec r)\cdot\vec p + c(\vec r)$$

$$\Rightarrow \boxed{f(\vec r,\vec p) = e^{a(\vec r)\frac{p^2}{2m} + \vec b(\vec r)\cdot\vec p + c(\vec r)} = C(\vec r)\, e^{-\beta(\vec r)\frac{(\vec p - m\vec\alpha(\vec r))^2}{2m}}}$$

$$\frac{p^2}{2m} - \vec\alpha(\vec r)\cdot\vec p + \frac{m}{2}\alpha^2(\vec r)$$

$$= e^{\underbrace{-\beta(\vec r)}_{=\,a}\frac{p^2}{2m} + \underbrace{\beta(\vec r)\vec\alpha(\vec r)}_{b}\cdot\vec p\ \underbrace{-\beta(\vec r)\frac{m}{2}\alpha^2(\vec r) + \log C(\vec r)}_{c}}$$

Junto con lo anterior ($\nexists$ fuerzas $\to$ $f = f(\bar{p})$),

$$\boxed{f(\bar{p}) = C\, e^{-\beta \frac{(\bar{p}-m\bar{\alpha})^2}{2m}}}$$

$$n = \int d^3p\, f(\bar{p}) = C \int d^3p\, \underbrace{e^{-\beta \frac{(\bar{p}-m\bar{\alpha})^2}{2m}}}_{= e^{-\beta \frac{1}{2m}\left[(p_x - m\alpha_x)^2 + (p_y - m\alpha_y)^2 + (p_z - m\alpha_z)^2\right]}}$$

$$= C \underbrace{\left[\int dp_x\, e^{-\beta \frac{(p_x - m\alpha_x)^2}{2m}}\right]}_{= \sqrt{\frac{2\pi m}{\beta}}} \left[\int dp_y\, e^{-\beta \frac{(p_y - m\alpha_y)^2}{2m}}\right] \left[\int dp_z\, e^{-\beta \frac{(p_z - m\alpha_z)^2}{2m}}\right]$$

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$$\Rightarrow \quad n = C \left(\frac{2\pi m}{\beta}\right)^{3/2} \quad \Rightarrow \quad \boxed{C = n \left(\frac{\beta}{2\pi m}\right)^{3/2}}$$

$$\vec{u} = \left\langle \frac{\vec{p}}{m} \right\rangle = \frac{1}{n} \int d^3p \, \frac{\vec{p}}{m} f(\vec{p}) =$$

$$= \frac{1}{\cancel{n}} \cancel{n} \left(\frac{\beta}{2\pi m}\right)^{3/2} \frac{1}{m} \int d^3p \, \vec{p} \, e^{-\beta \frac{(\vec{p} - m\vec{\alpha})^2}{2m}}$$

$$\underset{\uparrow}{=} \left(\frac{\beta}{2\pi m}\right)^{3/2} \frac{1}{m} \int d^3q \, (\underline{\vec{q}} + m\vec{\alpha}) \, e^{-\beta \frac{q^2}{2m}}$$

$\hookrightarrow 0$ (par x impar)

$$\vec{q} \equiv \vec{p} - m\vec{\alpha}$$

$$\Rightarrow \vec{p} = \vec{q} + m\vec{\alpha}$$

$$= \left(\frac{\beta}{2\pi m}\right)^{3/2} \bar{\alpha} \underbrace{\int d^3q \, e^{-\beta \frac{q^2}{2m}}}_{\left(\frac{2\pi m}{\beta}\right)^{3/2}} = \bar{\alpha} \Rightarrow \boxed{\bar{\alpha} = \bar{U}}$$

$$\frac{3}{2} kT = \left\langle \frac{(\bar{p} - m\bar{U})^2}{2m} \right\rangle =$$

$$= \frac{1}{n} \frac{1}{2m} \int d^3p \, (\bar{p} - m\bar{U})^2 \, f(\bar{p})$$

$$= \left(\frac{\beta}{2\pi m}\right)^{3/2} \underbrace{\int d^3p \, \frac{(\bar{p} - m\bar{U})^2}{2m} e^{-\beta \frac{(\bar{p} - m\bar{U})^2}{2m}}}$$

$$= -\frac{d}{d\beta}\int d^3p \; e^{-\beta\frac{(\bar{p}-m\bar{v})^2}{2m}}$$

$$= -\frac{d}{d\beta}\left(\frac{2\pi m}{\beta}\right)^{3/2}$$

$$\Rightarrow \quad \frac{3}{2}kT = -\left(\frac{\beta}{2\pi m}\right)^{3/2}\frac{d}{d\beta}\left(\frac{2\pi m}{\beta}\right)^{3/2} = -\frac{d}{d\beta}\log\left(\frac{2\pi m}{\beta}\right)^{3/2}$$

$$= -\frac{3}{2}\frac{d}{d\beta}\log\frac{2\pi m}{\beta} = +\frac{3}{2}\frac{d}{d\beta}\log\beta$$

$$= \frac{3}{2\beta} \quad \Rightarrow \quad \boxed{\beta = \frac{1}{kT}}$$

Otra forma:

$$\left(\frac{\beta}{2\pi m}\right)^{3/2}\int d^3p\ \underbrace{(\bar{p}-m\bar{u})^2}_{\shortparallel} e^{-\beta\frac{(\bar{p}-m\bar{u})^2}{2m}} = f_x(p_x)\, f_y(p_y)\, f_z(z) \qquad = \sigma_x^2+\sigma_y^2+\sigma_z^2$$

$\uparrow$ gaussiana

$$(p_x - mu_x)^2 + (p_y - mu_y)^2 + (p_z - mu_z)^2$$

$$\Rightarrow \boxed{f(\bar{p}) = \frac{n}{(2\pi m k T)^{3/2}}\, e^{-\frac{(\bar{p}-m\bar{u})^2}{2mkT}}}$$

Distribución de Maxwell-Boltzmann

$$\vec{q} = \vec{J}_{\frac{(\bar{p}-m\bar{u})^2}{2m}} \underset{\uparrow}{=} n\left\langle \frac{(\bar{p}-m\bar{u})^2}{2m}\,\frac{\bar{p}-m\bar{u}}{m}\right\rangle =$$

Vel. del

$\frac{}{2m}$

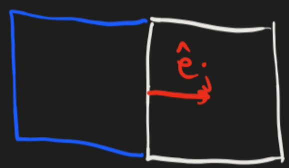
$$\bar{J}_x = n \left\langle x \frac{\bar{p} - m\bar{v}}{m} \right\rangle$$

Vel. del elemento de área

$$= \int d^3p \, \frac{(\bar{p} - m\bar{v})^2}{2m} \frac{\bar{p} - m\bar{v}}{m} \underbrace{f(\bar{p})}_{\propto\, e^{-\frac{(\bar{p}-m\bar{v})^2}{2m}}} = \boxed{0 = \bar{q}}$$

par x impar

$$P_{ij} = (j_{p_i})_j = \bar{J}_{p_i} \cdot \hat{e}_j = F_i \text{ por unidad de área}$$



$$P_{ij} = n \left\langle p_i \frac{p_j - mU_j}{m} \right\rangle = n \left\langle (p_i - mU_i) \frac{p_j - mU_j}{m} \right\rangle$$

$$\chi = p_i$$

$$\bar{v} = \bar{U}$$

$$= \frac{1}{m} \int d^3p \, (p_i - mU_i)(p_j - mU_j) f(\vec{p})$$

$$= \frac{1}{m} \frac{n}{(2\pi m kT)^{3/2}} \int d^3p \, (p_i - mU_i)(p_j - mU_j) \, e^{-\frac{(\vec{p} - m\vec{U})^2}{2mkT}}$$

$$P_{xy} = \frac{1}{m} \frac{n}{(2\pi m kT)^{3/2}} \left[\int dp_x \, (p_x - mU_x) \, e^{-\frac{(p_x - mU_x)^2}{2mkT}} \right] \times \ldots$$

$0 \leftarrow$ par $\times$ impar

0

$= 0$

En gral, $P_{ij} = 0 \quad i \neq j$

$$P_{ii} = \frac{1}{m} \frac{n}{(2\pi m k T)^{3/2}} \underbrace{\int d^3p \, (p_i - m u_i)^2 \, e^{-\frac{(\bar{p} - m\bar{u})^2}{2mkT}}}_{\parallel \leftarrow \vec{q} = \bar{p} - m\bar{u}}$$

$$\int d^3q \quad q_i^2 \quad e^{-\frac{q^2}{2mkT}} \quad \longleftarrow \text{Indep. de } i$$

$$\parallel$$

$$\frac{1}{3} \int d^3q \underbrace{\left(q_x^2 + q_y^2 + q_z^2\right)}_{\substack{\parallel \\ q^2}} e^{-q^2/2mkT}$$

$$\frac{1}{3}\int d^3q \; q^2 \; e^{-q^2/2mkT}$$

$$\Rightarrow P_{ii} = \frac{1}{m}\,\frac{n}{(2\pi m k T)^{3/2}}\,\frac{1}{3}\int d^3q \; q^2 \; e^{-\frac{q^2}{2mkT}}$$

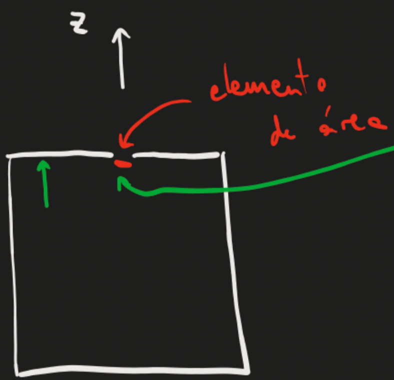
$$= \frac{2m}{3m}\,\underbrace{\frac{n}{(2\pi m k T)^{3/2}}\int d^3p \; \frac{(\bar{p}-m\bar{u})^2}{2m} \; e^{-\frac{(\bar{p}-m\bar{u})^2}{2mkT}}}_{= \; n\,\underbrace{\left\langle \frac{(\bar{p}-m\bar{u})^2}{2m} \right\rangle}_{= \; \frac{3}{2}kT}}$$

$$= \frac{2}{3}\,\frac{3}{2}\, nkT = nkT = P_{ii}$$

$$\Longrightarrow \boxed{P_{ij} = nkT\,\delta_{ij}}$$

Recuperamos ec. estado gas ideal

5. Un gas en equilibrio está a temperatura T y tiene densidad de partículas n. Su función de distribución es la de Maxwell-Boltzmann. Sobre una de las paredes del recipiente que contiene al gas, hay un pequeño orificio de área A. Asumiendo que pueda despreciarse el efecto del orificio sobre la distribución de equilibrio del gas, calcular el número de partículas que escapan por unidad de tiempo.



En este punto,

$$f(\vec{p}) = \begin{cases} MB & \text{si } p_z > 0 \\ 0 & p_z < 0 \end{cases}$$

flujo de partículos:

$$\Phi = \vec{J}\cdot\underbrace{\hat{n}}_{=\hat{z}} = \vec{J}\cdot\hat{z} = j_z = n\left\langle \frac{p_z}{m}\right\rangle = \int d^3p\ \frac{p_z}{m}\, f(\vec{p})$$

$$= \frac{1}{m}\frac{n}{(2\pi mkT)^{3/2}}\int d^3p\ p_z\, e^{-\frac{p^2}{2mkT}}$$

$$= \frac{1}{m}\frac{n}{(2\pi mkT)^{3/2}}\underbrace{\int_{-\infty}^{\infty} dp_x\, e^{-\frac{p_x^2}{2mkT}}}_{=\sqrt{2\pi mkT}}\ \underbrace{\int_{-\infty}^{\infty} dp_y\, e^{-\frac{p_y^2}{2mkT}}}_{=\sqrt{2\pi mkT}}\int_0^{\infty} dp_z\, p_z\, e^{-\frac{p_z^2}{2mkT}}$$

$$= \frac{1}{m} \frac{n}{\sqrt{2\pi m k T}} \int_0^\infty dp_z \underbrace{p_z \, e^{-\frac{p_z^2}{2mkT}}}$$

$$\underbrace{- mkT \frac{d}{dp_z} e^{-\frac{p_z^2}{2mkT}}}$$

$$- mkT \int_0^\infty dp_z \frac{d}{dp_z} e^{-\frac{p_z^2}{2mkT}}$$

$$- mkT \; e^{-p_z^2/2mkT} \Big|_0^\infty = + mkT$$

$$\Rightarrow \Phi = \frac{1}{\cancel{m}} \frac{n}{\sqrt{2\pi m k T}} \cancel{m} k T = \boxed{n \sqrt{\frac{kT}{2\pi m}} = \Phi}$$

$$\boxed{\# \text{ part q escapen por u. de tiempo} = \Phi A}$$

8. Escribir la función de distribución de equilibrio de un gas en un potencial externo $\phi(\boldsymbol{r})$. Sea n_0 la densidad del aire en la superficie terrestre. Determine la densidad $n(z)$ a una altura z suponiendo equilibrio. Despreciar la variación de g con la altura.

f constante del mov. $\Rightarrow f(\bar{r}, \bar{p}) = h\left(\frac{p^2}{2m} + \phi(\bar{r})\right)$

Aplicando balance detallado,

$$f(\bar{r},\bar{p}) = C\, e^{-\beta\left(\frac{p^2}{2m} + \phi(\bar{r})\right)}$$

$$n(\bar{r}) = \int d^3p\, f(\bar{r},\bar{p}) = C\, e^{-\beta\phi(\bar{r})} \underbrace{\int d^3p\, e^{-\beta\frac{p^2}{2m}}}_{= \left(\frac{2\pi m}{\beta}\right)^{3/2}}$$

$$\Rightarrow\ n(\bar{r}) = C\, e^{-\beta\phi(\bar{r})} \left(\frac{2\pi m}{\beta}\right)^{3/2} \longrightarrow C = \left(\frac{\beta}{2\pi m}\right)^{3/2} \underbrace{n(\bar{r})\, e^{\beta\phi(\bar{r})}}_{\text{cte}}$$

$$\Rightarrow\ \boxed{n(\bar{r}) = n_0\, e^{-\beta\phi(\bar{r})}}$$

n en el cero de potencial

$$\Longrightarrow \quad f(\bar{r}, \bar{p}) = n(\bar{r}) \, \cancel{e^{\beta \phi(\bar{r})}} \left(\frac{\beta}{2\pi m} \right)^{3/2} e^{-\beta \left(\frac{p^2}{2m} + \cancel{\phi(\bar{r})} \right)}$$

$$= n(\bar{r}) \left(\frac{\beta}{2\pi m} \right)^{3/2} e^{-\beta \frac{p^2}{2m}}$$

$$\frac{3}{2} kT = \left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{n} \int d^3p \, \frac{p^2}{2m} f(\bar{p})$$

$$= \left(\frac{\beta}{2\pi m} \right)^{3/2} \int d^3p \, \frac{p^2}{2m} e^{-\beta \frac{p^2}{2m}}$$

$$= \frac{3}{2\beta} \quad \Longrightarrow \quad \beta = \frac{1}{kT}$$

$$\Longrightarrow \boxed{\begin{aligned} f(\bar{r}, \bar{p}) &= \frac{n(\bar{r})}{(2\pi m k T)^{3/2}} \, e^{-\frac{p^2}{2mkT}} \\ n(\bar{r}) &= n_0 \, e^{-\frac{\phi(\bar{r})}{kT}} \end{aligned}}$$

$$P_{ij} = \frac{1}{n} \int d^3p \, \cdots \underset{\substack{\uparrow \\ \text{Igual} \\ \text{q MB}}}{=} n k T \, \delta_{ij}$$