Guía 6, clase 1

Monday, 31 May 2021 17:49

Bosones (clase 1 guía 5):

$$\boxed{\begin{aligned} E &= \frac{3}{2} kT\, g_s \frac{V}{\lambda^3} g_{5/2}(z) \\ N &= \cancel{g_s \frac{V}{\lambda^3} g_{3/2}(z)} \end{aligned}}$$

$$g_\nu(z) = I_\nu^-(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty dx \frac{x^{\nu-1}}{z^{-1}e^x - 1}$$

Integrales de Bose

$$z < e^{\beta \epsilon_{min}} = 1$$

Límite termo $N \to \infty$

$$N_0 = \frac{g_s}{z^{-1} - 1} = N$$

BE $N_i = \dfrac{1}{z^{-1}e^{\beta\epsilon_i} - 1}$ $\qquad$ $T=0$

$\Rightarrow$ En el límite termo, $z \leq 1$

$$g_\nu(1) = \frac{1}{\Gamma(\nu)} \int_0^\infty dx \, \frac{x^{\nu-1}}{e^x - 1}$$

$$\frac{x^{\nu-1}}{e^x-1} \approx \frac{x^{\nu-1}}{x} = x^{\nu-2} \quad x \ll 1$$

diverge para $\nu \leq 1$

finito para $\nu > 1$

$$g_\nu(1) = \sum_{m=1}^\infty \frac{1}{m^\nu} = \zeta(\nu)$$

función zeta de Riemann

Repaso teórico ec. (61)

$\nu > 1$

Ec. correcta para N:

$$N = \begin{cases} g_s \dfrac{V}{\lambda^3} g_{3/2}(z) & z < 1 \quad (1) \quad \longleftrightarrow \quad T > T_c \\ N_0 + g_s \dfrac{V}{\lambda^3} \zeta(3/2) & z = 1 \quad (2) \quad \longleftrightarrow \quad T \leq T_c \end{cases}$$

$(1): \quad N < \underline{g_s \frac{V}{\lambda^3} \zeta(3/2)} \quad \longleftrightarrow \quad T > T_c$

$(2): \quad N \geq g_s \frac{V}{\lambda^3} \zeta(3/2) \quad \longleftrightarrow \quad T \leq T_c$

$T_c: \quad \boxed{N = g_s \frac{V}{\lambda_c^3} \zeta(3/2)} \longrightarrow$ De acá despejan T_c

Fracción de partículas en el fundamental $\quad f = \frac{N_0}{N}$

$T > T_c \longleftrightarrow \underline{z < 1}: \quad N_0 = \frac{g_s}{z^{-1} - 1}$ finito

distr. BE $\quad N_i = \frac{1}{z^{-1} e^{\beta \epsilon_i} - 1}$

$N \to \infty$ (límite termo)

$\Rightarrow \quad f = \frac{N_0}{N} = 0$ (límite termo)

$(T > T_c)$

$T \leq T_c \longleftrightarrow z = 1$:

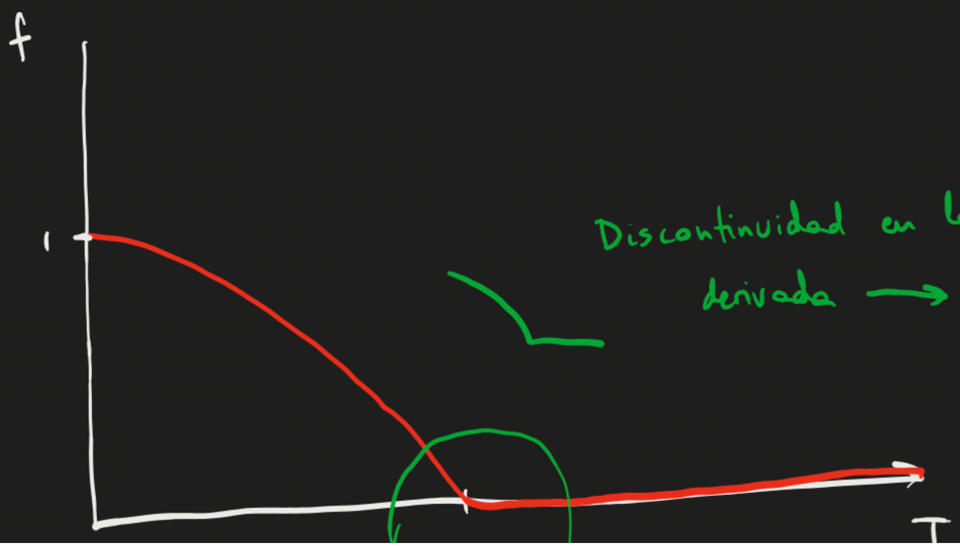
$$f = \frac{N_0}{N} = \frac{N - g_s \frac{V}{\lambda^3}\zeta(3/2)}{N} = 1 - g_s \frac{V}{N\lambda^3}\zeta(3/2)$$

$$= 1 - \left(\frac{\lambda_c}{\lambda}\right)^3 = \boxed{1 - \left(\frac{T}{T_c}\right)^{3/2} = f} \qquad (T \leq T_c)$$

$\uparrow$

$$N = g_s \frac{V}{\lambda_c^3}\zeta(3/2)$$

$$\Rightarrow g_s V \zeta(3/2) = N\lambda_c^3$$



Discontinuidad en la derivada $\longrightarrow$ Transición fase

$f \neq 0$ a $T < T_c$: <u>Condensación de Bose-Einstein</u>

Parts. distinguibles:

$$f = \frac{N_0}{N} \underset{\substack{\uparrow \\ \text{Distinguibles}}}{=} P_0 = \frac{e^{-\beta 0}}{Z_1} = \frac{1}{Z_1} = \frac{\lambda^3}{V} \xrightarrow[\substack{\text{límite} \\ \text{termo} \\ (V \to \infty) \\ \text{y } T>0}]{} 0$$

$\Rightarrow P(n_0$ parts. en el fund.$)$ binomial

$$\Rightarrow N_0 = \langle n_0 \rangle \underset{\substack{\uparrow \\ \text{guía 2}}}{=} N P_0$$

$\leftarrow$ prob. de q una dada part. esté en el fund

Ejemplo sencillo: tiro 2 monedas, prob. de q los 2 sean cara?

* Distinguibles : cc , cx , xc , xx

$$p = \frac{1}{4}$$

* Indistinguibles : 2c , 2x , 1c 1x

$$p = \frac{1}{3} > \frac{1}{4}$$

A E y P para $T \leq T_c$ tb hay que agregarle la contribución del fundamental ?

$E_0 = 0 \longrightarrow$ A E no hay q agregarle nada

$$P_0 = \frac{kT}{V} \log Z_0 = - g_s \frac{kT}{V} \log(1 - z)$$

$$\log Z_i = - \log(1 - z e^{-\beta \epsilon_i})$$

$p = -\frac{\boxed{\cdot}}{V} = \frac{kT}{V}\log z$

clase 1, guía 5

$$= -g_s\frac{kT}{V}\log\frac{g_s}{N_0} = g_s\frac{kT}{V}\log\frac{N_0}{g_s} \leq g_s\frac{kT}{V}\log\frac{N}{g_s} \xrightarrow[\text{límite termo}]{} 0$$

$$N_0 = \frac{g_s}{z^{-1}-1}$$

$$\rightarrow z = \frac{1}{1+\frac{g_s}{N_0}} \simeq 1 - g_s/N_0 \Rightarrow 1 - z = \frac{g_s}{N_0}$$

$T \leq T_c \Rightarrow N_0 >> 1$

$\Rightarrow P_0 = 0 \Rightarrow$ Tampoco hay q agregar nada a p.

$$E = \frac{3}{2}kT\, g_s\frac{V}{\lambda^3}\zeta(5/2) = \frac{3}{2}NkT\left(\frac{\lambda_c}{\lambda}\right)^3\frac{\zeta(5/2)}{\zeta(3/2)}$$

$T \leq T_c$

$$N = g_s\frac{V}{\lambda_c^3}\zeta(3/2)$$

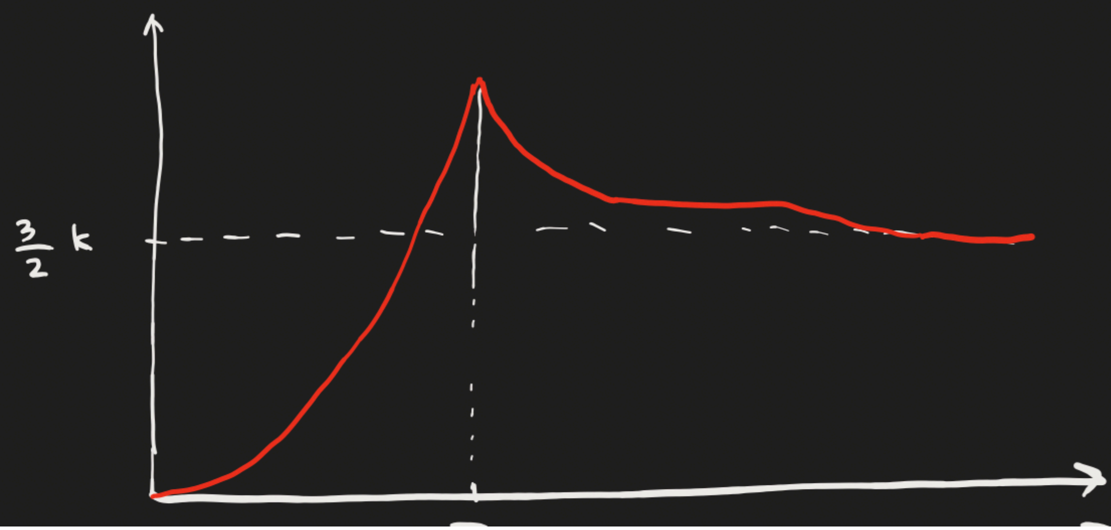
$$\Rightarrow g_s V = \frac{N \lambda_c^3}{\zeta(3/2)}$$

$$= \boxed{\frac{3}{2} NkT \left(\frac{T}{T_c}\right)^{3/2} \frac{\zeta(5/2)}{\zeta(3/2)} = E \qquad (T \le T_c)}$$

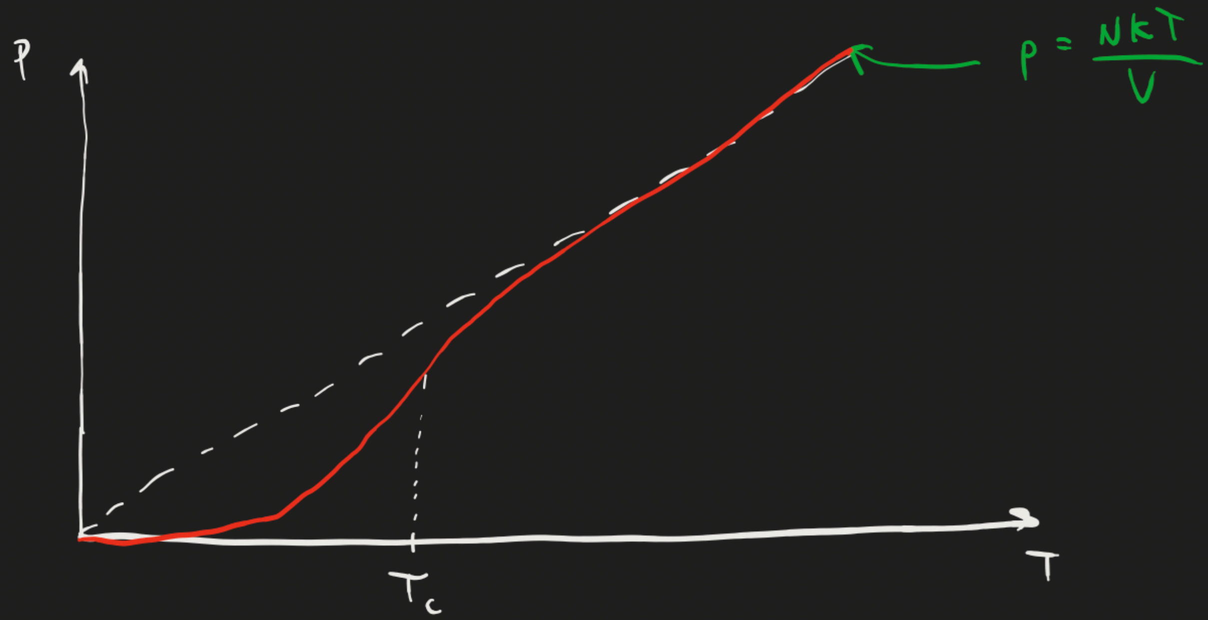
$$\Rightarrow c_v = \frac{1}{N} \frac{\partial E}{\partial T} = \boxed{\frac{3}{2} k \left(\frac{T}{T_c}\right)^{3/2} \frac{5}{2} \frac{\zeta(5/2)}{\zeta(3/2)} = c_v \qquad (T \le T_c)}$$

12

1.3



$$p = \frac{2}{3}\frac{E}{V} = \frac{NkT}{V}\left(\frac{T}{T_c}\right)^{3/2}\frac{\zeta(5/2)}{\zeta(3/2)}$$



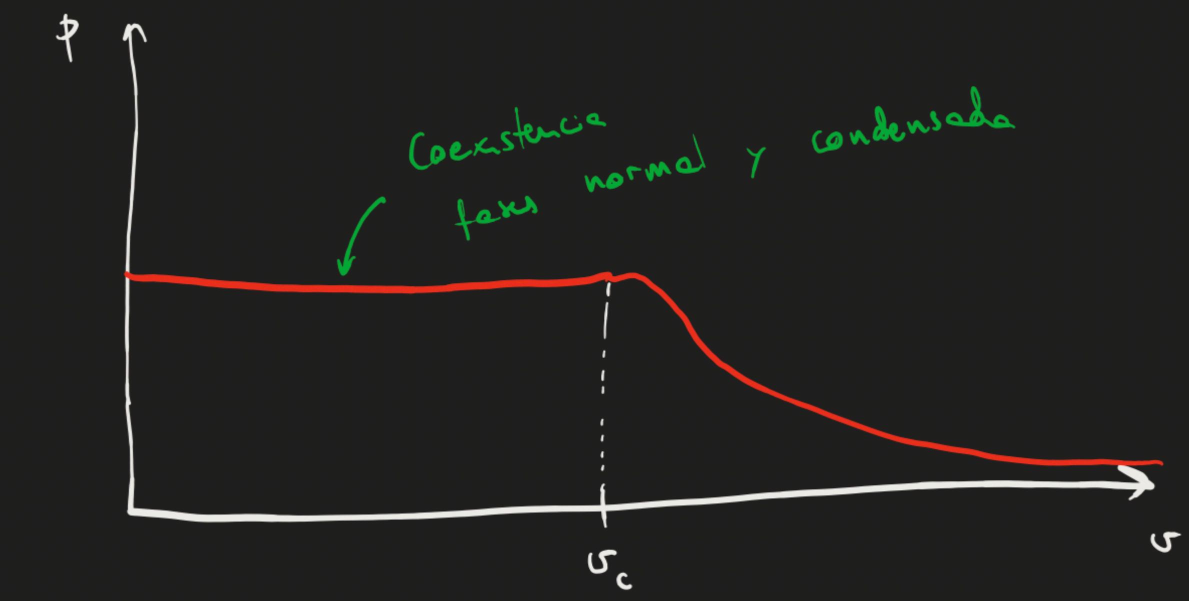
Condensación de BE ocurre cuando $N \geq g_s \frac{V}{\lambda^3}\zeta(3/2)$

$$\longleftrightarrow \quad g_s \frac{v}{\lambda^3} \zeta(3/2) \leq 1 \longleftrightarrow$$ A T fija,

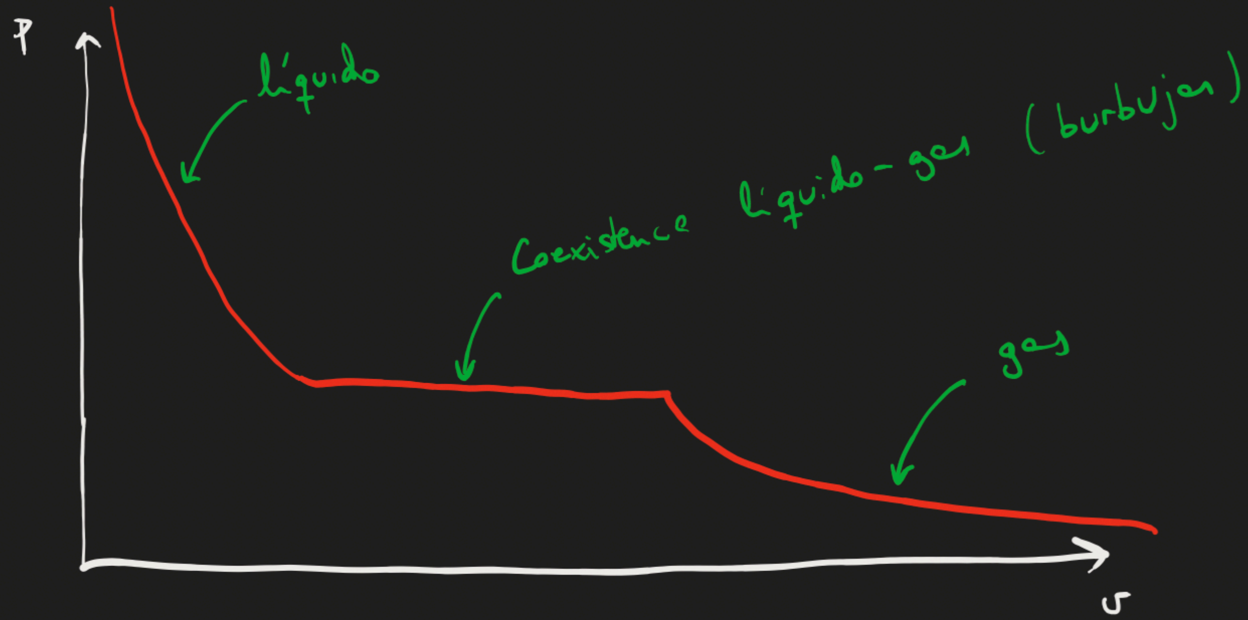
$$v \leq v_c = \frac{\lambda^3}{g_s \zeta(3/2)}$$

Para $v \leq v_c$, $\quad p = \frac{2}{3}\frac{E}{V} = kT\, g_s \frac{1}{\lambda^3} \zeta(5/2)$

No depende del volumen!



Comparemos con la isoterma de Van der Waals



=

Gas de bosones en $\underline{d}$ dimensiones ($\sim$ prob. 1)

$$\log Z = -\sum_i \log\left(1 - z e^{-\beta \epsilon_i}\right) = -\int_0^\infty d\epsilon \; g(\epsilon) \log\left(1 - z e^{-\beta \epsilon}\right)$$

$$\int_0^\infty d\epsilon \; g(\epsilon) f(\epsilon) = \sum_i f(\epsilon_i) = g_s \int \frac{d^d q \, d^d p}{h^d} f(p^2/2m)$$

$$= g_s \frac{V}{h^d} \int d^d p \; f(p^2/2m)$$

$$= g_s \frac{V}{h^d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dp \; p^{d-1} f(p^2/2m)$$

$\uparrow$

Área de la esfera de radio 1 en espacio de d dimensiones $\Omega_d = \dfrac{2\pi^{d/2}}{\Gamma(d/2)}$

$$d^d p = A \, dp$$

$$= \Omega_d \, p^{d-1} dp$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} p^{d-1} dp$$

$$= g_s \frac{V}{h^d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \left(\frac{2m}{2}\right)^{d/2} \int_0^\infty d\epsilon \; \epsilon^{\frac{d-1}{2}-\frac{1}{2}} f(\epsilon)$$

$$\uparrow$$

$$\epsilon = p^2/2m$$

$$p = \sqrt{2m\epsilon}$$

$$dp = \frac{\sqrt{2m}}{2\sqrt{\epsilon}} d\epsilon$$

$$\Longrightarrow \boxed{g(\epsilon) = g_s \; V \left(\frac{2\pi m}{h^2}\right)^{d/2} \frac{1}{\Gamma(d/2)} \epsilon^{\frac{d}{2}-1}}$$

$$\Longrightarrow \log Z = -\int_0^\infty d\epsilon \; g(\epsilon) \log\left(1 - z e^{-\beta\epsilon}\right) =$$

$$= -g_s \; V \left(\frac{2\pi m}{h^2}\right)^{d/2} \frac{1}{\Gamma(d/2)} \int_0^\infty d\epsilon \; \epsilon^{\frac{d}{2}-1} \log\left(1 - z e^{-\beta\epsilon}\right)$$

$$= -g_s V \left(\frac{2\pi m k T}{h^2}\right)^{d/2} \frac{1}{\Gamma(d/2)} \int_0^\infty dx \underbrace{x^{\frac{d}{2}-1}}_{= f'} \underbrace{\log(1 - z e^{-x})}_{= g}$$

$$\uparrow$$

$$x = \beta \epsilon$$

$$\left(\frac{2\pi m k T}{h^2}\right)^{d/2} = \left(\frac{h}{\sqrt{2\pi m k T}}\right)^{-d} = \frac{1}{\lambda^d}$$

$$= -g_s \frac{V}{\lambda^d} \frac{1}{\Gamma(d/2)} \frac{1}{d/2} \left\{ \cancel{x^{d/2} \log(1 - z e^{-x}) \Big|_0^\infty} - \int_0^\infty dx \frac{x^{d/2} \, z e^{-x}}{1 - z e^{-x}} \right\}$$

$$= + g_s \frac{V}{\lambda^d} \frac{1}{\frac{d}{2}\Gamma(d/2)} \int_0^\infty dx \frac{x^{d/2}}{z^{-1} e^{x} - 1}$$

$$\frac{d}{2}\Gamma(d/2) = \leftarrow \Gamma(\nu+1) = \nu\Gamma(\nu)$$

$\Gamma\left(\frac{d}{2}+1\right)$

" $g_{\frac{d}{2}+1}(z)$

$$\Rightarrow \boxed{\log Z = \overbrace{g_s \frac{V}{\lambda^d}\, g_{\frac{d}{2}+1}(z)}^{= pV/kT}} \longrightarrow E = \frac{d}{2} kT\, g_s \frac{V}{\lambda^d}\, g_{\frac{d}{2}+1}(z)$$

$$\boxed{E = \frac{d}{2} pV}$$

$$\Rightarrow N = z\, \partial_z \log Z = g_s \frac{V}{\lambda^d}\, g_{\frac{d}{2}}(z)$$

$g_\nu(1)$ diverge cuando

$\nu \leq 1$

$$d=2 \longrightarrow N = g_s \frac{V}{\lambda^2}\, g_1(z)$$

$$T_c : \quad N = g_s \frac{V}{\lambda_c^2}\, g_1(1) \longrightarrow T_c \propto \frac{1}{g_1(1)} = 0$$

$g_1(1) = \infty$

$$N = g_s \frac{V}{\lambda_c^d} \zeta(d/2) = g_s V \left(\frac{2\pi m k T_c}{h^2} \right)^{d/2} \zeta(d/2)$$

$$\Rightarrow T_c = \frac{h^2}{2\pi m k} \left(\frac{N}{V} \frac{1}{g_s \zeta(d/2)} \right)^{1/d}$$