Gas de Fermi

Lectura: R. K. Pathria & P. D. Beale, Cap. 8.

» Gas no-interactuante en el Gran Canónico

$$Z_{GC} = \sum_{N=0}^{\infty} \left[z^N {\sum_{\{n_i\}}}' e^{-\beta \sum_i n_i \epsilon_i} \right] = \sum_{N=0}^{\infty} \left[{\sum_{\{n_i\}}}' \prod_i \left(z e^{-\beta \epsilon_i} \right)^{n_i} \right] = \sum_{n_0} \sum_{n_1} \cdots = \prod_i \sum_{n_i} \left(z e^{-\beta \epsilon_i} \right)^{n_i}$$

Recordamos, si es Fermión $n_i = 0$ o 1, si es Bosón $n_i = 0, 1, \cdots$.

$$Z_{GC} = \begin{cases} \prod_i \dfrac{1}{1 - z e^{-\beta \epsilon_i}} & \text{B} \\ \prod_i \left(1 + z e^{-\beta \epsilon_i}\right) & \text{F} \end{cases}$$

Cerramos la termodinámica con

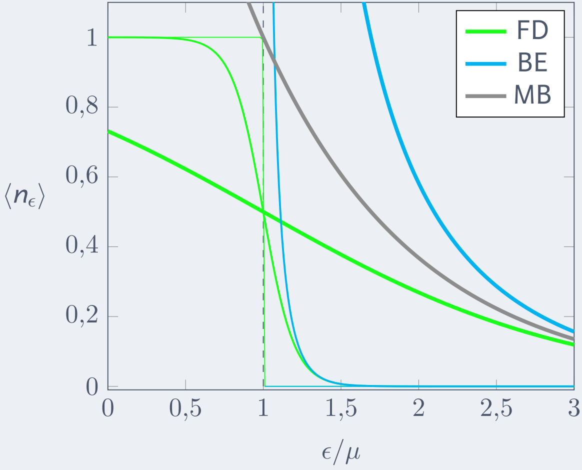
Gran Potencial $\quad -\beta\Omega = \ln Z_{GC} \qquad \beta p V = \ln Z_{GC}$

$$\Omega = -pV \qquad = -\eta \sum_i \ln\left(1 - \eta z e^{-\beta \epsilon_i}\right)$$

» Estadística de los números de ocupación

$$\langle n_\epsilon \rangle = \frac{1}{e^{\beta(\epsilon-\mu)} - \eta}$$

$$\langle n_\epsilon^2 \rangle - \langle n_\epsilon \rangle^2 = \left[\left(-\frac{1}{\beta} \frac{\partial}{\partial \epsilon} \right) \langle n_\epsilon \rangle \right] \Big|_{z,T}$$
$$= \langle n_\epsilon \rangle^2 \, z^{-1} \, e^{\beta\epsilon}$$



» El Gas de Fermi

Si son Fermiones idénticos ($\eta = -1$), $\langle n_\epsilon \rangle = \dfrac{1}{e^{\beta(\epsilon-\mu)}+1}$.

Si $\lambda^3 \gg V/N$ aparecen los efectos cuánticos (repulsivo si F),
$\mu \sim U(N+1) - U(N) > 0$

$$\beta p V = \sum_{1P} \ln\left(1 + z e^{-\beta\epsilon_{1P}}\right)$$

$$N = \sum_{1P} \langle n_{\epsilon_{1P}} \rangle = \sum_{1P} \frac{1}{z^{-1} e^{\beta\epsilon_{1P}} + 1}$$

$$U = \sum_{1P} \frac{\epsilon_{1P}}{z^{-1} e^{\beta\epsilon_{1P}} + 1}$$

Estados 1P

De mínima, los estados están caracterizados por la proyección de spín s y una parte espacial, e.g.,
$\{1P\} = \{s, \boldsymbol{p}\}$

$$\sum_{1P} = \sum_{s,\boldsymbol{p}} = \sum_{s}\sum_{\boldsymbol{p}} \cdots = \int g(\epsilon)\, d\epsilon$$

» El gas libre

Si tienen $H = \sum p^2/(2m)$, los autoestados de H no dependen del spin,

$$\sum_{1P} = \overbrace{(2S+1)}^{g_s} \sum_{\boldsymbol{p}} = g_s \frac{V}{h^3} \int d^3p$$

$$\beta p V = \int g(\epsilon) \ln\left[1 + z e^{-\beta\epsilon}\right] d\epsilon \qquad N = \int g(\epsilon) \frac{d\epsilon}{z^{-1} e^{\beta\epsilon} + 1} \quad \text{con} \quad g(\epsilon) = A\,\epsilon^{1/2}$$

y $A = (2S+1)\dfrac{V}{h^3} 4\pi m (2m)^{1/2}$, $A = \dfrac{V}{\pi^2 \hbar^3} m (2m)^{1/2}$ (electrones).

$$\beta p V = \int g(\epsilon) \ln\left[1 + z e^{-\beta\epsilon}\right] d\epsilon = -\int \frac{1}{1 + z e^{-\beta\epsilon}} z e^{-\beta\epsilon} (-\beta) A \frac{\epsilon^{3/2}}{3/2} d\epsilon$$

$$= \frac{2}{3}\beta \int A\,\epsilon^{3/2} \frac{1}{z^{-1} e^{\beta\epsilon} + 1} d\epsilon$$

$$N = \int A\,\epsilon^{1/2} \frac{1}{z^{-1} e^{\beta\epsilon} + 1} d\epsilon$$

Funciones de Fermi

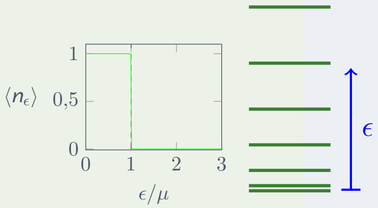
$$\boxed{f_\sigma(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty dx \frac{x^{\sigma-1}}{z^{-1} e^{x} + 1}}$$

» Las funciones de Fermi

$$\beta p V = g_s \frac{V}{\lambda^3} f_{5/2}(z), \qquad N = g_s \frac{V}{\lambda^3} f_{3/2}(z)$$

El límite ultradegenerado

$f_\sigma(z) =$



» ¿Qué pasa más en general?

Analicemos la función

$$f_\sigma(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \frac{x^{\sigma-1}}{z^{-1}e^x + 1}\, dx = \frac{1}{\Gamma(\sigma)} \int_0^\infty \frac{x^{\sigma-1} z e^{-x}}{1 + z\, e^{-x}}\, dx$$

Si z es pequeña, podemos integrar (convirtiéndolo en serie),

$$f_\sigma(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty x^{\sigma-1} z e^{-x} \sum_{n=0}^{\infty} \left(-ze^{-x}\right)^n = \frac{1}{\Gamma(\sigma)} \sum_{n=1} \int_0^\infty (-ze^{-x})^n (-1) x^{\sigma-1}\, dx$$

$$= \sum_{n=1} (-1)^{n+1} z^n \left[\frac{1}{\Gamma(\sigma)} \int_0^\infty x^{\sigma-1}\, e^{-xn}\right] = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n^\sigma}$$

$$= z - \frac{z^2}{2^\sigma} + \frac{z^3}{3^\sigma} - \cdots$$

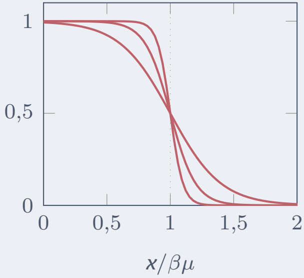
¿Cuándo es pequeña z?

» Expansión de Sommerfeld

Y si $z = e^{\beta\mu}$ es grande, estoy en límite de bajas T. Calculemos

$I(T) = \int_0^\infty \frac{h(x)}{e^{x-\beta\mu}+1}dx$. ($h$ crece más lento que exponencial.)

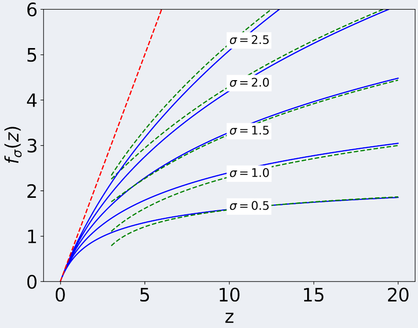
$$
\begin{aligned}
I(T) &= \int_0^{\beta\mu} \cdots + \int_{\beta\mu}^{\infty} \cdots \\
&= \int_0^{\beta\mu} h(x)\left[1 - \frac{1}{e^{\beta\mu-x}+1}\right]dx + \int_{\beta\mu}^{\infty} \frac{h(x)}{e^{x-\beta\mu}+1}dx \\
&= \int_0^{\beta\mu} h(x) - \int_0^{\beta\mu} \frac{h(\beta\mu-y)}{e^y+1}dy + \int_0^{\infty} \frac{h(y+\beta\mu)}{e^y+1}dy \\
&\simeq \int_0^{\beta\mu} h(x) + \int_0^{\infty} \frac{h(y+\beta\mu)-h(\beta\mu-y)}{e^y+1}dy \\
&= \int_0^{\beta\mu} h(x) + \int_0^{\infty} \frac{dy}{e^y+1} 2\, h'(\beta\mu) y + \int_0^{\infty} \frac{dy}{e^y+1} 2h'''(\beta\mu)\frac{y^3}{3!} + \cdots
\end{aligned}
$$



$$I(T) = \int_0^{\infty} h(x)\, \frac{1}{e^{x-\beta\mu}+1} \simeq \int_0^{\beta\mu} h(x) + \frac{\pi^2}{6}\, h'(\beta\mu) + \frac{7\pi^4}{360}\, h'''(\beta\mu) + \cdots$$

Y en particular,

$$f_\sigma(z) \simeq \frac{\ln(z)^\sigma}{\sigma\Gamma(\sigma)} \left[1 + + \frac{\pi^2}{6}\sigma(\sigma-1)\ln(z)^{-2}\right] = \frac{(\beta\mu)^\sigma}{\sigma\Gamma(\sigma)} \left[1 + \frac{\pi^2}{6}\sigma(\sigma-1)\left(\frac{k_B T}{\mu}\right)^2\right]$$



Veamos como lo usamos, calculemos la energía a baja T y de ahí el C_v