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Aspects of Well-Formed Scales

Norman Carey and David Clampitt

A single structural principle accounts for pentatonic, diatonic, and chromatic scales. The same structure, that of the *well-formed scale*, also underlies the tonic-subdominant-dominant relationship, the 17-tone Arabic and 53-tone Chinese theoretical systems, and other pitch collections in non-Western music. This article shows that the concept of a well-formed scale can serve as a principled basis for tonal music.

In the past, such a basis was sought in the physical phenomenon of the overtone series. This approach was found wanting in important respects: not only did the overtone hypothesis fail to generalize to non-triadic music, but it also inadequately and inconsistently explained features within the major-minor tonal system, such as the status of the minor triad as a consonance and as functionally equivalent to the major triad.[1] In recent years, diatonic set theory has provided an alternative perspective, which generally has proceeded from the assumption of an ideal equal division of the octave. For example, the diatonic scale has been examined in terms of its configuration against a chromatic background.[2] While our point of view intersects with diatonic set theory, we do not begin by assuming the diatonic and other pitch structures to be subsets embedded within a chromatic set.

In the theory presented here, every pitch relationship is defined in terms of the octave and fifth, which are treated as

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[1]The overtone hypothesis has been discussed at length elsewhere. We refer the reader to critical appraisals in the following: Milton Babbitt, "Past and Present Concepts of the Nature and Limits of Music" and "The Structure and Function of Music Theory," in *Perspectives on Contemporary Music Theory,* ed. Benjamin Boretz and Edward T. Cone (New York: Norton, 1972), 3–9 and 10–21; Robert Cogan and Pozzi Escot, *Sonic Design* (Englewood Cliffs, N.J.: Prentice-Hall, 1976), 139–141; Fred Lerdahl and Ray Jackendoff, *A Generative Theory of Tonal Music* (Cambridge, Mass: MIT Press, 1983), 290–293.

[2]John Clough and Gerald Myerson, "Variety and Multiplicity in Diatonic Systems," *Journal of Music Theory* 29 (1985), 249–270; Robert Gauldin, "The Cycle-7 Complex: Relations of Diatonic Set Theory to the Evolution of Ancient Tonal Systems," *Music Theory Spectrum* (1983), 39–55; Richmond Browne, "Tonal Implications of the Diatonic Set," *In Theory Only* 5/6–7 (1981), 3–21; Clough, "Diatonic Interval Sets and Transformational Structures," *Perspectives of New Music* 18/1–2 (1979–80), 461–482; Clough, "Aspects of Diatonic Sets," *Journal of Music Theory* 23 (1979), 45–61; Peter Westergaard, *An Introduction to Tonal Theory* (New York: Norton, 1975), 411–427; Benjamin Boretz, "Musical Syntax (II)," *Perspectives of New Music* 9/2–10/1 (1971), 232–270.

primitive terms. These intervals are parameters which may take on different values but which are fixed for the purposes of any given discussion. Given this degree of generality, equal-tempered systems are included in the theory as extreme or limiting cases.

The particular values assigned to the octave and fifth are of little importance for the formal theory. However, the theory will have a significant musical interpretation when these values are close to the overtone series values 2 and $\frac{3}{2}$. At the completely uninterpreted, purely mathematical level, the octave and the fifth play perfectly symmetrical roles: they are simply numbers which generate other numbers. At the level of the formal theory presented here, however, octave and fifth are assumed to play fundamentally dissimilar roles: the octave establishes a primary equivalence relation—octave equivalence—while the fifth determines the different pitch and interval classes. The fifth generates material which fills the frame provided by the octave.

We give a definition of a well-formed scale and several characterizations which involve elements of group theory and of the theory of continued fractions. Groups and continued fractions have been applied separately in musical contexts before, but the relationship between the two mathematical subjects and the importance of this relationship for tonal music have not been clearly understood. The two subjects are introduced informally in Part I, where the scale structure is defined first in terms of a *symmetry condition*, and then in terms of a *closure condition*. The logical equivalence of these two conditions is the central conclusion of this paper. The structural features shared by all well-formed scales, features which have considerable practical and concrete musical significance, may be derived from the symmetry condition. This might be called the local viewpoint, while the characterization in terms of closure provides global information about the set of well-formed scales.

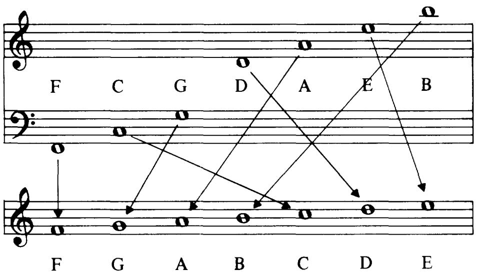
In Part II, these ideas are restated in a more formal and more general setting. We present a theorem which determines in principle all well-formed scales and organizes them into hierarchies. Following this is a brief treatment of another characterization which amplifies the local description. This aspect of the theory is our generalization of an approach developed by John Clough and Gerald Myerson. Finally, we turn to the particular case of the diatonic scale and to a discussion of the major-minor triadic tonal system.

I.

The Symmetry Condition

In one of the standard derivations of the diatonic scale, the seven pitch classes are obtained from a sequence of fifths and then ordered within an octave (Example 1). In a general sense, this is a satisfying, systematic derivation. We may be embarrassed, however, if asked to account for the number seven in this procedure, to justify halting at precisely this point. Part of the answer is suggested by the diagrams in Example 2, in which the sequence of fifths has been represented as seven points regularly spaced around a circle. In the first circle the tones have been connected by fifths, in the second by scale order. Both figures display the same degree of rotational symmetry. The regular heptagon on the left is an abstract geometrical representation of the fact that a diatonic pitch-class set has a realization as a chain of seven pitches linked consecutively by six identical intervals of a perfect fifth. The regular heptagon has seven degrees of rotational symmetry, which means there are seven distinct rotations of the figure which bring it into coincidence with itself. The figure formed by connecting adjacent scale elements also has seven degrees of rotational symmetry. We

Example 1.



express this relationship by saying that the scale figure preserves the symmetry of the circle of fifths.

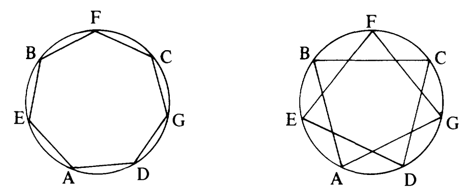
Although any number of fifths could be represented by some regular polygon, the preservation of symmetry is by no means the general rule. The diatonic hexachord formed by consecutive fifths provides a counterexample, as Example 3 makes clear. However, the familiar pentatonic scale formed by consecutive fifths does preserve symmetry (see Example 4). This symmetry condition makes meaningful and perhaps useful distinctions. Therefore we offer the following (informal) definition:

> Scales generated by consecutive fifths in which symmetry is preserved by scale ordering are called *well-formed scales*.

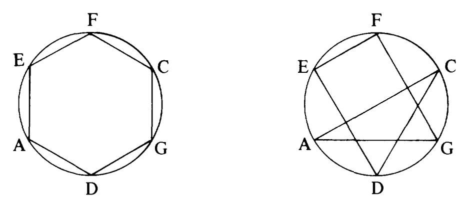
The term well-formed refers to the pitch collection as a whole, where the emphasis is on scale order and not on any particular mode. A formal definition will be given in Part II.

Continuing to test sets of up to 12 consecutive fifths, we discover six well-formed scales and six not well-formed. (See Table 1.) The collections distinguished by the symmetry condition correspond in a remarkable way to those affirmed by musical theory and by intuition. All 12 sets of consecutive

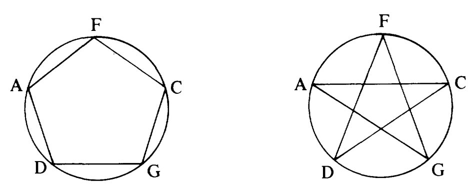
Example 2.



Example 3.



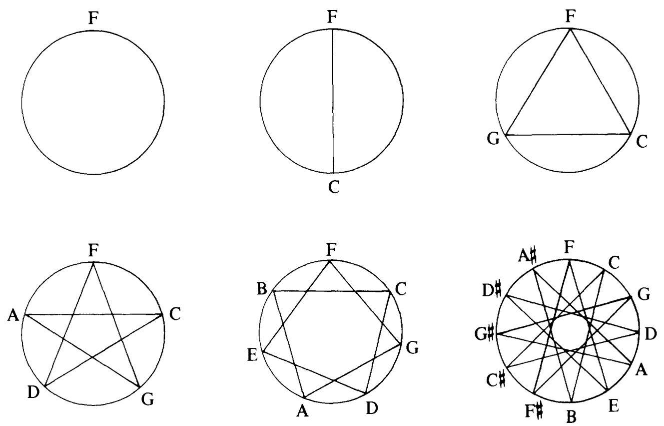
Example 4.



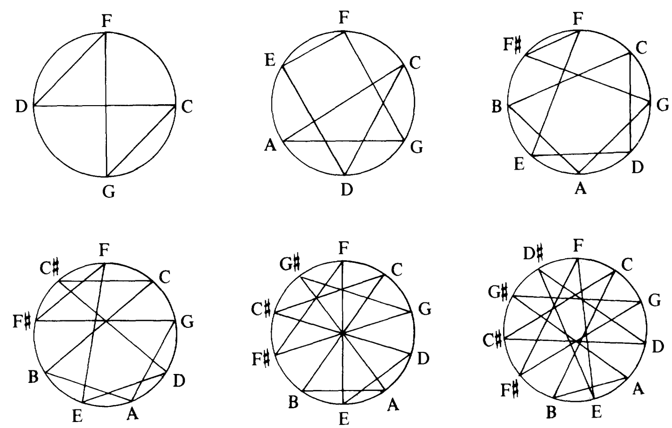
fifths share a certain degree of organization, reflected in the one degree of symmetry always present. However, the symmetry condition endows the musically significant scales with a higher degree of organization, which raises several questions: (1) What are the properties of well-formed scales,

Table 1. Scales formed by consecutive fifths

(a) well-formed (1, 2, 3, 5, 7, 12)



(b) not well-formed (4, 6, 8, 9, 10, 11)



and what musical advantages do these properties offer? (2) What determines the sequence 1, 2, 3, 5, 7, 12, and under what conditions would the sequence continue or another sequence altogether arise? These are the local and global questions, respectively, which we investigate here. To carry out this investigation, the geometric point of view is replaced by an algebraic one.

It is useful to designate pitch class number by position in the sequence of fifths. This is a non-standard notation, replacing the more common notation in 12 pitch classes which follows chromatic scale order:

pitch-class	F	C	G	D	A	E	B	F♯	C♯	G♯	D♯	A♯…
numbers	0	1	2	3	4	5	6	7	8	9	10	11…

With this notation, the elements of any scale derived from consecutive fifths, well-formed or not, may be associated with a set $Z_N = \{0,1,\ldots N\text{-}1\}$. For example, the pentatonic scale consists of the pitch classes F, C, G, D, and A whose pitch-class numbers are 0, 1, 2, 3, and 4, which are precisely the elements of Z_5. We will use the following names for the well-formed collections and associate each with the appropriate Z_N:

Well-Formed Scale	Z_N
octave	Z_1
octave-fifth	Z_2
structural	Z_3
pentatonic	Z_5
diatonic	Z_7
chromatic	Z_{12}

The circle diagrams above suggest modular arithmetic. Modular arithmetic is defined according to the *congruence* relation, which holds that two integers x and y are congruent modulo N (written $x \equiv y_{\mathrm{mod}\ N}$) if N divides $x - y$. For example, 9 is congruent to 2 modulo 7, since 7 divides $9 - 2$.[3] Congruence plays the role in modular arithmetic that equality plays in ordinary arithmetic. Congruence is the weaker of the two equivalence relations, but both are essential in what follows. To get a sense of the usefulness of the double description provided by congruence and equality, consider the diatonic set, represented by Z_7:

F		C		G		D		A		E		B		(F)	
0		1		2		3		4		5		6		(0)	
	+1		+1		+1		+1		+1		+1		−6		ordinary arithmetic
	+1		+1		+1		+1		+1		+1		+1		arithmetic modulo 7

In ordinary arithmetic, the difference between adjacent elements is either +1 or −6, where +1 corresponds to the interval of a perfect fifth and −6 to the interval of the diminished fifth. Considered in terms of modulo 7, however, all of the differences are equivalent, since $-6 \equiv 1_{\mathrm{mod}\ 7}$.

The rearrangement into scale order results in the sequence 0 2 4 6 1 3 5. Again, in ordinary arithmetic the difference between elements is either +2 or −5, where +2 indicates a whole step and −5 a half step. Nevertheless, considered in terms of modulo 7 the difference is once again a constant, since $-5 \equiv 2_{\mathrm{mod}\ 7}$:

F		G		A		B		C		D		E		(F)	
0		2		4		6		1		3		5		(0)	
	+2		+2		+2		−5		+2		+2		−5		ordinary arithmetic
	+2		+2		+2		+2		+2		+2		+2		arithmetic modulo 7

Thus the preservation of rotational symmetry in the diatonic set is represented algebraically by the fact that multiplication by 2 modulo 7 arranges the pitch classes into scale order:

	0	1	2	3	4	5	6	fifths order
$\times\ 2_{\mathrm{mod}\ 7}$:								
	0	2	4	6	1	3	5	scale order

For each well-formed scale of N pitch classes, we will be able to find an element b in Z_N which, operating on Z_N, arranges the pitch-class numbers into scale order. Consequently, $N - b$ yields reverse scale order. (See Table 2.) We will see in Part II that the elements of Z_N provided with addition mod_N form a group, and that the type of correspondence described above is a mapping of a group onto itself called an automorphism.

[3]It is easy to check that $a \equiv a_{\mathrm{mod}\ N}$, and that $a \equiv b_{\mathrm{mod}\ N}$ implies $b \equiv a_{\mathrm{mod}\ N}$. Furthermore, $a \equiv b_{\mathrm{mod}\ N}$ and $b \equiv c_{\mathrm{mod}\ N}$ implies that $a \equiv c_{\mathrm{mod}\ N}$ for all integers a, b, and c where N is any positive integer. Let Z represent the set of all integers. Then if z is in Z, $z \equiv Nq + r$ for unique integers q and r where $0 \leq r < N$, that is, r belongs to Z_N. Simply put, r is the least non-negative remainder upon division by N of z. Then $r \equiv z_{\mathrm{mod}\ N}$, so any integer z is congruent to one and only one element of Z_N. Therefore Z_N may be provided with an operation "addition modulo N": If z_1 and z_2 are in Z_N, their sum is defined to be the unique element r in Z_N such that $z_1 + z_2 \equiv r_{\mathrm{mod}\ N}$.

Table 2.

well-formed scale		b	N - b	order by fifths	$\times\ b_{modN} \longrightarrow$	scale order
Z_1	octave	0	0	0		0
Z_2	octave-fifth	1	1	0		0
				1		1
Z_3	structural	2	1	0		0
				1		2
				2		1
Z_5	pentatonic	2	3	0		0
				1		2
				2		4
				3		1
				4		3
Z_7	diatonic	2	5	0		0
				1		2
				2		4
				3		6
				4		1
				5		3
				6		5
Z_{12}	chromatic	7	5	0		0
				1		7
				2		2
				3		9
				4		4
				5		11
				6		6
				7		1
				8		8
				9		3
				10		10
				11		5

The Closure Condition

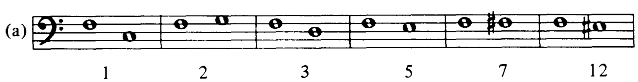
In the procedure defining well-formed scales, we evaluated the preservation of symmetry by taking advantage of a fictitious element. We formed a regular polygon even though the interval which completes the circle falls short of or exceeds a perfect fifth. Ignoring this difference is tantamount to introducing an equivalence relation, which in its algebraic guise is the relation "congruence modulo N." The equivalence relation suppresses information in order to reveal an important structural feature, the preservation of symmetry. However, it is precisely this initial element of asymmetry which enlivens the system, and which is expressed in more elaborate form in the asymmetry of the scale.

The well-formed collections may also be determined by a closure condition, which drops the equivalence relation and takes into account exact interval size. In Example 5, F is fixed in a given register and every other pitch class is represented by the member which is closest to F. On this basis, we may distinguish two types of intervals. The intervals in Example 6a below contain no pitches that occur earlier in the sequence of fifths, while at least one previous pitch (indicated by quarter-note heads) lies within the intervals in Example 6b. The intervals in Example 6a we define to be *primary intervals*, which provide the closure condition for well-formed scales. Each well-formed scale contains all of the pitch classes up to but not including the one associated with the primary interval. Therefore the numbers associated with the primary intervals correspond to the numbers of elements in the well-formed scales.

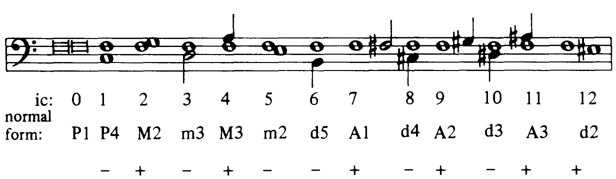
Some finer distinctions can be drawn by extending pitch-class notation to intervals. Interval-class numbers are designated in Example 7. Each interval class is represented by its smallest member, which is considered to be *normal form*. The normal form of interval class n is the smallest distance between pitch class n and pitch class 0 (= F). The normal

Example 5.

Example 6. Primary and non-primary intervals



Example 7. Intervals in normal form



form interval of interval class n > 0 is *positive* if it is upward from F, otherwise it is *negative*. Primary intervals are represented here in whole-note or half-note values according to the following distinction: A whole-note interval is smaller than any interval preceding it except unison, while half-note intervals are smaller than any preceding interval with the same sign. All other intervals are given quarter-note values.[4]

Again, each primary interval marks a well-formed scale by virtue of being the first interval excluded from that scale. However, the two scale steps in the well-formed scale are themselves primary intervals and the sum of their interval-class numbers is the class number of the determining primary interval.

The following diagram shows the case for N > 2:

well-formed scale	*stepwise intervals*		*determining interval*
Structural	1(P4) –	2(M2) +	3(m3)
Pentatonic	2(M2) +	3(m3) –	5(m2)
Diatonic	2(M2) +	5(m2) –	7(A1)
Chromatic	5(m2) –	7(A1) +	12(d2)

In each well-formed scale, the two step intervals have opposite signs. At least one of them is a whole-note interval according to the distinction drawn above. Furthermore, the intervallic difference between the two step intervals is exactly equal to the determining primary interval. Also, the interval numbers of this pair correspond to b and N – b. (See Table 2.)

[4]The normal form classification of three intervals is contingent upon an implicit assumption of Pythagorean tuning, which serves as our model. The diminished fifth is the normal form of interval class 6, since in this tuning it is smaller than the augmented fourth. Interval class 7 is given a half-note value, since the chromatic half step is larger than the diatonic half step of interval class 5. Finally, the polarity of interval class 12 is positive, since its normal form is the Pythagorean comma, where E♯ is higher than F. In Part II we introduce tuning in a general setting.

The hierarchy of well-formed scales is paralleled by a hierarchy of primary intervals which seems to be generated recursively. To understand the whole ensemble of interrelationships that has been suggested requires the introduction of more powerful mathematical theory. The theory of groups and the theory of continued fractions provide the appropriate languages for describing, respectively, the symmetry condition and the closure condition. In Part II we state a theorem which completely characterizes the relationship between these two aspects.

II

The heuristic presentation given in the first section left undefined the terms which all our procedures depended upon, the octave and fifth. Implicit in the whole discussion was a shared understanding of what was meant by octave and fifth. The question of the tuning of these intervals did arise in the classification of three intervals, the tritone, augmented prime, and diminished second. The more important point, however, is that we were able to forego explicit reference to tuning because the basic structures are left undisturbed by small variations in the values assigned to the octave and the fifth. The frequency ratios of the acoustically pure octave and fifth derived from the overtone series are $\frac{2}{1}$ and $\frac{3}{2}$ respectively, and it will turn out that as long as these intervals are assigned values sufficiently close to $\frac{2}{1}$ and $\frac{3}{2}$, the hierarchy of well-formed scales begins with the sequence N = 1, 2, 3, 5, 7, 12. The next step is to introduce the frequency ratios of the octave and the fifth into the theory.

We are positing what is effectively a generalized Pythagorean system, of which the paradigm is Pythagorean tuning itself.[5] The theory may be developed without loss of generality if the octave is given its usual value 2 and a *formal fifth* is introduced, which may be fixed at any value μ where

$$2^{\frac{1}{2}} \leq \mu \leq 2.$$

(Every possible unordered interval class has a unique representative between the equal-tempered tritone and the octave.)

The reader is cautioned that this degree of generality is introduced to underscore the structure of the theory, to illuminate the formal roles of the octave and fifth, not to postulate some uncountable infinity of musically interesting structures. On the contrary, many of the theoretically "well-formed" scales resulting from variation in the value of the formal fifth would be musical absurdities. Within the theory, the octave and fifth have the status of primitive terms. The paradigmatic status of the pure octave and fifth as they appear in the overtone series is an important issue but one which lies outside the formal theory.

Generalized Pythagorean Systems

A generalized Pythagorean system can be represented by the set $P = \{2^a\mu^b \mid a,b \in Z\}$. P can be thought of as representing all possible intervals formed by combining octaves and fifths, since the frequency ratios of the octave and fifth are assumed to be 2 and μ, and the frequency ratio of the combination of two intervals is the product of their frequency ratios. Unison is represented by $1 = 2^o\mu^o$, with numbers

[5]Eric Regener proposed studying such generalized Pythagorean systems in his article, "Layered Music-Theoretic Systems," *Perspectives of New Music* 6/1 (1967), 52–62. He carried out this project from the point of view of notation in *Pitch Notation and Equal Temperament* (Berkeley: University of California, 1973).

greater than 1 representing upward intervals, and numbers strictly between 0 and 1 representing downward intervals.

The elements of P can also represent pitches in the following way. If we assign a pitch X to the number 1, then the pitch associated with $2^a\mu^b$ is the pitch a octaves and b fifths from X, upwards or downwards depending on the signs on a and b. One must be careful to distinguish between two cases, according to whether μ is or is not a rational power of 2. In the more general case, μ is not a rational power of 2, and it can be shown that any such set P can be put into one-to-one correspondence with the notes of ordinary notation.[6] In the special case $\mu = 2^{\frac{M}{N}}$ for positive integers M and N, and some form of equal temperament obtains. For example, $2^{\frac{7}{12}}$ corresponds to the fifth of 12-tone equal temperament.

The set of all pitch classes is determined by a many-to-one mapping, $2^a\mu^b \rightarrow b$. This is a mapping from P to the set Z, if μ is not a rational power of 2, or to a set Z_N, if $\mu = 2^{\frac{M}{N}}$. If $\mu = 2^{\frac{M}{N}}$, then for any integer t, μ^{tN} is a power of 2: $\mu^{tN} = (2^{\frac{M}{N}})^{tN} = 2^{Mt}$, and a pitch represented by $2^a\mu^{tN+r}$ where $0 \le r < N$ belongs to pitch class r, since $2^a\mu^{tN+r} = 2^{a+Mt}\mu^r$. Therefore, $2^a\mu^b$ and $2^c\mu^d$ are in the same pitch class if and only if $b \equiv d_{\text{mod } N}$, and consequently Z_N is the set of all pitch classes.

The elements of P together with multiplication form a group, an algebraic structure of fundamental importance in mathematics and an essential concept for studying Pythagorean systems and the well-formed scales which they contain.[7]

[6]Regener, *Pitch Notation*.

[7]A generalized Pythagorean system may be usefully studied as an example of a Generalized Interval System (GIS) as defined by David Lewin in *Generalized Musical Intervals and Transformations* (New Haven: Yale University Press, 1987).

Groups and Symmetry

A *group* consists of a set of elements together with an operation, that is, a rule for combining any pair of elements. A set G with an operation * is a group if:

(1) the operation is closed, that is, if g_1 and g_2 belong to G, then $g_1 * g_2 \in G$;

(2) the operation is associative, that is, if $g_1, g_2, g_3 \in G$, then $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$;

(3) G contains an identity element, that is, there exists an element e in G such that for any $g \in G$, $e * g = g = g * e$; and

(4) every element in G has an inverse element, that is, if $g \in G$, there is an element $g^{-1} \in G$ such that $g * g^{-1} = e = g^{-1} * g$.

For example, the set of all integers Z with ordinary addition forms a group, because:

(1) the sum of any pair of integers is always an integer;

(2) addition is associative;

(3) the identity element is 0: $0 + n = n = n + 0$ for any integer $n \in Z$; and

(4) inverses exist, since $n + (-n) = 0 = (-n) + n$ for any $n \in Z$.

Every element in Z can be represented as the sum or difference of 1's (or −1's). Then we say that 1 is a *generator* of Z. A group which admits a generator is called a *cyclic group*, and Z is referred to as the *infinite cyclic group*. The *finite cyclic groups* are represented by Z_N with addition modulo N. The reader can check that Z_N with this operation satisfies the definition of a group, and that 1 or any element relatively prime to N is a generator of Z_N.

The set of ordered pairs $Z \times Z = \{(a,b) \mid a,b \in Z\}$ can be considered to be a group as follows: Two elements $x = (a,b)$ and $y = (c,d)$ are considered equal if and only if $a = c$ and $b = d$. If x and y are elements of $Z \times Z$, addition is defined by $x + y = (a + c, b + d)$. It is easy to verify that this addition is closed and associative, that (0,0) is an identity element, and given an element (a,b) in $Z \times Z$, the inverse element is $(-a,-b)$.

P provided with ordinary multiplication is a group: if $p_1,p_2 \in P$, then $p_1p_2 \in P$; multiplication is associative; 1 is a multiplicative identity element and belongs to P; and if $2^a\mu^b \in P$, then $2^{-a}\mu^{-b}$ is an inverse and is in P.

Group theory provides a way of defining the degree to which apparently dissimilar objects are alike. Two groups that are structurally identical are said to be *isomorphic*. Suppose G with the operation * and H with the operation ° are groups. If a mapping $i:G \rightarrow H$ sets up a one-to-one correspondence between G and H where, whenever $g_1,g_2 \in G$, $i(g_1 * g_2) = i(g_1) \circ i(g_2)$, then we say that G and H are isomorphic, written $G \cong H$, and is called an *isomorphism*. Since i sets up a one-to-one correspondence, this insures the existence of an inverse mapping $i^{-1}:H \rightarrow G$, which is also an isomorphism. That is, if $h_1,h_2 \in H$, then $i^{-1}(h_1 \circ h_2) = i^{-1}(h_1) * i^{-1}(h_2)$.

The isomorphic groups G and H may arise in very different contexts, yet every feature of one which may be expressed in terms of its elements and operation has its counterpart in the other. The isomorphisms i and i^{-1} provide the means for translating from the language of one group into that of the other.

For example, with the proviso that μ is not a rational power of 2, P is a group with multiplication isomorphic to the additive group $Z \times Z$: Let $i:Z \times Z \rightarrow P:(a,b) \rightarrow 2^a\mu^b$. Then if x and y are elements of $Z \times Z$ where $x = (a,b)$ and $y = (c,d)$, $i(x + y) = i(a + c, b + d) = 2^{a+c}\mu^{b+d} = (2^a \mu^b)(2^c\mu^d) = i(x)i(y)$. Furthermore, if $x \neq y$, $i(x) \neq i(y)$ since $2^a\mu^b \neq 2^c\mu^d$, here recalling the proviso. Finally, if $p \in P$ there is an $x \in Z \times Z$ such that $i(x) = p$, so i is an isomorphism and $P \cong Z \times Z$.

The statement that P *represents* a system of musical intervals is really the assertion of an isomorphism. Because an interval is determined by a frequency ratio, because there is a one-to-one correspondence between distinct elements of P and intervals formed by adding octaves and formal fifths, and because the frequency ratio of the sum of two intervals is the product of their frequency ratios, we may regard P as being the group of all such intervals.

The special case of an isomorphism of a group onto itself is called an *automorphism*. For example, the mapping of Z onto itself which multiplies every integer by -1 is an automorphism of Z. In the diatonic set, the permutation which rearranges the pitch-class numbers into scale order is an automorphism of Z_7. Recall that this rearrangement is effected by multiplying each element of Z_7 by 2 mod 7. If $z_1,z_2 \in Z_7$, then $2(z_1 + z_2) \equiv (2z_1 + 2z_2)_{\text{mod } 7}$, and if $z_1 \neq z_2$, $2z_1 \not\equiv 2z_{2(\text{mod } 7)}$, so the mapping $\Omega: Z_7 \rightarrow Z_7: z \rightarrow 2z_{\text{mod } 7}$ is an automorphism. The inverse automorphism Ω^{-1} is given by multiplication by 4 mod 7. The mapping Ω^{-1} takes pitch-class numbers in scale order and transforms them back into order by fifths:

	0	2	4	6	1	3	5
$\times 4_{\text{mod } 7}$							
	0	1	2	3	4	5	6

We can generalize the mapping Ω defined above to give the formal definition of a well-formed scale:

> Definition: Let $Z_N = \{0,1 \ldots N\text{-}1\}$ represent a set of pitch classes of P produced by consecutive fifths. These pitch classes are the elements of a well-formed scale if there exists an automorphism which arranges Z_N in scale order.

Recalling the earlier discussion, one can associate the value b with an automorphism of Z_N which places pitch-class numbers in scale order, and $N - b$ with an automorphism which places pitch-class numbers in reverse scale order.

CONTINUED FRACTIONS AND THE CLOSURE CONDITION

A mathematical statement of the closure condition is also possible in terms of continued fractions. The primary intervals are approximations to unison. Referring back to Example 7, the reader can see that the primary intervals generally become smaller, until the enharmonic interval of the diminished second is reached. In particular, the primary intervals designated by whole notes decrease in size, and in the sequences of both the positive primary intervals and the negative primary intervals the intervals become progressively smaller, that is, they approach unison.

In the general setting of P, it can be shown that if B determines a primary interval, then there is a normal form interval $\frac{2^A}{\mu^B}$ which is, in a well-defined sense, approximately unison, that is $2^A \sim \mu^B$. Equivalently, $\frac{A}{B} \sim \log_2\mu$, or $\frac{A}{B} \neq \log_2\mu$, and $\frac{A}{B}$ is a rational approximation to $\log_2\mu$. Unless μ is a rational power of 2, $\log_2\mu$ is irrational, that is, $\frac{A}{B} \neq \log_2\mu$ for all integers A and B. The theory of continued fractions is concerned with such rational approximations.

A simple continued fraction is an expression of the form

$$t_0 + \cfrac{1}{t_1 + \cfrac{1}{t_2 \; \ddots \; + \cfrac{1}{t_N}}}$$

where t_0 can be an integer which is positive, negative, or zero, and the other t_i are positive integers. For convenience the fraction is usually written $[t_0, t_1, t_2, \ldots t_N]$. For example,

$$1 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{3}}}$$

is notated [1,2,5,3]. Such a number can be evaluated in steps:

$$1,$$

$$1 + \frac{1}{2} = \frac{3}{2},$$

$$1 + \cfrac{1}{2 + \cfrac{1}{5}} = \frac{16}{11},$$

$$1 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{3}}} = \frac{51}{35}.$$

The successive values $[1] = 1$, $[1, 2] = \frac{3}{2}$, $[1, 2, 5] = \frac{16}{11}$, $[1, 2, 5, 3] = \frac{51}{35}$ are called the *convergents* of the continued fraction. The name is appropriate because if $x = [t_0, t_1, \ldots t_N]$ is a continued fraction, each convergent $c_k = [t_0, t_1, \ldots t_k]$ is closer to x than the previous convergents. The convergents can be computed recursively as follows: Let $a_{-2} = 0$, $b_{-2} = 1$, $a_{-1} = 1$, $b_{-1} = 0$. Then

$$c_k = \frac{a_k}{b_k} = \frac{t_k a_{k-1} + a_{k-2}}{t_k b_{k-1} + b_{k-2}}.[8]$$

[8]The proofs for this and other statements about continued fractions can be found in most introductory number theory texts. An elementary treatment is C. D. Olds, *Continued Fractions* (New York: Random House, 1963).

Infinite continued fractions can be shown to be well defined by using this fact to prove that the convergents approach a limit. Every irrational number can be expressed uniquely as a continued fraction. It is easy to see that

$$1 + \frac{1}{2} = 1 + \frac{1}{1 + \frac{1}{1}},$$

or in general that $[t_0, t_1, \ldots t_{N-1}, t_N] = [t_0, t_1, \ldots t_{N-1}, t_N - 1, 1]$. Every rational number can be expressed uniquely as a finite continued fraction $[t_0, t_1, \ldots t_N]$ with the proviso that the last term t_N be greater than 1. Conversely, every finite continued fraction determines a unique rational number, and every infinite continued fraction determines an irrational number; thus in each case there is a one-to-one correspondence between numbers and continued fractions.

The continued fraction which is crucial in the theory of well-formed scales is the continued fraction which represents $\log_2\mu$. There is a complicated division algorithm for determining the continued fraction representation of $\log_x y$. Surprisingly, a musician can use simple interval calculations to carry out this algorithm and compute the first five terms of the continued fraction for $\log_2 3 = [1, 1, 1, 2, 2, 3, 1, 5, \ldots]$. Table 3 illustrates the procedure. The first five convergents are $[1] = 1$, $[1, 1] = \frac{2}{1}$, $[1, 1, 1] = \frac{3}{2}$, $[1, 1, 1, 2] = \frac{8}{5}$, and $[1, 1, 1, 2, 2] = \frac{19}{12}$. Since $\log_2\frac{3}{2} = \log_2 3 - \log_2 2 = \log_2 3 - 1$, the continued fraction for $\log_2\frac{3}{2}$ is $[1, 1, 1, 2, 2 \ldots] - 1 = [0, 1, 1, 2, 2, \ldots]$, and the first five convergents of $\log_2\frac{3}{2}$ are therefore 0, 1, $\frac{1}{2}$, $\frac{3}{5}$, and $\frac{7}{12}$.

The convergents of the continued fraction x are the best approximations to x in the sense that no rational number with the denominator the same size or smaller is closer to x. The convergents alternately approach x from below and above. For example, the decimal representation of $\log_2\frac{3}{2}$ begins 0.5849 . . . , and the first five convergents in decimal form are 0.0, 1.0, 0.5, 0.6, and 0.5833 . . .

There are also weaker convergents, called semi-convergents, which are the best approximations from one side. If $\frac{A}{B}$ is a semi-convergent to x and $\frac{A}{B} < x$, there is no rational number between $\frac{A}{B}$ and x with denominator less than or equal to B, and similarly if $\frac{A}{B}$ is greater than x. If the continued fraction is $[t_0, t_1, \ldots, t_k, \ldots]$ and $t_k > 1$, we can form semi-convergents $[t_0, t_1, \ldots t_{k-1}, 1]$, $[t_0, t_1, \ldots t_{k-1}, 2]$, $\ldots [t_0, \ldots t_{k-1}, t_k - 1]$. For example, the semi-convergents to $\frac{7}{12} = [0, 1, 1, 2, 2]$ are $[0, 1, 1, 1] = \frac{2}{3}$ and $[0, 1, 1, 2, 1] = \frac{4}{7}$. Now the sequence of convergents and semi-convergents reads $[0] = 0$, $[0, 1] = 1$, $[0, 1, 1] = \frac{1}{2}$, $[0, 1, 1, 1] = \frac{2}{3}$, $[0, 1, 1, 2] = \frac{3}{5}$, $[0, 1, 1, 2, 1] = \frac{4}{7}$, $[0, 1, 1, 2, 2] = \frac{7}{12}$.

The correspondence between the denominators of these fractions and the sequence of primary intervals is not accidental. It is not obvious, but in fact an interval between 2^A and $(\frac{3}{2})^B$ is primary if and only if $\frac{A}{B}$ is a convergent or semi-convergent of $\log_2\frac{3}{2}$. That is, in a generalized system P, the interval between 2^A and μ^B is a primary interval exactly when $\frac{A}{B}$ is a convergent or semi-convergent of $\log_2\mu$.

The classification of primary intervals into whole-note and half-note and positive and negative categories corresponds to distinctions made in continued fraction terms. The primary intervals designated by whole notes in Example 7 correspond to convergents, and those designated by half notes correspond to semi-convergents of $\log_2\frac{3}{2}$.[9] The positive primary intervals correspond to the convergents and semi-convergents where $\frac{A}{B} < \log_2\frac{3}{2}$, while the negative primary intervals correspond to those where $\frac{A}{B} > \log_2\frac{3}{2}$.

[9]We refer to $\log_2\frac{3}{2}$ because Pythagorean tuning is our model. The same statements can be made for any system for which the continued fraction begins [0,1,1,2,2, . . .], finite or infinite.

Table 3.

k	smaller interval		larger interval	t_k		remainder interval
0	*octave*	goes into	*perfect twelfth*	1 time	leaving	*perfect fifth*
1	*perfect fifth*	goes into	*octave*	1 time	leaving	*perfect fourth*
2	*perfect fourth*	goes into	*fifth*	1 time	leaving	*major second*
3	*major second*	goes into	*fourth*	2 times	leaving	*minor second*
4	*minor second*	goes into	*major second*	2 times	leaving	*diminished second*

This entire development leads to the following theorem characterizing well-formed scales:

Characterization Theorem: In a generalized Pythagorean system P, a scale with pitch classes 0, 1, . . .B − 1 is a well-formed scale if and only if B is the denominator in a convergent or semi-convergent $\frac{A}{B}$ in the continued fraction representation of $\log_2\mu$. Moreover, the automorphism of Z_B which places pitch-class numbers in a scale order is the mapping

$$\Omega: Z_B \rightarrow Z_B : z \rightarrow zb_k\,(-1)^k{}_{\text{mod } b}$$

where $\frac{a_k}{b_k}$ is the full convergent immediately preceding $\frac{A}{B}$ in the continued fraction of $\log_2\mu$.[10]

The theorem asserts that each value μ gives rise to a unique hierarchy of well-formed scales, a hierarchy which is recursively organized by the continued fraction of $\log_2\mu$. Consider, for example, Pythagorean tuning. Then $P = \{2^a(\frac{3}{2})^b \mid a,b \in Z\}$, and $\log_2\frac{3}{2} = [0,1,1,2,2,3,1,5,2,23,2, \ldots]$, an infinite continued fraction. The sequence of (semi-)convergents $\frac{A}{B}$ beginning with c_1 is $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{7}{12}, \frac{10}{17}, \frac{17}{29}, \frac{24}{41}, \frac{31}{53}, \ldots$ and Z_1, Z_2, Z_3, Z_5, and so on represent well-formed scales. Z_{17} represents the Arabic theoretical scale of 17 notes to the octave,[11] while Z_{53} represents a system in Chinese theoretical writings.[12]

Consider the pentatonic scale, represented by Z_5. In this case, $\frac{A}{B} = [0,1,1,2] = \frac{3}{5}$. Then the previous full convergent is $\frac{a_2}{b_2} = [0, 1, 1] = \frac{1}{2}$. The characterization theorem states that the defining automorphism for the pentatonic scale is associated with the value $b_k(-1)^k{}_{\text{mod B}} = 2(-1)^2{}_{\text{mod 5}} \equiv 2$. This is closely related to the fact that the steps of the pentatonic scale belong to the interval classes 2 and the equivalent of 2 modulo 5, −3. Indeed, the proof of the characterization theorem shows that the step intervals are themselves primary

[10]A proof of this theorem is contained in our unpublished paper, "Two Theorems Concerning Rational Approximations: Number Theory and Music Theory."

[11]Curt Sachs, *The Rise of Music in the Ancient World: East and West* (New York: Norton, 1943), 279–280. Sachs points out that the 17-note scale is more properly considered a "set of elements," from which various seven-note scales are derived.

[12]Alain Daniélou, *Introduction to the Study of Musical Scales* (London: The India Society, 1943), 77. The 53-note division is also familiar in its equal-tempered guise as a tuning scheme for devotees of just intonation. See Hermann Helmholtz, *On the Sensations of Tone*, trans. Alexander J. Ellis (London: Longmans, 1885; reprint ed., New York: Dover, 1954), 479–481; Ben Johnston, "Scalar Order as a Compositional Resource," *Perspectives of New Music* 2/2 (1964), 56–76.

intervals associated with $\frac{a_k}{b_k}$ and $\frac{A-a_k}{B-b_k}$, in this case $\frac{1}{2}$ and $\frac{2}{3}$. In the Pythagorean example, the intervals are the whole step, $(2)^{-1}(\frac{3}{2})^2 = \frac{9}{8}$ and the minor third, $(2)^2(\frac{3}{2})^{-3} = \frac{32}{27}$.

Families

Each convergent of $\log_2\mu$ gives rise to a finite sub-hierarchy of well-formed scales which we call a *family*. If $\frac{a_k}{b_k}$ is a full convergent to $\log_2\mu = t_0, t_1, \ldots t_k \ldots]$ (where $k < N$ if there is a last term t_N), we define the b_k family to be $F(b_k) = \{Z_B \mid B = nb_k + b_{k-1}, 0 \leq n \leq t_{k+1}+1\}$. The most musically significant example is the whole-step family: The whole step is associated with interval class 2, and $F(2) = \{Z_B \mid B = 2n + 1, n = 0,1,2,3\} = \{Z_1, Z_3, Z_5, Z_7\}$, the octave, structural, pentatonic, and diatonic scales.[13] (See Table 4.) Each member of a family except the initial one has as one step an interval associated with the convergent $\frac{a_k}{b_k}$, and the final well-formed scale of the family is the last scale which contains this interval. For example, the last member of the Pythagorean diatonic half-step family F(5) is the Arabic 17-note scale.

Some accounts of Javanese and Balinese pelog and slendro describe them as embedded within a nine-tone equal division of the octave.[14] This corresponds to a system with formal fifth $\mu = 2^{\frac{5}{9}}$, associated with the continued fraction [0,1,1,4] which gives rise to the hierarchy Z_1, Z_2, Z_3, Z_5, Z_7, Z_9. In this case the family $F(2) = \{Z_1, Z_3, Z_5, Z_7, Z_9\}$.

[13]Membership in F(2) underlies the "remarkable property" Jay Rahn discerns in pentatonic and diatonic scales as well as in slendro, pelog, and others, wherein "odd members are adjacent and form a unit registrally distinct from that formed by even members . . ." Recall that the diatonic scale, for example, is represented 0 2 4 6 1 3 5 (0). See Jay Rahn, "Some Recurrent Features of Scales," *In Theory Only* 2/11–12 (1977), 47, 51.

[14]Ibid., 49–50.

Table 4. Whole-step family

Z_1	0							F						
Z_3	0	2			1			F	G			C		
Z_5	0	2	4		1	3		F	G	A		C	D	
Z_7	0	2	4	6	1	3	5	F	G	A	B	C	D	E

Degenerate Well-Formed Scales

Every equal-tempered scale is the last scale of one or more finite hierarchies. We call the equal-tempered scales the *degenerate well-formed scales*, degenerate in the same sense that the geometer can consider a straight line to be a degenerate circle. Straight lines have their own beauty and usefulness, but they are uninteresting circles. The degree of asymmetry present in the general well-formed scale has been smoothed out in the special case of the symmetrical equal-tempered scale.

Every hierarchy can be considered to be equivalent in some degree to a finite hierarchy. We will say that two hierarchies are *nth-order equivalent* if the first n (semi-) convergents of each are identical. Thus the Pythagorean tuning hierarchy and the $\frac{1}{4}$-comma mean-tone temperament hierarchy are sixth-order equivalent, and are both equivalent to the finite hierarchy of ordinary chromatic equal temperament.

III

Another characterization of non-degenerate well-formed scales can be given in terms of scale-step measure. In the diatonic scale, an interval can be described either in terms of the number of scale steps it spans or by its exact size. Each span is represented by two intervals. Seconds and thirds, sixths and sevenths come in two sizes—major or minor—while fourths are perfect or augmented and fifths are perfect or diminished. A generalization of this two-to-one property applies to all non-degenerate well-formed scales, as we will show.

The pitch classes and interval classes of P are represented by Z, in general, or by Z_N, in the case where $\mu = 2^{\frac{M}{N}}$. If $2^a\mu^b$ is a pitch or interval, it belongs to pitch or interval class b in the first case, and to class b modulo N in the second case. Two pitch classes x and y determine at most two distinct ordered interval classes $x - y$ and $y - x$. (We will see that in special cases it may be that $x - y$ and $y - x$ refer to a single interval class.)

When we say that a well-formed scale is represented by Z_B, we mean first of all that the elements of Z_B are the pitch-class numbers of the scale. The ordered interval classes possible with these pitch classes are then contained in the set of all differences $a - b$ where a and b are elements of Z_B. (The subtraction here is ordinary arithmetic, not modular arithmetic.) The possible interval classes are therefore $1 - B$, $2 - B, \ldots -1, 0, 1, \ldots B - 2, B - 1$. We will call these the *specific* ordered interval classes of the well-formed scale. There are at most $2B - 1$ of these since they may not all be distinct if P is an equal-tempered system. For example, if Z_B is a degenerate well-formed scale, the distinct classes are 0, 1, . . . $B - 1$.

Even in a non-degenerate scale, such as the diatonic scale in ordinary equal temperament, there may be some duplication: Class 6 contains the augmented fourth and its octave equivalents, and class −6 contains the diminished fifth and its octave equivalents, which in this tuning are indistinguishable classes. However, if Z_B is non-degenerate well-formed, it must be that classes n and $n - B$ are distinct, for $n = 1, 2, \ldots B - 1$. Otherwise we would have $2^{A'}\mu^{n-B} = 2^{A''}\mu^n$, for some integers A′ and A″, that is $2^{A'-A''} = \mu^B$. Setting $A = A' - A''$, than $2^A = \mu^B$, that is, $\frac{A}{B} = \log_2\mu$ and Z_B is degenerate, contrary to the assumption.

The non-zero specific interval classes can be collected into $B - 1$ pairs of congruent interval classes: $n \equiv n - B_{\text{mod } B}$, but n and $n - B$ represent distinct interval classes for $n = 1, 2, \ldots B - 1$. We define the *generic* interval classes for the well-formed scale to be these $B - 1$ classes, each of which contains two distinct specific interval classes, plus the zero class.[15] (See Table 5.) Then Z_B represents a set of generic interval classes. Since generic ordered interval classes have been defined in terms of the congruence relation, it makes sense to define the sum of two generic interval classes x and y to be the generic interval class $(x + y)_{\text{mod } B}$. So the group Z_B with addition modulo B represents the group of generic interval classes with addition. This is the meaningful sense in which it can be said that a well-formed scale is a group.

If Ω is the automorphism of Z_B which places pitch-class numbers of the well-formed scale in scale order, then $\Omega(1)$, $\Omega(2), \ldots \Omega(B - 1)$ is that scale order, which one can relabel $0, 1, \ldots B - 1$. Automorphism Ω also relabels generic interval classes according to scale order: $\Omega(1)$ is the generic scale step, $\Omega(2)$ is the generic "third" in the scale, third meaning "skip one step," just as in the diatonic scale, "fourth" and "fifth" similarly defined, up to the generic "Bth," $\Omega(B- 1)$. In a well-formed scale, then, the notion of a generic interval is consistent with, indeed equivalent to, scale-step interval measure. Since there are two distinct

[15]The terms "generic" and "specific" are those of John Clough and Gerald Myerson; see "Variety and Multiplicity." We have extended the meaning of specific interval to interval size in a generalized Pythagorean system. Clough and Myerson restrict their study to diatonic structures embedded in chromatic universes.

Table 5.
Diatonic ordered interval classes (≠0)

generic ic	1	2	3	4	5	6
specific ic	+1	+2	+3	+4	+5	+6
interval	P5	M2	M6	M3	M7	A4
specific ic	−6	−5	−4	−3	−2	−1
interval	d5	m2	m6	m3	m7	P4

specific interval classes in each non-zero generic ordered interval class, we can say that within the octave exactly two specific intervals correspond to each non-zero generic interval. Following Clough and Myerson, we call this two-to-one property Myhill's property.

A stronger assertion can be made, namely that Myhill's property characterizes non-degenerate well-formed scales. In an arbitrary scale we can define a generic interval in terms of scale-step measure. If the scale possesses Myhill's property, that is, if seconds, thirds, and so forth come in exactly two distinct sizes, it can be shown that the scale is a non-degenerate well-formed scale.[16]

In most cases, there are exactly $2B - 2 = 2(B - 1)$ distinct, non-zero, specific ordered interval classes in a well-formed scale Z_B. That is, there are twice as many non-zero specific interval classes as there are non-zero generic interval classes, and the generic interval classes partition the set of specific interval classes into pairs. If there are fewer than $2B - 2$ specific classes, it is an anomaly due to tuning. We will say that a well-formed scale Z_B is in a *normal tuning* when there are $2B - 2$ non-zero specific ordered interval classes defined by the scale. It can be proved that a well-formed scale is in a normal tuning as long as it is in a universe P where μ is not a rational power of 2, or, if $\mu = 2^{\frac{M}{N}}$, as long as $B \leq [\frac{N+1}{2}]$.[17]

The usual diatonic scale in ordinary equal temperament is not in a normal tuning, because partitioning fails in the isolated case of the tritone. Whenever N is even and $B = \frac{N+2}{2}$ there is such a singularity. Two points are noteworthy in these cases. First, that partitioning may be rescued by considering unordered interval classes. For example, in the diatonic scale, in any tuning, there are three non-zero generic unordered interval classes, which partition the six non-zero specific classes into disjoint pairs. Second, that because of this duplication at the tritone, these scales have what Gamer refers to as depth of transposition.[18] The number of specific ordered interval classes is $2B - 2 = N$ in these scales. These are exactly the N interval classes of the equal-tempered system P; thus the scale can be transposed to any pitch in the system by one of the intervals of the well-formed scale. This is also the case when N is odd and $B = \frac{N+1}{2}$.[19]

In view of the equivalence of non-degenerate well-formed scales and scales with Myhill's property, the results obtained by Clough and Myerson for scales which have Myhill's property carry over to well-formed scales, although our definition of specific interval is more general in that it does not appeal to the assumption of equal division. Unordered pitch-class sets (chords) and ordered pitch-class sets (lines) in a non-degenerate well-formed scale can be described both generically and specifically, according to their constituent intervals. Lines which share a generic description are "tonal"

[16] Carey and Clampitt, "Two Theorems."

[17] $[\frac{(N+1)}{2}]$ means the integral part of $\frac{(N+1)}{2}$ — that is, the largest integer less than or equal to $\frac{(N+1)}{2}$.

[18] Carlton Gamer, "Some Combinational Resources of Equal-Tempered Systems," *Journal of Music Theory* 11 (1967), 32–59.

[19] Here also see Clough and Myerson, "Variety and Multiplicity," 267–268.

or "diatonic" transpositions of each other, while those sharing a specific description are "real" transpositions. The number of specific types of lines which share a common generic description is equal to the number of distinct pitch classes per line. Moreover, the number of "real" transpositions of each specific type is determined by the shared generic description. For example, in a well-formed scale represented by Z_B there are B distinct modes, each one a diatonic transposition of the others. That is, all share the same generic description, but each has a unique specific description.

The property of partitioning, under which the double description of intervals is completely consistent and unambiguous in that a specific interval always has the same generic description, is a property of any well-formed scale in a normal tuning. This partitioning property is inherited by chords and lines when the scale is in a normal tuning. That is, chords or lines of a given specific description always have the same generic description. While partitioning is a rare property from Clough and Myerson's point of view, in this more general setting it is the rule rather than the exception.

The Diatonic Set

The symmetry condition, subsequently reinterpreted as a group automorphism condition, provided the point of departure for our theory. This defining property results from the intersection of two conditions, symmetry preservation and scale order, which easily could be met independently, but when met simultaneously characterize the well-formed scale. In this section we consider the special case of the diatonic set and the implications for the major-minor tonal system of the existence of a third symmetry transformation.

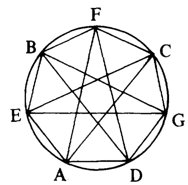
If Z_B represents a well-formed scale, two automorphisms of Z_B are involved by definition: the identity automorphism, which maps each pitch class onto itself; and the automorphism defined by $b_k(-1)^k{}_{\text{mod B}}$ which arranges the pitch classes into scale order. (There are trivial cases where $b_k(-1)^k{}_{\text{mod B}} \equiv 1$ and there is a single mapping.)

If we drop the scale order condition, there are other symmetry-preserving mappings possible. The number of automorphisms of a cyclic group Z_B is equal to the number of elements of Z_B relatively prime to B. Thus, if B is prime, there are $B - 1$ automorphisms. In general, if $\Phi(B)$ represents the number of elements less than and relatively prime to B, then the number of automorphisms is $\Phi(B)$. It is easy to show that $\Phi(B)$ is even if $B > 2$. For our purposes, only half of these mappings are essentially different. For example, if $b_k(-1)^k{}_{\text{mod B}}$ determines the automorphism arranging pitch classes into scale order, $-b_k(-1)^k{}_{\text{mod B}}$ determines the complementary mapping placing the elements in reverse scale order. The essential sameness of these mappings is reflected geometrically in that the circle diagram representations of these mappings are indistinguishable, unless arrows marking clockwise and counterclockwise order are added. Therefore we will consider $\frac{\Phi(B)}{2}$ different automorphisms of Z_B.[20]

In the case of the diatonic set Z_7, $\frac{\Phi(7)}{2} = \frac{6}{2} = 3$, so a third automorphism, distinct from those defined by fifths order and scale order, exists in this system. In this mapping the pitch classes are in order by thirds. The three essentially different symmetry transformations of the diatonic set are displayed below; the geometric representation of the thirds automorphism appears in Example 8.

[20]The reader with a background in group theory will recall that the automorphisms of a group themselves form a group with ordinary composition of mappings. For the cyclic group Z_B, $B - 1$ determines an involution, an automorphism of order 2, and we are concerned with the quotient group of the automorphisms modulo the subgroup $\{1, B - 1\}$.

Example 8.



	0	1	2	3	4	5	6	
$\times 1_{\text{mod } 7}$								(fifths)
	0	1	2	3	4	5	6	
	0	1	2	3	4	5	6	
$\times 2_{\text{mod } 7}$								(seconds)
	0	2	4	6	1	3	5	
	0	1	2	3	4	5	6	
$\times 4_{\text{mod } 7}$								(thirds)
	0	4	1	5	2	6	3	

In the double description given by ordinary arithmetic and arithmetic modulo 7, we see the following:

0		4		1		5		2		6		3	
	+4		−3		+4		−3		+4		−3		ordinary arithmetic
	+4		+4		+4		+4		+4		+4		arithmetic modulo 7

The diatonic set is generated by 4 mod 7, or by the threefold repetition of the cell [4 − 3]. These cells correspond to the three primary triads, FAC, CEG, and GBD:

F		A		C		E		G		B		D
	[4		−3]		[4		−3]		[4		−3]	

The importance of the major mode in triadic harmony can be inferred from the central position of the tonic triad, flanked by the dominant and subdominant.

The minor triad, represented by the cell [−3 4], generates the diatonic set in an analogous way, starting from D:

D		F		A		C		E		G		B
	[−3		4]		[−3		4]		[−3		4]	

The complementary role of the minor mode is illustrated here, where the central position is held by the tonic triad. Thus the triad emerges as a structural feature of the diatonic set.[21]

Any pitch-class set generated by N consecutive fifths has an axis of inversional symmetry. This axis is a pitch class if N is odd, and is an interval class if N is even.[22] In any such set, the imperfect fifth which completes the cycle of fifths is also such an axis of symmetry. (See Table 1.) In the diatonic set, the pitch-class axis of symmetry is D, and the imperfect fifth is the diminished fifth B to F.

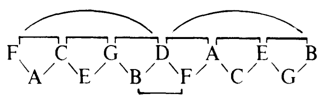
The pivotal position of the tritone (and of the diminished triad) flanked by major and minor triads, and the interrelationship of the cycle of thirds and the cycle of fifths, is displayed in Example 9.[23] One aspect of the polarity between

[21]This is, of course, an old observation, presented here in yet another context. See also Lewin, "A Formal Theory of Generalized Tonal Functions," *Journal of Music Theory* 26 (1982), 23–60, and Ramon Fuller, "A Structuralist Approach to the Diatonic Scale," *Journal of Music Theory* 19 (1975), 182–210.

[22]Gauldin, "The Cycle-7 Complex," 41.

[23]The structural importance of the tritone in the diatonic set is discussed in Browne, "Tonal Implications of the Diatonic Set." Experimental evidence

Example 9.



the tritone and the tonic major and minor triads is the fact that the C-major and the A-minor triads are the only triads which are "tritone-free," in the sense that every other triad contains either an F or a B.[24]

The intrinsic complementarity of the major-minor contrast makes it unnecessary to justify the minor triad in terms of the overtone hypothesis. In this theory, then, the triad is derived from the diatonic set, rather than the other way around. There has been a failure to recognize the correct logical status of the triad not only on the part of theorists who have embraced variants of the overtone hypothesis, such as Schenker, but also among those who have considered the formal structure of the diatonic scale against a notional background of equal division of the octave.[25] The logical error both stems from and leads to an ethnocentric view of music history, which overlooks the fact that triadic tonal music is only a small subset of world diatonic music. On the other hand, the acoustical resemblance between the major triad and the fourth, fifth, and sixth harmonics is not to be denied. Undoubtedly there is a correlation between the physical phenomenon and the perception of consonance and dissonance, although the correlation is by no means a simple one.

Tonality is an extremely complex phenomenon, to which the static, formal view given here does not do justice. For example, a satisfactory theory of tonality must embrace rhythm, which we have not considered. The point of our analysis is simply that the diatonic set does possess a third dimension which permits the strong definition of a tonic, which diatonic music may or may not exploit. By contrast, no matter what the tuning, any well-formed scale of 12 tones contains only the minimal two distinct automorphisms, since $\Phi(12) = 4$. In this regard, the 12-tone set is poorer than the diatonic. In the pentatonic scale as well there are just two automorphisms. On the other hand, many theoretical well-formed scales are much richer in automorphisms.

Gregory Bateson, in his book *Mind and Nature,* insists upon the value of double description in contexts ranging from binocular vision to theory formation.[26] An opposition which has been thematic in our discussion is the double description which arises from the interplay of symmetry and asymmetry. At the top of the logical hierarchy there is an opposition between octave and fifth. These terms may be regarded mathematically as playing symmetrical roles, but an initial

for the importance of the tritone in establishing a tonal context is found in Helen Brown and David Butler, "Diatonic Trichords as Minimal Cue-Cells," *In Theory Only* 5/6–7 (1981), 39–55. Browne's dialectic of "pattern matching" versus "position finding" parallels our "generic" versus "specific" and "symmetry" versus "asymmetry." Pattern matching is further explored in Edwin Hantz, "Recognizing Recognition: A Problem in Musical Empiricism," *In Theory Only* 5/6–7 (1981), 22–38.

[24]Diatonic complementation is discussed in Browne, with clarifications in Clough and Myerson. Assuming set classes determined by transpositional equivalence only, the tonic major and minor triads are the only triads with unique complements. That is, BDFA is the only half-diminished seventh chord and GBDF the only dominant seventh in the C-major collection. The other triads are not uniquely determined by their complements.

[25]For example, Gerald Balzano, "The Group-Theoretic Description of 12-Fold and Microtonal Pitch Systems," *Computer Music Journal* 4/4 (1980), 66–84.

[26]Gregory Bateson, *Mind and Nature* (New York: Dutton, 1979).

element of asymmetry enters with the introduction of octave equivalence. This primary equivalence gives rise to the notions of pitch class and interval class and to the notion of scale as an ordered set of pitches spanning an octave. A secondary equivalence relation, based on the sequence of fifths, together with the notion of scale order gives rise to the well-formed collections, in each of which is inherent a consistent double description, a generic description which applies within the well-formed scale, and a specific description which applies throughout the given generalized Pythagorean system.

Associated with each Pythagorean system is a continued fraction, which organizes all the well-formed scales of the system into a hierarchy. Continued fraction theory has previously been applied to aspects of tuning. Although tuning is important in our theory, in that the value of the formal fifth determines the continued fraction, our interest lies primarily in the organization of well-formed scales, expressed in group-theoretical terms. The connection between continued fractions and the appropriate application to music of group theory has been hitherto unacknowledged. We hope that an understanding of familiar musical entities within the well-formed scale context will lead to a greater awareness of the way pitch is organized and of the way the mind organizes musical pitch.

ABSTRACT

Pentatonic, diatonic, and chromatic scales share the same underlying structure, that of the *well-formed scale.* Well-formedness is defined in terms of a relationship between the order in which a single interval generates the elements of a pitch-class set and the order in which those elements appear in a scale. Another characterization provides a recursive procedure for organizing all well-formed scales into hierarchies. Finally, well-formed scales are defined in terms of scale-step measure, and aspects of the diatonic set are examined.