AMERICAN JOURNAL of PHYSICS

The Oscillating Spring and Weight—An Experiment Often Misinterpreted

H. L. Armstrong

Citation: Am. J. Phys. **37**, 447 (1969); doi: 10.1119/1.1975602
View online: http://dx.doi.org/10.1119/1.1975602
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v37/i4
Published by the American Association of Physics Teachers

NOTES AND DISCUSSIONS

The Oscillating Spring and Weight—An Experiment Often Misinterpreted

H. L. Armstrong
Stirling Hall, Queen's University, Kingston, Ontario, Canada
(Received 27 November 1967; revision received 7 November 1968)

Generations of students of mechanics have experimented with weights hung on springs. They have measured the extension of the spring, and the period of the vertical oscillation of the weight. The results are supposed to illustrate Hooke's law and simple harmonic motion. Certainly the experiment is a simple and straightforward one, but there are difficulties in the interpretation, of which the students are rarely even aware.

First of all, the action of a coil spring is a complicated thing, and one which the students do not understand when they are doing the experiment. From the information which they have, it cannot be concluded that the result bears out Hooke's law. Consider an arrangement of links, with some sort of spring action at the joints, as shown in Fig. 1. If the extension should be proportional to the stretching force, the torque at the joints (which is the origin of the elastic effect) would be proportional, not to the angle turned through, but rather to the sine of twice that angle.

In fact, the force to stretch a coil spring is not strictly proportional to the amount of stretch; it involves the angle which the wire of the spring makes with the plane perpendicular to the axis of the spring,[1,2] which angle increases with the amount of stretch. But, in fact, it is easy to find by experiment that this effect is quite negligible under the conditions commonly met.

Incidentally, it is often not made clear to the students that the action of a coil spring has nothing to do with stretching of the material, but that the wire is rather being twisted.

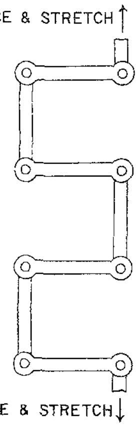

Fig. 1. In this link-work, if the extension is "Hookean," the turning at the joints is not, as is explained in the text.

In the second place, it is not seen that the motion is really simple harmonic, i.e., sinusoidal. All that is seen is that the period varies in such and such a way with the loading. An experiment in which the harmonic nature of the motion is revealed has been suggested recently.[3]

In the third place, the effect of the weight of the spring, which is hardly ever negligible, is rarely considered very thoroughly. The main present purpose, in fact, is to investigate that matter.

The problem is complicated by the fact that a given length of the spring (measured while it is stretched) has a mass which depends on the amount of stretching. The complications which this causes can be eliminated, however, by using a difference equation.

Let the spring, of mass m and coefficient k, be held at its upper end and hang down. On the lower end let there be fastened a weight M. In order to get a difference equation, consider the spring as if it were divided into N parts, and then let it be approximated by a chain of N weights, each of magnitude m/N, joined by weightless springs, each of coefficient Nk. Let the displacement of the elemental weight number n downward from the position which it would have were the spring not stretched be called s_n. Note that now the masses and coefficients of the elemental parts are not affected by the stretching. Moreover, since N will eventually be supposed to increase without limit, it will do no harm to replace N by $N+1$ or $N-1$ if that should be convenient.

The spring forces acting on element number n, (count downward positive), will be $Nk(s_{n+1}+s_{n-1}-2s_n)$. There will also be the force of gravity, mg/N. Thus the equation of motion for part number n will be:

$$(m/N)(d^2s_n/dt^2) = (mg/N) + Nk(s_{n+1}+s_{n-1}-2s_n). \quad (1)$$

At the lower end, the spring force acting on the weight M up will be $Nk(s_N-s_{N-1})$; there is also the force of gravity Mg, so

$$M(d^2s_N/dt^2) = Mg - Nk(s_N - s_{N-1}). \quad (2)$$

At the upper end $s_0=0$.

A solution, as may be found by trial or otherwise, (actually, I guessed what to try by considering the corresponding differential equation), is:

$$s_n = A \sin\alpha n \sin\omega t - (mg/2N^2k)n(n+1) + [(m+M)/Nk]gn. \quad (3)$$

Here A is arbitrary, and α and ω are to be determined. This solution will satisfy the Eq. (1) provided

$$m\omega^2/N = 2Nk(1-\cos\alpha), \quad (4)$$

and will satisfy the boundary condition Eq. (2) provided

$$M\omega^2 \sin\alpha N = 2Nk \sin(\alpha/2) \cos\alpha(N+\tfrac{1}{2}); \quad (5)$$

s_0 is zero anyway.

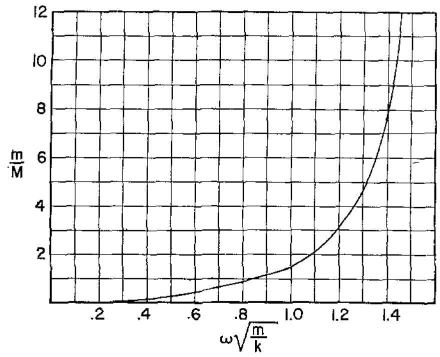

FIG. 2. From this graph, if m/M is known, $\omega(m/k)^{1/2}$ can be found; and thus ω if k is known.

Now let N increase without limit. In Eq. (4), $1-\cos\alpha$ will become small, so α must approach zero, and

$$1-\cos\alpha \approx \alpha^2/2 = m\omega^2/2N^2k. \tag{6}$$

So $\alpha=(\omega/N)(m/k)^{1/2}$, and is very small. Put this into Eq. (5). On the right-hand side $\sin(\alpha/2)$ can be replaced by $\alpha/2$, and $\cos\alpha(N+\frac{1}{2})$ by $\cos\alpha N$. Thus Eq. (5) becomes

$$M\omega^2 \sin\omega(m/k)^{1/2} = 2Nk(\omega/2N)(m/k)^{1/2}\cos\omega(m/k)^{1/2} \tag{7}$$

which gives

$$\omega(m/k)^{1/2}\tan\omega(m/k)^{1/2}=m/M. \tag{8}$$

Now for the interpretation. Consider first static stretching, without oscillation; then the A in Eq. (3) is zero. The weight goes down by $s_N=(m+M)gN/Nk-mgN(N+1)/2N^2k$, which becomes $(M+m/2)(g/k)$. Thus half of the weight of the spring is effective in the static stretching. However, if two different weights, M_1 and M_2, are used, the difference in stretch is $(M_2-M_1)(g/k)$, and this agrees with the usual way of finding the coefficient of a spring by stretching. Thus no criticism of the static part of the experiment is needed, except in the small points already mentioned.

It might be noticed that letting N increase without limit amounts to replacing n by a continuous independent variable, which would indicate the fraction of the spring between the point of support and the point in question, or something proportional to that fraction. Then instead of a difference equation there would be a differential equation, and Eq. (3) would be modified in ways that are apparent. However, to do so would not improve the solution of the present problem.

As for Eq. (8), it seems to have no exact solution. But m, M, and k will be all known; then ω, which is the thing of interest, could be found, e.g., by a graph such as that shown in Fig. 2.

If m were truly negligible, the solution, of course, would be $\omega^2=k/M$. It is sometimes said that a finite m is allowed for by replacing M by $M+m/3$. But consider how this arises. If m is small, the tangent may be expanded in a series, and only two terms retained, to give:

$$\omega\left(\frac{m}{k}\right)^{1/2}\left\{\omega\left(\frac{m}{k}\right)^{1/2}+\tfrac{1}{3}\omega^3\left(\frac{m}{k}\right)^{3/2}\right\}=\frac{m}{M}, \tag{9}$$

and the solution for this, considered as a quadratic equation, is:

$$\omega^2 m/k=\tfrac{1}{2}\{-3\pm[9+12(m/M)]^{1/2}\}. \tag{10}$$

The positive sign will be wanted. Expand the surd and retain only three terms to get:

$$\omega^2(m/k)=(m/M)(1-m/3M). \tag{11}$$

By the binomial theorem, $1-m/3M\approx(1+m/3M)^{-1}$, so

$$\omega=\left(\frac{k}{M}\right)^{1/2}\left(1-\frac{m}{3M}\right)^{1/2}$$

$$\approx\left(\frac{k}{m}\right)^{1/2}\left(1+\frac{m}{3M}\right)^{-1/2}\approx\left(\frac{k}{M+m/3}\right)^{1/2}. \tag{12}$$

So this formula has involved three approximations, all of which assumed that m/M is small.

Suppose, on the other hand, that m/M is large. Then, in Eq. (8), the tangent must be large, and that will be so if its argument is nearly $\pi/2$. Put Eq. (8) into the form

$$\omega(m/k)^{1/2}=\arctan(m/M\omega)(k/m)^{1/2}. \tag{13}$$

Now if x is large, $\arctan x$ is approximated by $\pi/2-1/x$, in the sense that the functions, and x^2 times their first derivatives, agree as x approaches infinity. So, in this approximation

$$\omega(m/k)^{1/2}=\tfrac{1}{2}\pi-(M\omega/m)(m/k)^{1/2}, \tag{14}$$

which gives:

$$\omega=(\pi/2)(k/m)^{1/2}(1+M/m)^{-1},$$

$$=\tfrac{1}{2}\pi\left\{\frac{k}{m[1+(2M/m)]}\right\}^{1/2},$$

$$=\frac{\pi}{2^{3/2}}\left(\frac{k}{M+(m/2)}\right)^{1/2}, \tag{15}$$

which is certainly different from Eq. (12). However, the factor $\pi/2^{3/2}=$ about 1.1, and $m/3$ has been replaced by $m/2$, so the results are really not all that different.

Now this experiment seems commonly to be done with springs which weigh from 100–200 g, while the loads used on them may range from 50–500 g or so .Thus it certainly is not always true that m/M is small. Indeed, in this respect, the fact is that an inadequate theory, by a series of coincidences, gives an adequate result. But if the students do not see this—and in fact they do not—they do not understand the experiment; and if they do not understand it, it is hard to see what point there is in their doing it.

As a matter of fact, this is one of those experiments in which students whom a teacher knows to be good experimenters often get poorer results than others who are known to be a bit careless. It may be that a little carelessness helps in getting a theoretical result which is not really in accordance with the facts. If this is so, it is surely, as it stands, not a good experiment for teaching, for it does the most careful experimenters an injustice.

There could be another disturbing effect in this experiment, if the up-and-down motion should be coupled to a twisting motion, as in Wilberforce's pendulum. However, it seems that the matter that has been considered here is usually more important.

[1] F. V. Warnock and P. P. Benham, *Mechanics of Solids and Strength of Materials* (Sir Isaac Pitman and Sons, Ltd., London, 1965), pp. 258–260.

[2] C. J. Smith, *The General Properties of Matter* (Edward Arnold, Publ., Ltd., London, 1960), 2nd ed., pp. 390–393.

[3] H. L. Armstrong, The School Sci. Rev. **XLVI**, 368 (1965).

Electromagnetic Mass of Spheroidal Bodies

A. M. Luiz

Istituto di Fisica, University of Pisa, Italy

(Received 21 October 1968)

Let us consider a uniform charge distribution in the surface of a moving conductor. The body is moving with a constant velocity v in empty space. We shall suppose that v is so low that we can use the quasistatic approximation. As it has been pointed out by Phillips,[1] the total linear momentum of the field is given by

$$p=\frac{v}{6\pi c^2}\iiint E^2 dV=mv, \tag{1}$$

where c is the velocity of the light in a vacuum, E is the quasistatic electric field, and m is the so-called electromagnetic mass. The integral is taken over all space outside the conductor.

We want to calculate the electromagnetic mass of axisymmetric spheroids moving in a direction parallel to the axis of symmetry of the body. Usually the electromagnetic mass is obtained by the calculation of the integral (1). However, it is more convenient to transform this integral in a surface integral. By an elementary procedure we obtain

$$p=-\frac{v}{6\pi c^2}\iint \psi \operatorname{grad} \psi \cdot \hat{n}\, dS, \tag{2}$$

where the integral is taken over the surface of the body, ψ is the electric potential, and $\hat{n}$ is the unit normal vector at the surface of the body.

Let us consider the case of a uniformly charged sphere. By using the potential e/R and substituting in the relation (2) we obtain

$$m=(2/3)(e^2/R\,c^2), \tag{3}$$

which is a well-known result.

Let us consider the case of the oblate spheroid. We shall use oblate spheroidal coordinates. By the relations derived by Landau[2] and Moon[3] we obtain the following electrical potential:

$$\psi=\frac{e}{(a^2-b^2)^{1/2}}\tan^{-1}[b^1(a^2-b^2)^{1/2}]\frac{\cot^{-1}(\sinh\eta)}{\cot^{-1}(\sinh\eta_0)}, \tag{4}$$

where η_0 is the surface of the oblate spheroid, a and b are the semiaxis of the spheroid $(a>b)$.

By the relations (2) and (4) we obtain the electromagnetic mass of the oblate spheroid:

$$m=\frac{2e^2a}{3c^2}\frac{\{\tan^{-1}[b^{-1}(a^2-b^2)^{1/2}]\}^2}{(a^2-b^2)\cosh\eta_0\cot^{-1}(\sinh\eta_0)}. \tag{5}$$

When the oblate spheroid degenerates into a sphere of radius R, we have: $a=b=R$, and η_0 becomes infinite. Considering that

$$\lim_{\eta_0\to\infty}[\cosh\eta_0\cot^{-1}(\sinh\eta_0)]=1,$$

and taking the limit of the relation (5), we can verify that this result is in agreement with the relation (3).

Let us consider the case of the prolate spheroid. We shall use prolate spheroidal coordinates. We shall give only the final result. The electromagnetic mass of the prolate spheroid is given by

$$m=\frac{2e^2b}{3c^2}\frac{\{\tanh^{-1}[a^{-1}(a^2-b^2)^{1/2}]\}^2}{(a^2-b^2)\sinh\eta_0\ln[\coth(\eta_0/2)]}, \tag{6}$$

where η_0 is the surface of the prolate spheroid, and a and b are the semiaxis of the spheroid.

When the prolate spheroid degenerates into a sphere we have: $a=b=R$. Taking the limit of the relation (6) it is easy to see that this equation is in agreement with the relation (3).

The financial assistance received from Conselho Nacional de Pesquisas do Brasil is acknowledged by the author. He wishes to extend his gratitude to Istituto di Fisica, University of Pisa.

[1] M. Phillips, *Handbuch der Physik* (Springer-Verlag, Berlin, 1962), Vol. 4, p. 80.

[2] L. D. Landau and E. M. Lifshitz, *Electro-dynamics of Continuous Media* (Pergamon Press, Inc., London, 1960), p. 24.

[3] P. Moon and D. E. Spencer, *Field Theory for Engineers* (D. Van Nostrand Co., Inc., New York, 1961), p. 275.