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The spring-mass system revisited

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(Received 15 April 1983; accepted for publication 6 November 1983)

A vertically oscillating spring of mass m and spring constant k suspended from its upper end and with a mass M attached to its lower end is a system often used for demonstrations and experiments in introductory physics courses. We discuss the motion of this system for arbitrary values of $\epsilon = m/M$, $0 \leqslant \epsilon < \infty$ and show explicitly why theory predicts that the amplitude of the lowest normal-mode frequency makes the major contribution to the motion of M (or of the lower end of the spring) for *all* values of ϵ. Although a complete understanding of this fact involves detailed mathematical analysis, the results themselves are simply stated and readily verified even by students in an introductory calculus-based physics course. The various predictions of the theory are easily demonstrated with simple equipment and lend themselves nicely to an introductory physics laboratory. These applications are discussed in some detail, and an analog electric circuit is given which exhibits similar behavior.

I. INTRODUCTION

This journal has a long history of papers[1–12] dealing with the problem of a spring of mass m fixed at one end with a mass M attached to its other end. Although two of these papers deal with the torsional[1] and pendular[7] modes of oscillation of such a spring-mass system, a common thread of the rest is the correction to be made to the usual simple-harmonic oscillator result for the period

$$\tau = \frac{2\pi}{\omega} = 2\pi\left(\frac{M}{k}\right)^{1/2}. \tag{1.1}$$

This expression is valid when the spring is assumed massless ($m = 0$) and is taken to obey Hooke's law with a spring constant k. Much of the discussion has focused on the range of validity of the commonly stated correction[13–14]

$$\tau = 2\pi\left(\frac{M + (m/3)}{k}\right)^{1/2}, \tag{1.2}$$

which is supposed to make allowance for the mass m of the spring. While these papers have recognized the fact that the motion of this coupled system is *not* simple harmonic, most were content to consider the lowest root α_1 of the transcendental equation

$$\tan \alpha = \epsilon/\alpha, \tag{1.3}$$

where $\epsilon = m/M$. Here $\alpha(\epsilon)$ depends on the ratio m/M and the corresponding normal mode frequency has the limiting values

$$\omega(\epsilon) = \left(\frac{k}{m}\right)^{1/2} \alpha_1(\epsilon) \begin{cases} \xrightarrow[\epsilon\to\infty]{} & \dfrac{\pi}{2}\left(\dfrac{k}{m}\right)^{1/2} \\ \xrightarrow[\epsilon\to 0]{} & \left(\dfrac{k}{M}\right)^{1/2}. \end{cases} \tag{1.4}$$

Since, for a given spring, k and m are fixed, the limit $\epsilon \to \infty$ means $M \to 0$, which corresponds to a freely oscillating spring. It is typically argued that Eq. (1.2) arises from a first-order correction to the $\epsilon = 0$ limit of Eq. (1.4) and that Eq. (1.2) is valid for $\epsilon \ll 1$. Heard and Newby[8] even observed that Eq. (1.4) [with $\alpha_1(\epsilon)$ being the lowest root to Eq. (1.3)] holds over a wide range of values of ϵ. That is, even though the motion of the coupled system is *not* simple harmonic and does not consist of just one harmonic, the angular frequency of the end of the spring (or of M) is *in fact* (i.e., experimentally) given quite reliably to Eq. (1.4). However, no explanation was offered of why the *amplitudes* of the other modes associated with Eqs. (1.3) should not become really significant as $0 \to \epsilon \to \infty$.

A quantitative explanation of the smallness of the *individual* higher-frequency amplitudes was given in a beautiful paper by Weinstock.[9] He was able to obtain a bound on the individual amplitudes of the higher normal modes compared to the lowest normal-mode amplitude. Subsequently Bowen[12] employed a novel technique to construct a simple ("algebraic") solution to the free spring case ($M = 0$) for a special set of initial conditions on the spring.

The purpose of this paper is to employ both Weinstock's technique and Bowen's idea to show explicitly that the $\epsilon \ll 1$ case [Eq. (1.2)] passes over to the $\epsilon \to \infty$ limit in such a way that the lowest normal-mode amplitude is *always* the dominant one so that Eq. (1.4) does lead to a reliable prediction of the observed period of motion of the end of the spring.

II. THE PROBLEM DEFINED

Since the equations of motion and the appropriate boundary conditions have been given in the literature previously (see especially Refs. 5, 8, 9, and 12), we simply state the necessary equations in a unified notation.[15] The uniform helical spring of natural length l_0 is characterized by a constant linear modulus γ_0 which is related to the Hooke's law constant k as

$$\gamma_0 = kl_0. \tag{2.1}$$

We take as our independent variable the coordinate ξ ($0 \leqslant \xi \leqslant l_0$) measured from the fixed end of the spring to a point P on the unstretched spring. If the free end of the spring is stretched beyond its normal length l_0, the point P will then be located at a position ζ as measured from the fixed end of the spring. Since ζ gives the position of a particular point on the spring, ζ will be a function of ξ. If the spring moves in time, ζ must also depend upon time so that $\zeta = \zeta(\xi,t)$. Let us denote by $y(\xi,t)$ the displacement of P away from its original position ξ as

$$\zeta(\xi,t) = \xi + y(\xi,t). \tag{2.2}$$

The variable of dynamical interest is $y(\xi,t)$. The spring (with mass M attached to its lower end) is then suspended (from its upper end) in a uniform gravitational field. Let $y_0(\xi)$ denote the *static* equilibrium displacement of P away from its original position (ξ). Since we wish to discuss the motion of the spring about this new equilibrium configuration, we introduce a variable $w(\xi,t)$ which represents the displacement of P away from $y_0(\xi)$ as

$$y(\xi,t) = y_0(\xi) + w(\xi,t). \tag{2.3}$$

Newton's second law of motion for an element of the spring becomes the homogeneous wave equation

$$c_0^2 \frac{\partial^2 w}{\partial \xi^2} - \frac{\partial^2 w}{\partial t^2} = 0 \tag{2.4}$$

and for M (at the lower end of the spring)

$$\left.\frac{\partial w}{\partial \xi}\right|_{\xi = l_0} + \kappa \left.\frac{\partial^2 w}{\partial \xi^2}\right|_{\xi = l_0} = 0, \tag{2.5}$$

where

$$c_0 = (kl_0^2/m)^{1/2}, \tag{2.6}$$

$$\kappa = l_0/\epsilon. \tag{2.7}$$

Here c_0 is the speed of wave propagation *relative to the intrinsic coordinate* ξ. The boundary condition at the fixed ($\xi = 0$) end is

$$w(\xi = 0,t) = 0. \tag{2.8}$$

The mathematical problem is now to solve Eq. (2.4) subject to the boundary conditions (2.5) and (2.8) and to some initial data at $t = 0$ on $0 \leqslant \xi \leqslant l_0$:

$$w(\xi,t = 0) = f(\xi), \tag{2.9}$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=0} = g(\xi). \tag{2.10}$$

In particular, we shall be concerned with those initial conditions characteristic of a typical demonstration,

$$w_0(\xi,t = 0) = \frac{a}{l_0}\xi, \tag{2.11}$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=0} = 0, \tag{2.12}$$

in which the spring is initially at rest but its lower end ($\xi = l_0$) is pulled down an amount a below its static equilibrium position. [Equation (2.11) is obviously a static solution to Eq. (2.4) meeting the conditions (2.8) and (2.12) and having $w(\xi = l_0) = a$. It does *not* satisfy (2.5) since an additional force is required for $t < 0$ to maintain the displacement a.]

Although use of the variable $w(\xi,t)$ transforms away the effect of the uniform gravitational field on the equations for w,[16] there is an interesting and easily demonstrated effect gravity has on wave propagation in a soft, freely hanging spring, such as a slinky. While the velocity of propagation c_0 of Eq. (2.6) is constant with regard to the intrinsic coordinate ξ, c measured with respect to a fixed reference frame (such as a meter stick) does vary since the slinky is *not* uniformly stretched over its length.[17] One can show[18] that for a soft spring the wave speed at the top is much greater than at the bottom

$$c_{\text{top}} \simeq \left(1 + \frac{mg}{kl_0}\right)c_0 \gg c_0 = c_{\text{bottom}}. \tag{2.13}$$

This is a very dramatic effect which can be demonstrated by compressing a portion of the stretched slinky near the bottom and watching the compression wave travel up and back down the slinky. The speed increases on the way up and decreases on the way back down.

III. MOTION OF THE SPRING

One standard way to construct a solution $w(\xi,t)$ to the problem defined by Eqs. (2.4) to (2.10) is via an eigenfunction expansion

$$w(\xi,t) = \sum_{n=1}^{\infty} A_n \sin\left(\alpha_n \frac{\xi}{l_0}\right)\cos(\omega_n t + \phi_n). \tag{3.1}$$

If we require that Eq. (3.1) be a solution to Eq. (2.4) and satisfy the boundary conditions (2.5) and (2.8) as well as the initial conditions (2.11) and (2.12), the series reduces to[19]

$$w(\xi,t) = \frac{2a}{\epsilon} \sum_{n=1}^{\infty} \frac{\tan^2\alpha_n \sin\alpha_n}{\epsilon + \sin^2\alpha_n} \sin\left(\alpha_n \frac{\xi}{l_0}\right)\cos(\omega_n t), \tag{3.2}$$

where the α_n are the roots of

$$\tan(\alpha_n) = \epsilon/\alpha_n \tag{3.3}$$

with

$$\omega_n = c_0(\alpha_n/l_0). \tag{3.4}$$

The graph of the smallest root $\alpha_1(\epsilon)$ vs $\sqrt{\epsilon}$ is plotted in Fig. 1 and $\tau(\epsilon)/\tau_0$ vs $1/\sqrt{\epsilon}$ in Fig. 2, where

$$\tau(\epsilon) = \frac{2\pi}{\omega_1(\omega)} = \tau_0 \frac{\pi}{2\alpha_1(\omega)} \tag{3.5}$$

with $\tau_0 = 4(m/k)^{1/2}$ being the period of the free spring. No-

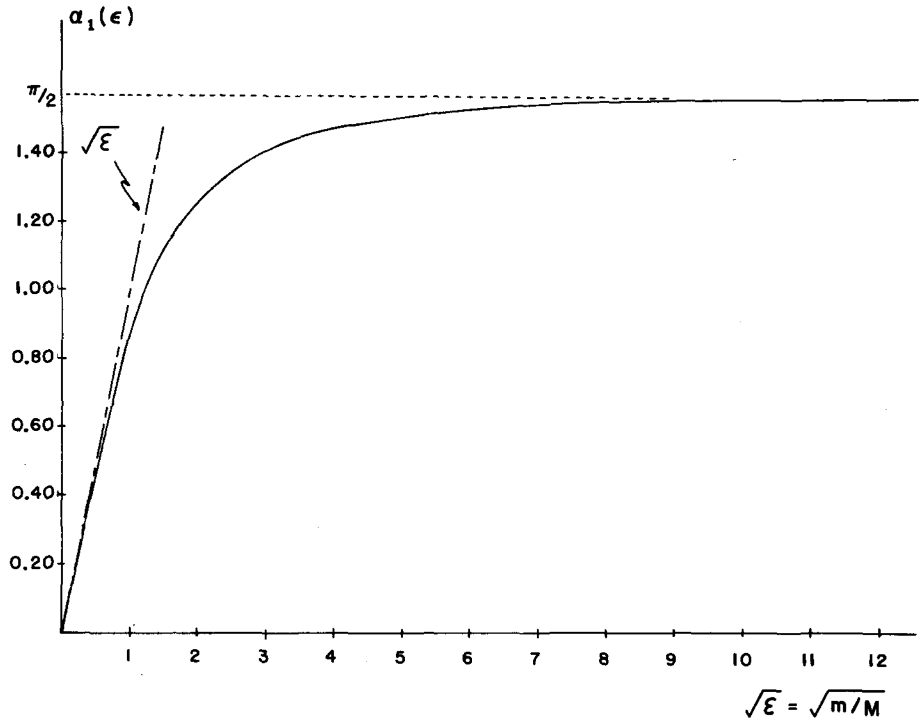

Fig. 1. The variation of $\alpha_1(\epsilon)$ vs $\sqrt{\epsilon}$.

tice that Eq. (1.2) follows as the small-angle approximation ($\tan\alpha_1 \simeq \alpha_1 + \frac{1}{3}\alpha_1^3$) to Eq. (3.3). Figure 2 makes it apparent that Eq. (1.2) is an extremely reliable approximation to the exact expression of Eq. (3.5). In fact, any appreciable departure from Eq. (1.2) can be detected only for *small* values of $1/\sqrt{\epsilon}$ (less than about 0.50 or $0 < M \leqslant m/4$, the case of a nearly free spring).

In a typical laboratory experiment, a student usually observes the motion of the end of the spring and measures the time taken for it to execute a large number of up and down oscillations in order to compute an average period. We can get a good indication that the fundamental term in Eq. (3.2) provides the dominant contribution to the exact motion of the lower end of the spring $[w(\xi = l_0, t)]$ by observing that the *amplitude* of that term is bounded *from below* as[20]

$$\frac{2a}{\epsilon}\frac{\tan^2\alpha_1\sin^2\alpha_1}{(\epsilon+\sin^2\alpha_1)} = \frac{2a(\sin\alpha_1/\alpha_1)^2}{1+(\sin\alpha_1/\alpha_1)\cos\alpha_1} \geqslant \frac{8}{\pi^2}a = 0.8106a, \tag{3.6}$$

for $0 \leqslant \alpha_1 \leqslant \pi/2$. However, for arbitrary values of ϵ, the mo-

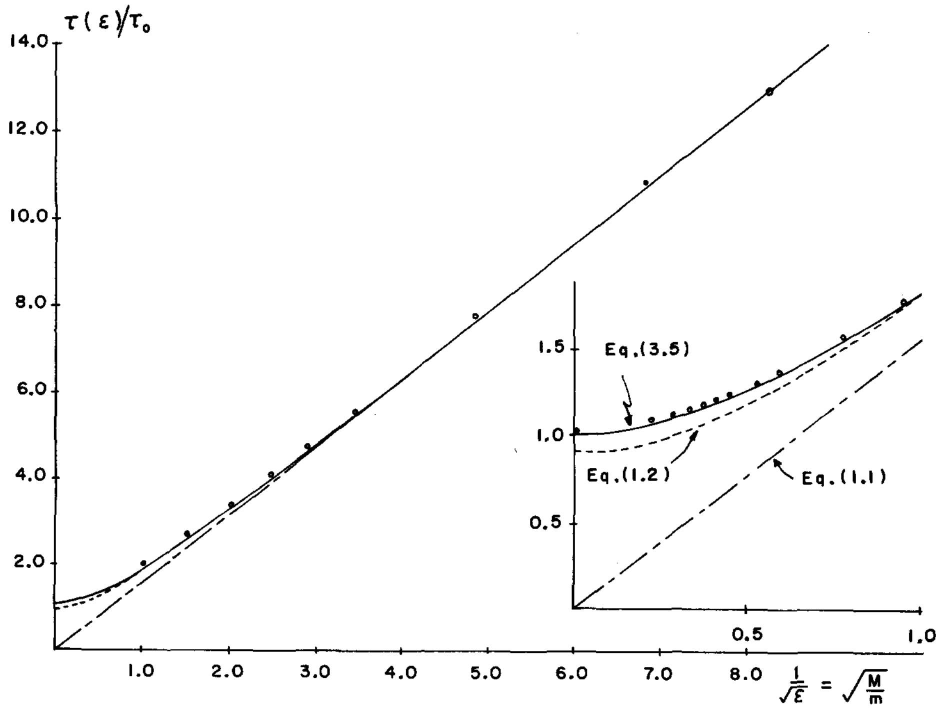

Fig. 2. The dependence of τ/τ_0 predicted by Eq. (3.5).

tion is not strictly periodic (let alone simple harmonic) so that we must expend some effort to obtain a bound on the quantity

$$\begin{aligned}\Delta(t;\epsilon) &\equiv \frac{1}{a}\left| w(\xi = l_0,t) - \frac{2a}{\epsilon}\frac{\tan^2\alpha_1 \sin^2\alpha_1}{(\epsilon + \sin^2\alpha_1)}\cos(\omega_1 t)\right| \\ &= \frac{1}{a}\left|\frac{2a}{\epsilon}\sum_{n=2}^{\infty}\frac{\tan^2\alpha_n \sin^2\alpha_n}{(\epsilon+\sin^2\alpha_n)}\cos(\omega_n t)\right| \\ &\leqslant \frac{2}{\epsilon}\sum_{n=2}^{\infty}\frac{\tan^2\alpha_n \sin^2\alpha_n}{\epsilon+\sin^2\alpha_n} = 2\sum_{n=2}^{\infty}\frac{\epsilon\sin^2\alpha_n}{\alpha_n^2(\epsilon+\sin^2\alpha_n)} \\ &\leqslant 2\sum_{n=2}^{\infty}\frac{\sin^2\alpha_n}{\alpha_n^2} \leqslant \sum_{n=2}^{\infty}\left(\frac{\sin\bar{\alpha}_n}{\bar{\alpha}_n}\right)^2 \\ &\leqslant 2\sum_{n=2}^{\infty}\frac{1}{\bar{\alpha}_n^2} = \frac{2}{\pi^2}\sum_{n=2}^{\infty}\frac{1}{(n-1+\delta_n)^2} \\ &= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(n+\delta_n)^2} < \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(n+\delta_2)^2} \\ &= \frac{2}{\pi^2}\Psi'(\delta_2) = \frac{2}{\pi^2}\Psi'(0.43) = 0.2019. \end{aligned} \tag{3.7}$$

Here the $\bar{\alpha}_n = [(n-1)+\delta_n]\pi$, $n > 2$, are the roots of

$\tan\bar{\alpha}_n = \bar{\alpha}_n$.

These values of $\bar{\alpha}_n$ maximize $(\sin\alpha_n/\alpha_n)$. A comparison of the graphs of the functions $\tan\alpha$ and α makes it clear that the δ_n *increase* with increasing n. For $n = 2$ we find $\delta_2 = 0.4303$. The final series can be related to the second derivative of the logarithm of the gamma function and evaluated numerically[21] to obtain the value given in (3.7). This is essentially the same result we would have gotten if we had simply *assumed* that the bound will get larger as ϵ increases[22] and taken the limit $\epsilon \to \infty$ on the first bound given in (3.7) for $\Delta(t;\epsilon)$

$$\begin{aligned}\Delta(t;\epsilon) &\leqslant 2\epsilon^2\sum_{n=2}^{\infty}\frac{1}{\alpha_n^2[\alpha_n^2+\epsilon(\epsilon+1)]} \\ &\underset{\epsilon\to\infty}{\longrightarrow}\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2} = \frac{8}{\pi^2}\left(\frac{\pi^2}{8}-1\right) = 0.1894.\end{aligned}$$

Here we have used the fact that, for $n \geqslant 2$,

$$(n-1)\pi \leqslant \alpha_n \leqslant (n-\tfrac{1}{2})\pi, \tag{3.8}$$

and summed the resulting series.[23] Similarly, we can obtain the useful bound as $\epsilon \to 0$,

$$\Delta(t,\epsilon)\underset{\epsilon\to 0}{\longrightarrow}\frac{2\epsilon^2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\epsilon^2}{45}.$$

Direct numerical calculation produces bounds on $\Delta(t;\epsilon)$ of 0.14 for $\epsilon = 10$, 0.077 for $\epsilon = 4$, 0.072 for $\epsilon = 3$, 0.062 for $\epsilon = 2$, and 0.018 for $\epsilon = 1$. All this together shows why the actual motion of the end of the spring is very well approximated *at all times* by the first term in Eq. (3.2).

Bowen[12] has pointed out that, in at least one case, some general results from the theory of partial differential equations can be used to express $w(\xi,t)$ in elementary form for the free spring. This technique can be developed[24] to treat the general problem defined by Eqs. (2.4), (2.5), (2.8)–(2.10). However, we now give a simple argument which allows us to write down in elementary form the solution $w(\xi,t)$ for the *free* spring ($M = 0$) subject to the initial conditions of Eqs. (2.11) and (2.12).

First, we can see that the period of the free spring will be just

$$\tau_0 = 4l_0/c_0 = 4(m/k)^{1/2}, \tag{3.9}$$

no matter what the initial conditions. Since one end ($\xi = 0$) of the spring is fixed and the other free ($\xi = l_0$), any pulse will be simply reflected at the ends, with a phase change of π at the fixed end and with no phase change at the free end. The motion of the spring can be considered as produced by a superposition of pulses each moving with a speed c_0. Figure 3 illustrates the state of motion and phase of one such pulse at times $t = 0$, l_0/c_0, $2l_0/c_0$, $3l_0/c_0$ and $4l_0/c_0$. After a time $\tau_0 = 4l_0/c_0$ every pulse in the spring will be back at its initial position, traveling with its initial velocity (not just speed) and with its initial phase. Therefore the free spring (no matter what its initial configuration and state of motion) will oscillate with period τ_0. One can also argue from the fact that the standing waves must have wavelengths λ such that an odd multiple of $(\lambda/4)$ equals l_0 or $\tau_0 = \lambda_0/c_0 = 4l_0/c_0$.

For the initial conditions of Eqs. (2.11) and (2.12) we see that the exact solution $w(\xi,t)$ remains just $a\,\xi/l_0$ until $t = (l_0 - \xi)/c_0$ since the relaxation wave travels with a speed c_0 and begins from the lower end ($\xi = l_0$) of the spring. Similarly, after a time $2\xi/c_0$ later, a wave reflected from the fixed end, with a phase change of π, will arrive back at ξ so that $w(\xi,t)$ becomes $-a\,\xi/l_0$. This is valid for $(l_0+\xi)/c_0 \leqslant t \leqslant (3l_0 - \xi)/c_0$, at which latter time a wave reflected from the free end engulfs the point ξ. Finally, at $(3l_0 + \xi)/c_0$ the wave is back in its initial phase (i.e., two fixed-end and one free-end reflections) so that the solution $w(\xi,t)$ is again $a\,\xi/l_0$ and remains so until $(l_0-\xi)/c_0 + \tau_0 = (5l_0 - \xi)/c_0$. That is, prior to the time $(4l_0 + l_0 - \xi)/c_0$ the spring must be in the same state as it was until $(l - \xi)/c_0$.

Let us take for granted the uniqueness of the solution $w(\xi,t)$ and its continuity in ξ and t.[25] We then have a solution which satisfies Eqs. (2.4), (2.8), (2.11), and (2.12), essentially because it is linear in ξ. We must also satisfy Eq. (2.5) for $M = 0$, namely, $(\partial w/\partial\xi)|_{l_0} = 0$ for all $t > 0$ (i.e., *after* the lower end of the spring has been released). This requires us to find a solution valid for $\xi = l_0$ when $t > 0$, including the intervals $(l_0 - \xi)/c_0 \leqslant t \leqslant (l_0+\xi)/c_0$ and $(3l_0-\xi)/c_0 \leqslant t \leqslant (3l_0+\xi)/c_0$. *If* $w(\xi,t)$ were linear in t here, then both Eqs. (2.4) and (2.5) would be satisfied. We can now take $w(\xi,t) = A + Bt$ and match it to the pieces of the solution

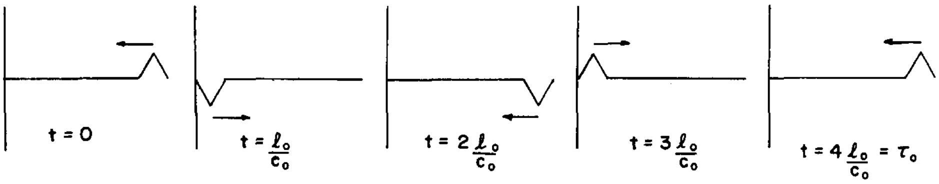

Fig. 3. Elementary argument for the period $\tau_0 = 4l_0/c_0$ for $M = 0$.

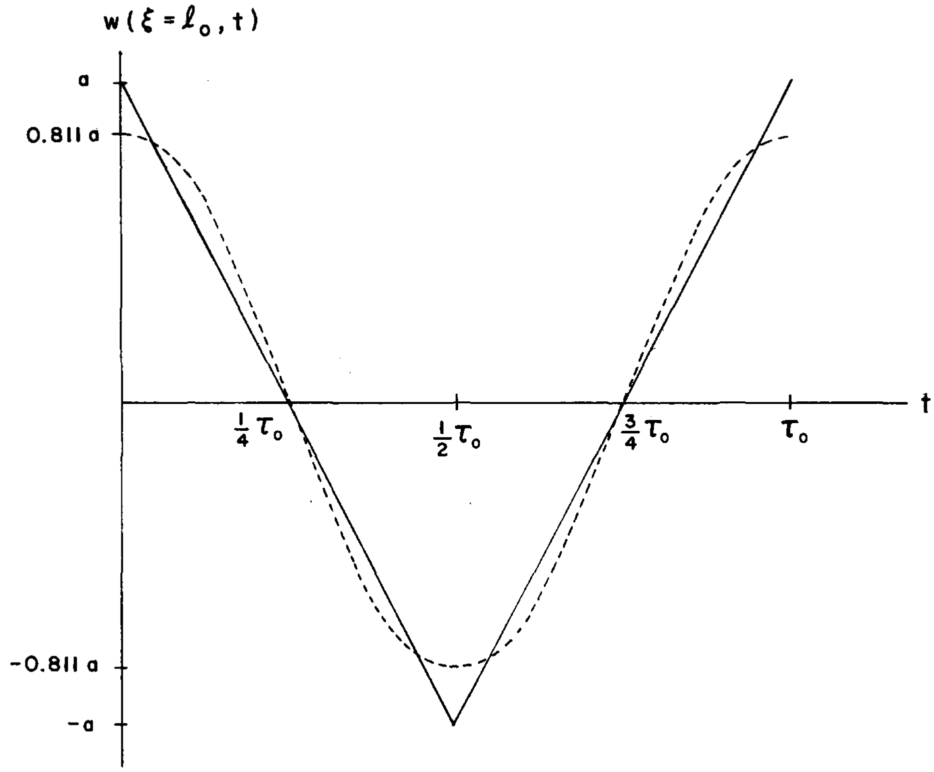

Fig. 4. Comparison of the exact and approximate solutions for $M = 0$.

we have found in the previous paragraph. In this way we obtain a valid solution which can be expressed as

$$w(\xi,t) = \begin{cases} \dfrac{a}{l_0}\xi, & 0 \leqslant t \leqslant \dfrac{l_0}{\omega} - \dfrac{\xi}{c_0} \quad \text{I} \\ a - \dfrac{a}{l_0}c_0 t, & \dfrac{l_0}{c_0} - \dfrac{\xi}{c_0} \leqslant t \leqslant \dfrac{l_0}{c_0} + \dfrac{\xi}{c_0} \quad \text{II} \\ -\dfrac{a}{l_0}\xi, & \dfrac{l_0}{c_0} + \dfrac{\xi}{c_0} \leqslant t \leqslant \dfrac{3l_0}{c_0} - \dfrac{\xi}{c_0} \quad \text{III} \\ -3a + \dfrac{a}{l_0}c_0 t, & \dfrac{3l_0}{c_0} - \dfrac{\xi}{c_0} \leqslant t \leqslant \dfrac{3l_0}{c_0} + \dfrac{\xi}{c_0} \quad \text{IV} \\ \dfrac{a}{l_0}\xi, & \dfrac{3l_0}{c_0} + \dfrac{\xi}{c_0} \leqslant t \leqslant \dfrac{5l_0}{c_0} - \dfrac{\xi}{c_0} \quad \text{V} \end{cases} \tag{3.10}$$

with

$$w(\xi,t + \tau_0) = w(\xi,t). \tag{3.11}$$

The simplicity of the expression of Eq. (3.10) for $w(\xi,t)$ is all the more remarkable when one appreciates that the Fourier series solution for the free spring case is

$$w(\xi,t) = \frac{2a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n-\frac{1}{2})^2} \times \sin\left((n-\tfrac{1}{2})\pi\frac{\xi}{l_0}\right)\cos\left((n-\tfrac{1}{2})\frac{\pi c_0}{l_0}t\right). \tag{3.12}$$

It is not immediately evident that the series in Eq. (3.12) can be summed to Eq. (3.10), although the task can be accomplished with some effort (especially when one knows the answer in advance). In fact, even a student in an introduc-

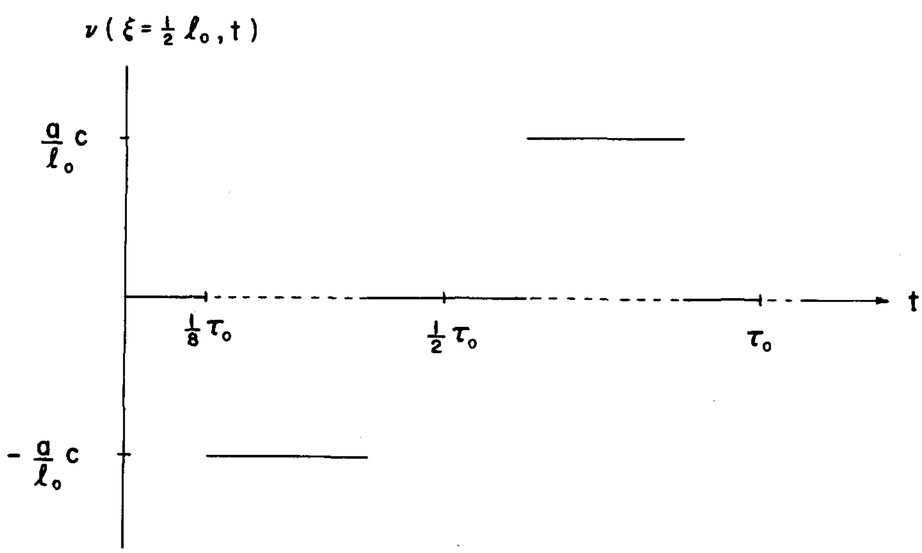

Fig. 5. Velocity of midpoint of freely oscillating spring.

tory course can rather easily verify directly that Eq. (3.10) does provide a solution to the problem which meets all the initial and boundary conditions. This is not as easy to do for the solution as represented by Eq. (3.12).

Figure 4 shows a comparison between the exact solution for the motion of the end of the free spring, $w(\xi = l_0, t)$ of Eq. (3.10), and the fundamental term (with $M = 0$) of Eq. (3.2). If we compute the velocity of the spring as $v(\xi,t) = \partial w(\xi,t)/\partial t$, we find that the velocity of the end of the spring is discontinuous, being a step function. Because of this discontinuity, the acceleration of the end of the spring is infinite at $t = 0, \frac{1}{2}\tau_0, \tau_0, \ldots$. This must be so since the *end* of the spring (which has *zero* mass) was pulled down (below its equilibrium position) and released. The spring, when released exerts a finite force. This impulsive acceleration does not occur when a mass M is attached to the end of the spring since then the finite force exerted by the spring produces a finite acceleration on the finite mass M.

The velocity profile of the end of the spring is not what a student would expect. It is certainly *not* simple harmonic. Even more surprising is the motion of the middle (that is at $\xi = \frac{1}{2}l_0$) of the spring.[26] Equation (3.10) allows us to construct the velocity curve of Fig. 5. [Realize from Eqs. (2.2), (2.3), (2.11), and Ref. 16 that $\xi = \frac{1}{2}l_0$ is the point

$$\zeta = \frac{l_0}{2} + \frac{3}{8}\frac{mg}{k} + \frac{a}{2}$$

on the stretched spring before it is released. That is, for a very soft spring, given a small displacement a, it is the point approximately $\frac{3}{4}$ down from the top of the spring.] This behavior, like that of the motion of the end of the spring, is directly understandable in terms of the initial impulse from the bottom end of the spring as it propagates up the spring at a constant speed c_0 (measured in the coordinate ξ).

IV. LABORATORY AND DEMONSTRATION APPLICATIONS

We have seen that a vertically oscillating massive spring with a mass M attached to its free end provides an interesting complex of phenomena for study. This system lends itself nicely to a laboratory demonstration for an introductory physics course. The uneven *linear* uncoiling of the spring and the variation in speed of the propagation of a pulse are easily illustrated with a long slinky spring. Similarly, since students can readily verify that the solution (3.10) satisfies the equations of motion, initial, and boundary conditions, they can understand the predicted behavior of the velocity-time curves (e.g., Fig. 5). Those curves have simple explanations in terms of an impulse which propagates along the spring and changes the state of motion as it engulfs successive elements of the spring.

Students can also verify directly the variation of the period $\tau(\epsilon)$ vs $1/\sqrt{\epsilon}$ predicted in Fig. 2.[27] The experimental data can be plotted directly on a piece of graph paper on which the curve of Fig. 2 has already been drawn. Not only can the students verify empirically that the effective period of oscillation is reliably given by Eqs. (3.3) and (3.5), but they will also appreciate that this result can be expected theoretically because the lowest-fundamental mode amplitude gives by far the largest contribution to the motion of the mass M for *all* values of ϵ.

The value of the spring constant k can be determined directly as follows. Suppose a spring of mass m is suspended and comes to static equilibrium under its own weight.[28] A mass M can then be attached to its free end and the new equilibrium position of the spring is at a position z, its initial equilibrium position (before M had been attached) being z_0. From Ref. 16 we have

$$z - z_0 \equiv y_0(M,m;\xi = l_0) - y_0(M = 0,m;\xi = l_0) = \frac{Mg}{k}. \tag{4.1}$$

Equation (4.1) is just what we would expect for the spring stretched horizontally by a force Mg (since the weight mg of the spring would not contribute to the stretching then). The result (4.1) is what most students would guess without reflection. A little more thought, taking account of the fact that $m \neq 0$, raises some question about the validity of (4.1). However, a careful analysis indeed justifies (4.1). Successively larger masses should be attached to the spring to find the range over which k remains essentially constant.

Each mass M can then be set in oscillation about its equilibrium position by pulling it down a small[29] distance a below its equilibrium position and releasing it from rest. Data from a typical experiment with different springs are shown as the open circles in Fig. 2. Since a student usually judges the period as the time between successive instants of rest of M at its lowest point of motion and since the motion is not *strictly* periodic for $0 < \epsilon < \infty$, the time should be measured for a large number (say 50–100) of successive oscillations to obtain an average value for the "period" τ.

It must be appreciated that all of the discussion and analysis above has been predicated upon a uniform spring which, in a horizontal (that is, $g = 0$) configuration, would respond *linearly* to an applied force. However, a spring is often wound in such a way that in its unstretched state there is a compressive force tending to draw the coils together. The actual physical thickness of the coils prevents this. When these coils are first separated, a larger force must be applied than would be required were the coil "loose" when it had its natural length l_0. This compressive force could, in principle, be measured by finding how large a force F_0 has to be applied before the spring begins to uncoil. This would have to be done with the spring in a *horizontal* position, of course, and such a direct determination is not easy to make accurately. It is easier to attempt to "prestretch" the spring to eliminate this unwanted compressive force, but again, this can be difficult to do *uniformly* for a long soft spring. Therefore we discuss how one eliminates these unwanted effects produced by a tightly wound spring.

Suppose that the natural length of a spring of Hooke's constant k would be l_0 if it were not for the thickness of the coil wire and that its actual physical length is l ($> l_0$). Once the spring begins to separate (in a horizontal position), the external force required to bring its free end to a position z is

$$F = k(z - l_0), \qquad z > l. \tag{4.2}$$

Here z is measured from the fixed end of the spring. Until a certain minimum force F_0 is applied, however, z remains at the value l as indicated in Fig. 6, where

$$F_0 = k(l - l_0). \tag{4.3}$$

Just as Eq. (4.2) holds only when $F > F_0$, so Eq. (4.1) (or Ref. 16) is valid only for those $M > M_0$, where

$$F_0 = M_0 g. \tag{4.4}$$

That is, we could think of first applying the force $F_0 = M_0 g$ to the horizontal spring (so that it would be "loosely"

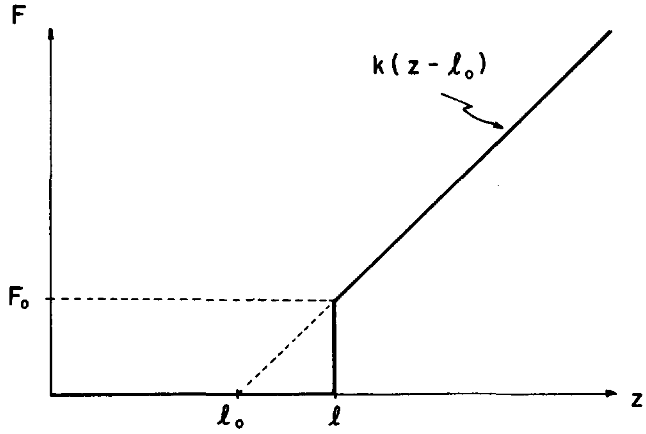

Fig. 6. The applied force versus horizontal elongation for a tightly wound spring.

coiled) and then suspending it vertically and attaching more weights. For $M < M_0$, the spring will uncoil nonuniformly, but the bottom portion will remain compressed since the weight of that portion will remain less than F_0. Once M exceeds M_0, then (4.1) applies. A typical plot of M vs z for a tightly wound spring is shown in Fig. 7. The extrapolation of the linear part of the curve back to $M = 0$ yields the equilibrium length for a loosely coiled spring of mass m (see Ref. 16):

$$z_0 = l_0 + M_0 g/k + mg/2k. \tag{4.5}$$

As a practical matter, students suspend a series of weights from the spring, measure z, and construct a graph like that of Fig. 7. The value of k is gotten from the slope of the straight-line segment of the graph. Of course, it is only for masses M corresponding to this straight part of the graph that Eq. (3.5) and the curve of Fig. 2 apply. Values of M_0 and z_0 can also be taken from Fig. 7 for a given spring and Eq. (4.5) used as a check on the value of k obtained from the slope. However, the values of M_0 and z_0 gotten from the graph are often too small to be useful. With m and k known, τ_0 is found from Eq. (3.9). Then, the experimental values measured for τ can be normalized to τ_0 for comparison with Fig. 2. Since the curve of Fig. 2 is a universal curve valid for any spring, data for τ/τ_0 from several springs (in their *linear* regions) can be plotted on the same graph and should fall on a common curve.

There is one other factor which can affect the period for large elongations of the spring. As a spring stretches, the radius r_0 of the helix remains nearly constant but the spring unwinds around its axis. Even though the restoring force may remain linear in the axial elongation, this unwinding effect can become large enough that the torsional motion of M becomes significant. Consider a spring whose uncoiled length would be L and whose lower end is a distance z beyond its upper, fixed end. The total angle ϕ turned through as this helix is wound is

$$\phi = 2\pi \frac{L \cos\theta}{2\pi r_0} = \frac{L}{r_0}\left[1 - \left(\frac{z}{L}\right)^2\right]^{1/2}, \tag{4.6}$$

where θ is the pitch angle of the helix. As the spring is stretched an amount Δz the change $\Delta\phi$ in this angle is

$$\frac{\Delta\phi}{2\pi} \simeq \frac{d\phi}{dz}\frac{\Delta z}{2\pi} = \frac{(z/L)(\Delta z/2\pi r_0)}{[1 - (z/L)^2]^{1/2}}. \tag{4.7}$$

Even for modest displacements Δz the ratio $(\Delta z/2\pi r_0)$ will be of order unity. However, as long as $(z/L) \ll 1$, the coupling between the torsional and axial motions remains negligible. Once z becomes an appreciable fraction of L, coupling to the torsional mode varies the period τ so that the prediction of Eq. (3.5) (or of Fig. 2) no longer remains valid.

Finally, there are two more instructive points related to the mechanical system we have studied. For the easily realizable initial condition of Eqs. (2.11) and (2.12), we have seen that the system (2.4)–(2.8) has a solution in which essentially only the lowest normal-mode frequency is excited. However, it is possible to use resonance to excite other normal modes. Suppose that the upper end of the spring is driven at a frequency ω

$$w(\xi = 0, t) = a \sin(\omega t). \tag{4.8}$$

Although a dissipation mechanism must be taken into account to provide a completely consistent treatment of the motion at resonance, we can simply assume on physical grounds that a steady-state motion of frequency ω *will* set in and write a solution of the form

$$w(\xi, t) = v(\xi)\sin(\omega t). \tag{4.9}$$

When Eq. (4.9) is substituted into the wave equation (2.4) and the resulting equation for $v(\xi)$ solved subject to the boundary condition (4.8), the solution becomes

$$w(\xi, t) = A \cos\left(\frac{\omega}{c_0}\xi + \phi\right)\sin(\omega t) \tag{4.10}$$

with

$$A \cos\phi = a. \tag{4.11}$$

Equation (2.5) at $\xi = l_0$ reduces to (since $\omega \neq 0$)

$$\tan\left(\frac{\omega}{c_0} l_0 + \phi\right) = -\frac{\omega}{c_0}\frac{l_0}{\epsilon}. \tag{4.12}$$

For a given spring (c_0, l_0) and a given M $(\epsilon = m/M)$, Eq. (4.12) defines ϕ as a function of ω. Therefore the displacement of the lower end of the spring $(\xi = l_0)$ is given as

$$w(\xi = l_0, t) = \frac{a \cos[(\omega/c_0) l_0 + \phi(\omega)]}{\cos[\phi(\omega)]}\sin(\omega t). \tag{4.13}$$

The amplitude resonates when ω is chosen to be one of the values ω_n such that

$$\phi(\omega_n) \to (n - \tfrac{1}{2})\pi, \quad n = 1, 2, 3, \ldots. \tag{4.14}$$

(The amplitude becomes infinite here only because we have neglected dissipation.) Equations (4.14) and (4.12) com-

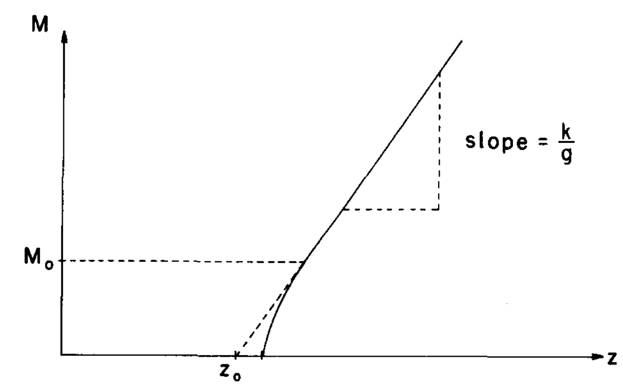

Fig. 7. The mass M versus vertical elongation for a tightly wound spring.

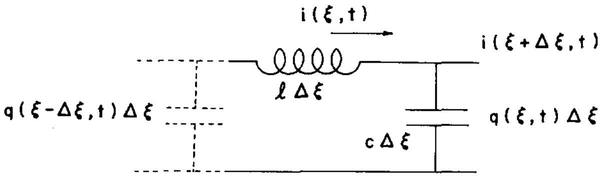

Fig. 8. The electric circuit analog of a massive spring.

bined give as the condition for the ω_n

$$\tan\left(\frac{\omega_n}{c_0}l_0\right)=\frac{\epsilon}{(\omega_n l_0/c_0)}, \tag{4.15}$$

which is the same as Eq. (3.3). That is, the resonant frequencies are just the normal modes we have found earlier. This resonance phenomenon is easily demonstrated by suspending the mass-spring system from a variable-frequency vibrator (or from a rod attached to an eccentric cam on a small variable-speed electric motor). The observed resonant frequencies can be compared with the predicted values for ω_n.

Second, the electric-circuit analog of the spring-mass system also provides a simple illustration of the difference between lumped and distributed circuit elements. A uniform coaxial cable or line with a capacitance c per unit length and an inductance l per unit length is the analog of the massive spring. The total capacitance C and inductance L of this line are given as $C = cl_0$ and $L = ll_0$, where l_0 is the length of the line. A length $\Delta\xi$ of this line can be represented by the circuit of Fig. 8. Here $q(\xi,t)$ is the charge per unit length and $i(\xi,t)$ is the current flowing in the segment. Kirchhoff's laws applied to the loop of Fig. 8 yield

$$-\frac{1}{c}\frac{\partial^2 i}{\partial\xi^2}+l\frac{\partial^2 i}{\partial t^2}=0. \tag{4.16}$$

This is of the same form as Eq. (2.4). If one end of the cable is driven with a sinusoidal input

$$V(t)=V_0\sin(\omega t) \tag{4.17}$$

from a variable-frequency generator and the other end of the cable is loaded with an inductance L', then we have an analog of the mass-spring system with

$$\epsilon = L/L'. \tag{4.18}$$

The analog of Eq. (2.5) comes from requiring that the emf at the end of the cable be the same as that appearing across the inductance L'. The resonance condition (3.3) is then an impedance-matching one. If the voltage $V_{L'}$ across the inductance L' is displayed on an oscilloscope, the resonant frequencies ω_n of Eqs. (3.3) and (3.4) can again be found. The cable, which is a distributed circuit element, does not behave as a simple lumped LC circuit, which would have just *one* resonant frequency.

For example, in one demonstration a 100-ft (30.48 m) length of coaxial cable with a specific capacitance $c=9.5\times10^{-11}$ F/m and a specific inductance $1=2.70\times10^{-7}$ H/m was used. The load inductance L' was 4.0×10^{-6} H (known to only $\pm$ 10%). From Eq. (4.18) we have

$$\epsilon = L/L' = 2.05 \tag{4.19}$$

so that according to Eq. (3.4) the normal-mode frequencies should be

$$\nu_n=\frac{\omega_n}{2\pi}=\frac{c_0}{l_0}\frac{\alpha_n}{2\pi},\quad n=1,2,3\ldots.$$

Here $c_0=(cl)^{-1/2}$ so that

$$\frac{c_0}{l_0}=\frac{1}{(LC)^{1/2}}=6.47\times10^6\ \mathrm{s}^{-1}$$

and the α_n are given as

$$\alpha_n=(n-1)\pi+\delta_n,\quad n=1,2,3\ldots,$$

$$[(n-1)\pi+\delta_n]\tan\delta_n=\epsilon=2.05.$$

The first few predicted normal-mode frequencies are $\nu_1=1.12$ MHz, $\nu_2=3.76$ MHz, $\nu_3=6.78$ MHz, $\nu_4=9.92$ MHz, and $\nu_5=13.1$ MHz. Resonance was observed when the frequency of the sine-wave signal generator was set at $\nu_1=1.1$ MHz, $\nu_2=3.6$ MHz, $\nu_3=6.6$ MHz, $\nu_4=9.6$ MHz, and $\nu_5=12.6$ MHz.

An exact mechanical analog was constructed using the soft spring employed to obtain the data of Fig. 2. Since $m=11.6$ g and $k=1.20$ N/m, for that spring, the value of ϵ from Eq. (4.19) required $M=5.66$ g for the mass suspended on the lower end of the spring. With

$$\frac{c_0}{l_0}=\left(\frac{k}{m}\right)^{1/2}=10.17\ \mathrm{s}^{-1},$$

the lowest predicted normal-mode frequencies are $\nu_1=1.76$ Hz, $\nu_2=5.91$ Hz, $\nu_3=10.66$ Hz, $\nu_4=15.6$ Hz, and $\nu_5=20.6$ Hz. These resonant frequencies were also observed, although one must increase the frequency of the vibrator *slowly* since it takes a while for the mechanical oscillations to settle down.

ACKNOWLEDGMENTS

It is a pleasure to thank Norman Gaiser for assisting in gathering the data for Fig. 2, Donald McLane for setting up the normal-mode resonance demonstrations for the coaxial cable and the loaded spring, and Neil Nash for drawing some of the figures. Also, Paul Chagnon pointed out to me that there should be a transmission-line analog to the massive spring. Robert Weinstock and an anonymous referee provided helpful comments on an earlier draft of the manuscript.

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[15]A much more complete version of this paper, which was judged too lengthy for publication in this journal, is available from the author upon request.
[16]The equation of motion in terms of the variable $y(\xi,t)$ of Eq. (2.3) can be obtained from Eq. (2.4) by writing $-g$ (the acceleration due to gravity)

on the right-hand side of Eq. (2.4), while Eqs. (2.5) and (2.8) remain unchanged for $y(\xi,t)$. From this the static solution $y_0(\xi)$ is readily obtained as

$$y_0(\xi)=\left(\frac{(M+m)g}{kl_0}-\frac{1}{2}\frac{mg}{kl_0^2}\,\xi\right)\xi.$$

Since this elongation is *not* linear in ξ (for $m\neq 0$), the spacing between neighboring coils will not be uniform for a spring suspended at rest in a uniform gravitational field. Students can easily test this for the $M=0$ case in the laboratory. However, the *additional* elongation of the spring as M is increased is proportional to M so that k can still be found in terms of the proportionality constant between ΔM and Δy_0.

[17]This qualitative feature has been noted previously in the literature (see Ref. 8).

[18]With respect to the reference or background variable $\zeta_0=\xi+y_0(\xi)+w_0(\xi)$, where $y_0(\xi)$ is given in Ref. 16 (with $M=0$ here) and $w_0(\xi)$ in (2.11), the local pulse speed is just

$$c(\zeta_0)=\left|\frac{d\zeta_0}{dt}\right|=\left|\frac{d\zeta_0}{d\xi}\right|\left|\frac{d\xi}{dt}\right|=\left|\frac{d\zeta_0}{d\xi}\right|c_0.$$

[19]Reference 9, Eq. (3.6).

[20]I thank an anonymous referee for suggesting this bound. The function defined in Eq. (3.6) can be shown to be a nonincreasing function of α_1 for $0<\alpha_1<\pi/2$.

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[23]L. B. W. Jolley, *Summation of Series* (Dover, New York, 1961), p. 64.

[24]This method will be treated in detail in a subsequent paper in this journal.

[25]F. John, *Partial Differential Equations* (Springer-Verlag, Berlin, 1982), 4th ed., pp. 40 and 41; R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1972), Vol. II, p. 41.

[26]This type phenomenon was also pointed out by Bowen (Ref. 12, Fig. 3, p. 1147).

[27]This comparison has also been suggested in Ref. 8.

[28]It is assumed here that the spring chosen not be a tightly wound one. If it is tightly wound, then in its natural unstretched state, there is a compressive force tending to press one coil against another. This is discussed further below.

[29]The amplitude of oscillation should not be too large in order to keep the torsional mode of oscillation small. This mode is driven by the unwinding of the spring as it is stretched. (See Refs. 1 and 2 for comments on this.) There is also an instability in the coupling of the pendular and vertical oscillatory modes when the equilibrium length is about $\frac{4}{3}l_0$ (Ref. 7).

The method of characteristics applied to the massive spring problem

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(Received 15 April 1983; accepted for publication 6 November 1983)

The method of characteristic curves applied to the wave equation is employed to obtain explicit solutions to the problem of a spring of mass m oscillating with a mass M attached to one end. For the case of the free spring ($M=0$), the method produces simple solutions expressible in terms of functions which are linear in the displacement and in the time.

I. INTRODUCTION

In the previous article in this journal,[1] the problem of a vertically oscillating spring of mass m and spring constant k suspended from its upper end and with a mass M attached to its lower end was discussed in some detail for arbitrary values of the parameter $\epsilon=m/M$. In this paper we extend a suggestion made by Bowen[2] to apply some general results from the theory of partial differential equations to that problem. The mathematical problem to be solved is the following.[3] We seek a solution $w(\xi,t)$ to the homogeneous wave equation for $0\leqslant\xi\leqslant l_0$,

$$c_0^2\frac{\partial^2 w}{\partial\xi^2}-\frac{\partial^2 w}{\partial t^2}=0, \tag{1.1}$$

subject to the boundary conditions

$$w(\xi=0,t)=0, \tag{1.2}$$

$$\left.\frac{\partial w}{\partial\xi}\right|_{\xi=l_0}+\kappa\left.\frac{\partial^2 w}{\partial\xi^2}\right|_{\xi=l_0}=0, \tag{1.3}$$

and to the initial conditions

$$w(\xi,t=0)=f(\xi), \tag{1.4}$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=0}=g(\xi). \tag{1.5}$$

Here $f(\xi)$ and $g(\xi)$ are given functions on $0\leqslant\xi\leqslant l_0$, c_0 is a constant, and κ is the parameter

$$\kappa=l_0/\epsilon. \tag{1.6}$$

II. SOLUTION BY CHARACTERISTICS

As is easily seen,[4] the most general solution of Eq. (1.1) is of the form

$$w(\xi,t)=F(\xi+c_0t)+G(\xi-c_0t), \tag{2.1}$$

where $F(\eta)$ and $G(\eta)$ are *arbitrary* functions of η. This shows that any disturbance present at $t=0$ propagates as an undistorted pulse (or pulses depending upon the value of $\partial w/\partial t$ at $t=0$) with a speed c_0 (until some boundary is encountered). In fact, the pulse or wave form $F(\eta)$ propagates (undistorted) along the characteristic curve $\xi=-c_0t$ while $G(\eta)$ propagates along the characteristic $\xi=c_0t$.

If Eq. (2.1) is substituted into Eqs. (1.4) and (1.5), the