The static effectiveness mass of a slinky™

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The static elongation of a loaded spring is derived taking into account its own weight and the fact that a finite nonzero force is required to pull the spring apart. A simple experiment is described to verify that the static effective mass is equal to half of the total mass of the spring using a slinky. The experimental result is in good agreement with the theory.

I. INTRODUCTION

Hook's law is a topic that appears in every elementary mechanics course. If the mass of a loaded vertical spring is not negligible, its elongation is equivalent to that produced by a similar but massless spring with a further load equal to half the weight of the original spring added at its end. The law may then be expressed in the form[1]

$$\Delta l = w/2k + W/k, \qquad (1)$$

where Δl is the extension of the spring, w is its weight, W is the external load at its end, and k is its force constant.

Equation (1) is derived under the assumption that the contact force between any two neighboring turns is zero when the spring is unloaded. This assumption is only correct for compression springs whose turns are often widely separated in the unloaded state. Springs used in Hook's law experiments are, however, of extension types and are usually constructed in such a way that a finite force is required to pull the spring apart. Taking the effect of this internal stress into account, we shall show in the next section that Eq. (1) should be modified to write as

$$\Delta l = \frac{w}{2k} + \frac{W - F_0}{k}, \qquad \text{for } W \geqslant F_0, \qquad (2a)$$

$$= \frac{w}{2k} - \frac{F_0 - W}{k} + \frac{(F_0 - W)^2}{2wk},$$
$$\text{for } (F_0 - w) \leqslant W < F_0, \qquad (2b)$$

and

$$= 0, \qquad \text{for } W < (F_0 - w),$$

where F_0 is the minimum force required to pull the spring apart. Since F_0 may have values comparable to w and can hardly be determined accurately, the use of conventional experimental method by plotting elongation against the applied load is unlikely to yield satisfactory results in the determination of static effective mass. Consequently, experiments to verify that the effective mass is equal to half of its own mass are seldom mentioned in textbooks or in the literature. In this article, we present a very simple and reasonably accurate method to measure the static effective mass of a slinky spring. Essentially, the only piece of apparatus required is the slinky itself plus a meter rule. Slotted weights are not needed because while part of the slinky is taken as the "spring" under consideration, the part below the "spring" may be taken as the "load."

II. THE MODIFIED HOOK'S LAW

In this section, we first derive Eq. (2) for the general situation of a loaded suspended spring in static equilibrium. We then rewrite the equation into a form that facilitates discussion in our particular setting.

Let l be the natural length and k be the force constant of an arbitrary spring. Suppose the spring is pulled horizontally and gives rise to an extension δ. Consider a segment of the spring at a distance y from one end, of length Δy [Fig. 1(a)].

For a uniform spring, the extension of the segment is, by direct proportion,

$$\Delta l' = \delta(\Delta y/l).$$

Let k' be the force constant of the segment. Then, by definition,

$$k' = \frac{F}{\Delta l'} = \frac{F}{\delta}\frac{l}{\Delta y} = k\frac{l}{\Delta y};$$

or in differential terms, replacing Δy by dy, we get

$$k'\,dy = kl. \qquad (3)$$

When the spring is suspended vertically and loaded by an external weight W at its lower end, dy will suffer an elongation dl, under the action of both the load W and the weight of the spring below the segment $w[(l-y)/l]$ [Fig.

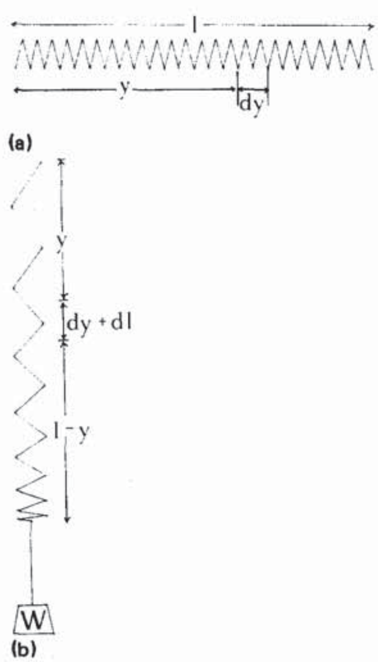

Fig. 1.(a) The unloaded spring in a horizontal position. (b) The extension of dl for the segment dy of the loaded vertical spring.

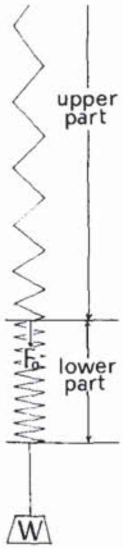

Fig. 2. The division of the spring into two parts to demonstrate the equivalence of Eqs. (2a) and (2b).

1(b)]. According to Hook's law

$$dl = \left[W + w\left(\frac{l-y}{l}\right) - F_0\right]\frac{1}{k'}$$

$$= \left[W + w\left(\frac{l-y}{l}\right) - F_0\right]\frac{dy}{kl},$$

$$\text{for } W + w\left(\frac{l-y}{l}\right) \geqslant F_0$$

and

$$dl = 0, \quad \text{for } W + w\left(\frac{l-y}{l}\right) < F_0. \tag{4}$$

Integrating for dy between 0 and l, we get back (2a) and (2b) for the total elongation.

For $W \geqslant F_0$,

$$\Delta l = \int_0^l dl = \frac{1}{kl}\int_0^l \left[W - F_0 + w\left(\frac{l-y}{l}\right)\right]dy$$

$$= w/2k + (W - F_0)/k, \tag{2a'}$$

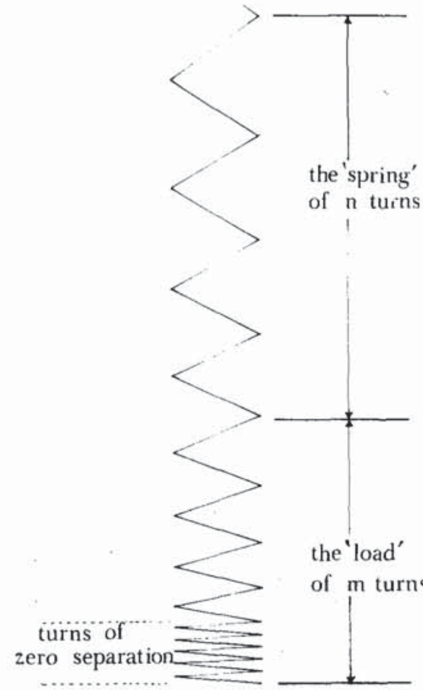

Fig. 3. The division of the slinky into two parts with the upper part taken as the "spring" and the lower part taken as the "load."

for $(F_0 - w) \leqslant W < F_0$,

$$\Delta l = \frac{1}{kl}\int_0^{l - l(F_0 - W)/w}\left[W - F_0 + w\left(\frac{l-y}{l}\right)\right]dy$$

$$= \frac{w}{2k} - \frac{F_0 - W}{k} + \frac{(F_0 - W)^2}{2kw}, \tag{2b'}$$

and for $(F_0 - w) \geqslant W$,

$$\Delta l = 0.$$

We shall restrict our discussion to the case with $W \geqslant F_0$. The case with $(F_0 - w) \leqslant W < F_0$ can be considered a special case of (2a) if we imagine the spring to be composed of two parts with the upper part having a nonzero extension

Table I. Positions of different turns of a vertically suspending slinky.

Turn no.	1	2	3	4	5	6	7	8	9	10
Position z (in cm with $\Delta z = \pm 0.05$ cm)	0	3.3	6.6	9.9	13.2	16.4	19.6	22.7	25.7	28.6
	11	12	13	14	15	16	17	18	19	20
	31.5	34.3	37.1	39.8	42.4	44.9	47.4	49.9	52.3	54.7
	21	22	23	24	25	26	27	28	29	30
	57.0	59.3	61.5	63.7	65.8	67.8	69.8	71.7	73.5	75.2
	31	32	33	34	35	36	37	38	39	...63
	76.8	78.4	80.0	81.5	83.0	84.4	85.7	86.9	88.1	...~100

between any two neighboring turns and a natural length of $l[1-(F_0-W)/w]$ while the lower part has a natural length $l[F_0-W)/w]$ but zero extension (Fig. 2). As the lower part may be treated as a load added to the upper part in addition to W, we can obtain (2b) from (2a) by making the transformation

$$w \rightarrow w[1-(F_0-W)/w],$$

$$W \rightarrow F_0,$$

and

$$k \rightarrow k[w/(w+W-F_0)].$$

Similarly, for a freely suspended unloaded slinky, it is legitimate to choose an arbitrary portion of the total length as our "spring" and the portion below the "spring" as the "load" (Fig. 3). We shall denote the number of turns of the "spring" by n and the number of turns of our "load" by m. Provided that the weight of our "load" is greater than F_0, we can use Eq. (2a) and rewrite it in the form

$$\Delta l(n,m)=\frac{nw_1}{2k_n}+\frac{mw_1-F_0}{k_n}, \quad (5)$$

where k_n is the spring constant of the "n-turn spring" and w_1 is the weight per turn.

Since the "springs" so defined have different lengths but are otherwise identical, we have, according to (3),

$$nk_n=k_1,$$

where k_1 is the spring constant of the "one-turn spring." With k_1 replacing k_n, (5) may be further simplified to the form

$$\Delta l(n,m)=\frac{n^2 w_1}{2k_1}+\frac{(mw_1-F_0)n}{k_1}. \quad (6)$$

Hence, the elongation is expressed explicitly as a function of two variables, n and m, with the dependence being quadratic in the first but linear in the second. Also, all coefficients of the quadratic form are determined by constants w_1, k_1, and F_0.

III. EXPERIMENTAL PROCEDURE

A slinky[2] is clamped in such a way that an extended length of approximately 1 m, which corresponds to about 60 turns (in our case 63), is suspended vertically in space. Vertical positions z of about 40 turns, not near to either end, are recorded (Table I). Both the weight and the natural length of the slinky are measured and the corresponding weight per turn and thickness per turn are calculated giving, respectively, $w_1=2.00\times10^{-2}$ N and $l_1=0.055$ cm. For any n and m in Eq. (6), Δl can be obtained from the information given in Table I, e.g.,

$$\Delta l(10,25)=z_{38}-z_{28}-10l_1=86.9-71.7-10\times0.055$$
$$=14.7 \text{ cm}. \quad (7)$$

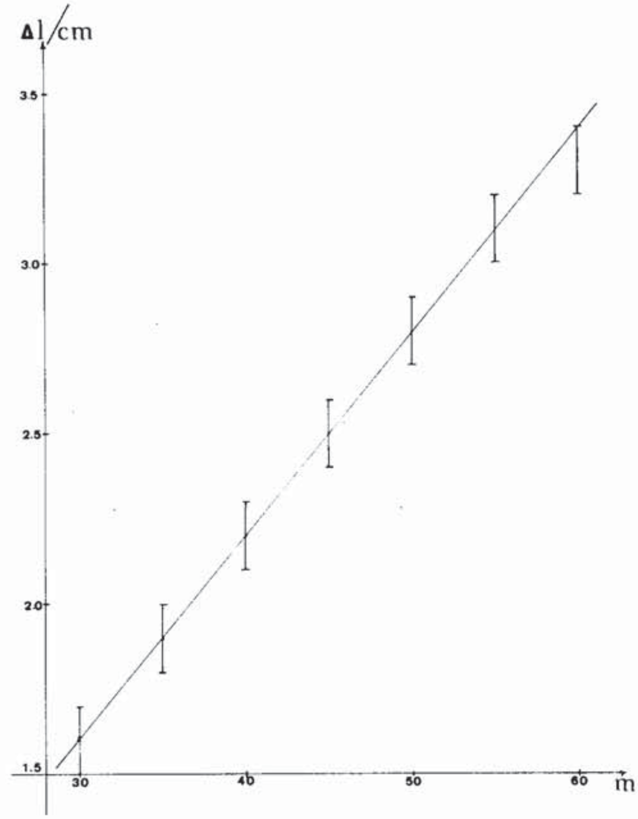

Fig. 4. The determination of k_1—the force constant of the "one-turn spring."

IV. DATA ANALYSIS

In order to verify that the contribution to the elongation due to the weight of the spring has a coefficient equal to one half, we first have to determine k_1 for the "one-turn spring." From (6), if we put $n=1$,

$$\Delta l=(l,m)=m(w_1/k_1)+\text{constant}. \quad (8)$$

Note that (8) is valid irrespective of the value of the effective mass and can also be obtained directly from Hook's law without reference to (6). For a slinky, if m is sufficiently large, $\Delta l(1,m)$ will be sufficiently long that it can be measured with a random percentage error kept below 5%. A rough inspection of Table I shows that this criterion can be satisfied for $m \geqslant 30$. Values of $\Delta l(1,m)$ can be calculated in a way similar to that demonstrated in Eq. (7). For the sake of easy reference, the relation between $\Delta l(1,m)$ and m is presented in Table II. A graph is plotted with $\Delta l(1,m)$ against m and a straight line is obtained (Fig. 4) with a slope equal to

$$w_1/k_1=6.0\times10^{-2}\text{ cm}^{-1},$$

Table II. The relation between $\Delta l(1,m)$ and m.

m	30	35	40	45	50	55	60
Δl/cm	1.6 ± 0.1	1.9	2.2	2.5	2.8	3.1	3.3

Table III. The relation between $\Delta l(n,25)/n$ and n.

n	10	15	20	25	30	35
$(\Delta l/n)$cm	1.47 ± 0.02	1.64	1.79	1.94	2.09	2.24

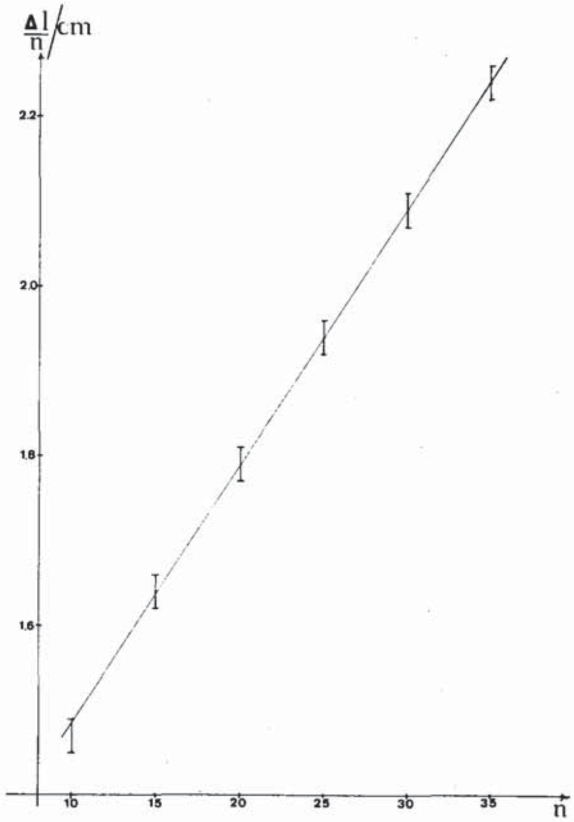

Fig. 5. The verification of the coefficient ½ for the static effective mass.

giving

$k_1 = 0.33 \text{N cm}^{-1}$.

We now turn back to our main issue; for any fixed m, (6) may be written as

$$\frac{\Delta l}{n} = \frac{w_1}{2k_1} n + \text{ constant independent of } n. \qquad (8')$$

Bearing in mind that the above equation is actually an alternate form of (2), the statement that "static effective mass is one half of the total mass of the spring" is equivalent to saying "the average elongation per turn increases linearly with the total number of turns of the 'spring' with a slope equal to $w_1/2k_1$." To minimize experimental errors, two conditions have to be met in choosing m. On the one hand, it must be sufficiently large to ensure that $mw_1 \gg F_0$; on the other hand, it must not be too large to limit the variation in the range of n. In our experiment we set $m = 25$. Again, for the sake of easy reference, the relation between $\Delta l(n,25)/n$ and n is presented in Table III.

A graph is plotted with $\Delta l/n$ against n, a straight line is obtained (Fig. 5), and the slope is measured giving

slope $= 0.030 \pm 0.002$ cm.

Theoretically, this slope is just $w_1/2k_1$. By direct substitution, we get

$w_1/2k_1 = 0.030$ cm.

Hence, our experimental result is in good agreement with that calculated from theory.

V. CONCLUSION

A simple theory describing the nature of real objects is uncommon in physics courses. This article presents an example of such a theory that can also be verified by a very simple experiment. Furthermore, the art of extracting useful information from a set of raw data, including the invention of *ad hoc* notations for particular settings, the choice of suitable constant values for controlled variables, the elimination of unknown/unwanted quantities plus the conversion of a quadratic relation into a linear plot provide good teaching opportunities in an elementary laboratory course. Finally, we feel that meaningful but interesting experiments related to the topic of statics are rare. The inclusion of this experiment in the laboratory course will relieve such an imbalance.

[1]E. E. Galloni and M. Kohen, Am. J. Phys. **47**, 1076 (1979).

[2]The slinky used is made in Japan with 100 turns of diameter 6.8 cm and a natural length of 5.5 cm.