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2006 Eur. J. Phys. 27 1349

(http://iopscience.iop.org/0143-0807/27/6/009)

Eur. J. Phys. **27** (2006) 1349–1355 doi:10.1088/0143-0807/27/6/009

# Energy propagation of thermal waves

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Received 10 July 2006, in final form 11 July 2006
Published 18 September 2006
Online at stacks.iop.org/EJP/27/1349

**Abstract**
Although waves are ubiquitous in nature it is difficult to give a precise and unambiguous definition of what a wave is. Actually the distinction between wave-like and non-wave-like behaviour can be fuzzy, as it is the case of a solid sample excited by a periodic heat source. The resulting temperature oscillations inside the sample have the same mathematical expression as highly damped waves, the so-called thermal waves. The aim of this paper is to stress the energy propagation as the key to affirm whether there is wave motion. In this way it is demonstrated that there is no wave nature in these improperly called thermal waves by showing that they do not transport energy. This result has been obtained not only in the frame of the parabolic heat conduction equation that evidences the diffusive nature of the heat conduction process, but also in the frame of the hyperbolic heat conduction equation, that is a wave equation.

## 1. Introduction

Physicists have extended the concept of a wave to a large number of phenomena corresponding to physical situations described by a time-varying field that propagates in both space and time. For instance, sound waves or electromagnetic waves are launched whenever the equilibrium is broken in the pressure field or in the electromagnetic field. This situation gives rise to the question of whether perturbations in the temperature field can propagate as thermal waves (also designed as heat waves or temperature waves). This term has been widely used in classical books on heat transfer [1–3] to designate the temperature oscillations produced by a periodic heat source, since they have the same mathematical expression as a highly damped wave, similar to that found in electromagnetic waves propagating through metals [4]. However heat conduction is a diffusive process governed by a parabolic differential equation which lacks second-order derivative with respect to time, characteristic of a wave equation.

Waves are present everywhere in nature. However, the concept of waves is very hard to define and the distinction between wave-like and non-wave-like behaviour can be fuzzy [5]. As an example most textbooks in introductory physics lack a precise and unambiguous definition of what a wave is. Perhaps the most acknowledged feature of wave motion is that energy is transferred through space without the transport of mass. This paper is addressed to wave physics lecturers with the aim of stressing the role of energy propagation as the key to affirm whether there is wave motion, instead of the type of differential equation it is the solution of. To do this we discuss a limiting problem as it is the case of heat conduction in solid samples that are excited by periodic light sources. We demonstrate that there is no wave nature in the improperly called thermal waves by showing that the energy they transport is zero. It must be pointed out that it has already been shown that thermal waves cannot be considered as real travelling waves because they show neither wave fronts nor reflection and refraction phenomena [6, 7]. However these demonstrations involve complicated mathematics for undergraduate students. The method proposed here is simple since it uses basic physics (energy) and mathematics.

Just for the sake of simplicity calculations are performed on an opaque and semi-infinite material which is illuminated by a periodically modulated light beam. In the first part of this paper we work in the frame of the parabolic heat conduction equation (with a first-order derivative with respect to time), derived from the classical Fourier's law, which establishes that the heat and temperature gradients are proportional to each other, with minus the thermal conductivity as the constant of proportionality. Its main drawback is that it does not take into consideration any propagation speed. This means that if, for instance, a heat source is applied to one end of a rod, the temperature of the other end begins to change instantaneously! James Clerk Maxwell is credited to be the first to realize that Fourier's law cannot be a complete description of the physical processes involved in heat conduction [8]. To overcome this limitation he introduced a relaxation time between temperature gradient and heat flux. That is the reason why we take into consideration, in the second part of this work, the influence of this relaxation time that leads to a hyperbolic heat equation (with both first- and second-order derivatives with respect to time). For the purpose of this paper it is worth noting that this last one is a wave equation similar to that found for electromagnetic waves propagating through conducting media. Consequently it is expected that heat could propagate as a real wave. However, we demonstrate that even in the frame of the hyperbolic heat conduction equation, periodic illumination of a sample does not launch thermal waves because they do not carry energy.

## 2. The parabolic heat conduction equation

In the classical approach, whenever there is a temperature gradient ($\vec{\nabla} T$) into a material a heat flow ($\vec{j}$) is instantaneously established, which in the case of homogeneous and isotropic materials follows Fourier's law

$$\vec{j} = -K\vec{\nabla} T, \tag{1}$$

where $K$ is the thermal conductivity of the material. Equation (1) together with the law of energy conservation leads to the parabolic heat diffusion equation, which in the absence of internal heat sources is written as

$$\nabla^2 T - \frac{1}{D}\frac{\partial T}{\partial t} = 0, \tag{2}$$

where $D$ is the thermal diffusivity, which is related to the thermal conductivity through the equation $K = \rho c D$, with $\rho$ and $c$ being the density and the specific heat, respectively.

Let us consider an opaque and semi-infinite solid whose surface is uniformly illuminated by a light beam of periodically modulated intensity $I_o(1 + \cos(\omega t))/2 = \mathrm{Re}[I_o(1 + \mathrm{e}^{\mathrm{i}\omega t})/2]$, where $I_o$ is the intensity of the beam and $\omega = 2\pi f$, with $f$ being the modulation frequency.

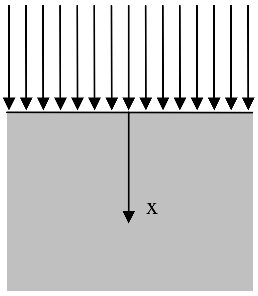


**Figure 1.** Geometry of an opaque and semi-infinite solid whose surface is uniformly illuminated by a modulated light beam.

The geometry of the problem is shown in figure 1. The temperature at any point of the material is given by

$$T(x,t) = T_{amb} + T_{dc}(x) + T_{ac}(x,t), \tag{3}$$

where $T_{amb}$ is the ambient temperature, $T_{dc}$ is a time-independent temperature rise above the ambient and $T_{ac}$ is a periodic temperature oscillation of the same frequency as the illumination that can be expressed as $T_{ac}(x,t) = \mathrm{Re}[\theta(x)\,\mathrm{e}^{\mathrm{i}\omega t}]$. Substituting equation (3) into equation (2) the spatial distribution of this time-dependent component satisfies Helmholtz's equation

$$\frac{\mathrm{d}^2\theta(x)}{\mathrm{d}x^2} - q^2\theta(x) = 0, \tag{4}$$

where $q = \sqrt{\mathrm{i}\omega/D}$.

By solving equation (4) and using as boundary condition the heat flux continuity on the sample surface

$$-K\left.\frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\right|_{x=0} = \frac{I_o}{2}, \tag{5}$$

the time-dependent component of the temperature is obtained as

$$T_{ac}(x,t) = \mathrm{Re}\left[\frac{I_o}{2Kq}\,\mathrm{e}^{-qx}\,\mathrm{e}^{\mathrm{i}\omega t}\right] = \frac{I_o}{2\varepsilon\sqrt{\omega}}\,\mathrm{e}^{-x/\mu}\cos\left(\frac{x}{\mu} - \omega t + \frac{\pi}{4}\right). \tag{6}$$

Here $\varepsilon = K/\sqrt{D}$ is the thermal effusivity of the sample and $\mu = \sqrt{2D/\omega}$. In figure 2(a) we show the amplitude of $T_{ac}$ as a function of the normalized depth $(x/\mu)$ for a semi-infinite stainless steel sample ($D = 4$ mm$^2$ s$^{-1}$, $\varepsilon = 7500$ J m$^{-2}$ K$^{-1}$ s$^{-1/2}$) illuminated by a light beam of intensity $I_o = 2\times10^6$ W m$^{-2}$ modulated at 100 Hz. Calculations have been performed at $t = 3.7 \times 10^{-3}$ s. As time elapses the temperature oscillates between the two dashed lines. As can be seen the parameter $\mu$ represents the depth at which the temperature amplitude is reduced by a factor $e$ and is usually called the thermal diffusion length. At a depth of 5 $\mu$ or $6\mu$ the temperature oscillation in the sample is completely damped.

Equation (6) has the same mathematical shape as a plane, harmonic and highly damped wave propagating along the $x$ axis. This is the reason why these temperature oscillations are usually designed as thermal waves, although they are the solution of a diffusion equation (with first-order time derivative) instead of a wave equation (with second-order time derivative).

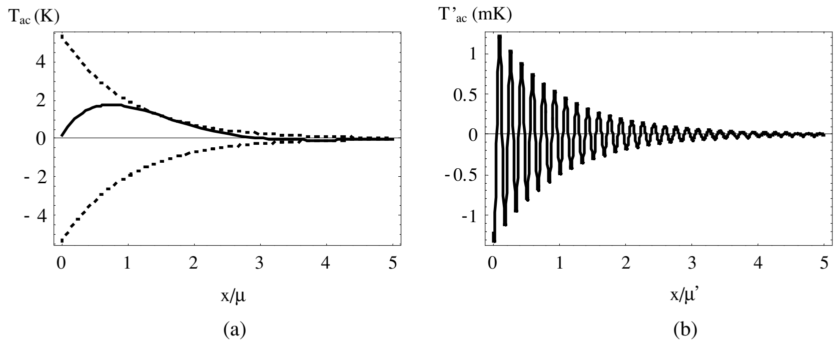


**Figure 2.** Amplitude of the oscillation temperature as a function of the normalized depth for a semi-infinite stainless steel sample illuminated by a light beam of intensity $2 \times 10^6$ W m$^{-2}$. (a) Simulation of the parabolic solution (6) with $f = 100$ Hz and $t = 3.7 \times 10^{-3}$ s. As time elapses the temperature oscillates between the broken lines. (b) Simulation of the hyperbolic solution (13) with $\tau = 10^{-10}$ s, $f = 5 \times 10^{10}$ Hz and $t = 1.9 \times 10^{-11}$ s.

Here we are interested in calculating the energy carried by this thermal wave. Following with the semi-infinite sample of figure 1, the heat flux corresponding to $T_{ac}$ is obtained by using equation (1):

$$\vec{j}_{ac} = -K\vec{\nabla}T_{ac} = \frac{I_o}{2}\,e^{-x/\mu}\cos\left(\frac{x}{\mu} - \omega t\right)\hat{\imath}, \tag{7}$$

whose time average taken over a period is $\langle\vec{j}_{ac}\rangle = \vec{0}$. This means that, unlike acoustic or electromagnetic waves, thermal waves do not carry energy. This result is related to the fact that heat flux is proportional to the temperature gradient, and therefore to the temperature itself. In contrast, in the case of real waves the intensity is proportional to the square of the physical quantity involved in each particular wave (e.g. the square of the electric field in electromagnetic waves, the square of the gas pressure in acoustic waves, the square of the displacement in waves propagating through a tight string, etc) and therefore proportional to $\cos^2(\omega t)$, whose average value is not zero. We can conclude that thermal waves are not travelling waves carrying energy, but just an oscillation of the temperature field. This result seems paradoxical because we are illuminating the sample surface and therefore energy is actually been transported through the sample. However, this apparent paradox is overcome if we realize that energy is only carried by $T_{dc}$. To clarify this point let us consider the experimental configuration in which the surface of the semi-infinite sample is in contact with alternating cold water (temperature lower than ambient) and hot water (temperature higher than ambient). In fact, this is the experimental configuration used by Ångström in 1861 to measure the thermal diffusivity along a metallic bar [9]. In this case the same amount of energy is delivered to the sample by the hot water as it is extracted from it by the cold water, and therefore no heat transfer through the sample takes place. The temperature of the sample is similar to that given in equations (3) and (6) but without $T_{dc}$. This means that only a temperature oscillation with respect to the ambient temperature appears (the improperly called thermal wave). Now there is a perfect correlation between the fact that the average energy delivered to the sample is zero and the fact that the thermal wave does not transport energy. Note that this situation is different from what happens in real waves. Let us consider the simple case of a horizontal rope that is tied at one end while the other end is shaken up and down. Now energy is delivered continuously, both when shaken up and when shaken down, so a net amount of energy is delivered to the rope in each period which is propagated through the rope as a travelling wave.

## 3. The hyperbolic heat conduction equation

As equation (1) does not take into account any propagation speed it cannot be a fundamental description of the transportation of heat. To overcome this problem Cattaneo proposed a delay between the temperature gradient and the heat flux [10]:

$$\vec{j}(x, t+\tau) = -K\vec{\nabla}T(x, t), \tag{8}$$

where $\tau$ is the thermal relaxation time, whose value ranges for metals between $10^{-9}$ s and $10^{-12}$ s. These very small values of the relaxation time indicate that its physical effects are negligible unless the time scale of the temperature variation is lower than nanoseconds or the modulation frequency is higher than gigahertzs. After expanding the heat flux in Taylor series around $\tau = 0$ equation (8) is written as

$$\vec{j}(x, t) + \tau\frac{\partial\vec{j}(x, t)}{\partial t} = -K\vec{\nabla}T(x, t). \tag{9}$$

Equation (9) along with the law of energy conservation leads to a hyperbolic heat conduction equation, also known as Cattaneo's equation, which in the absence of internal heat sources is written as

$$\nabla^2 T - \frac{1}{D}\frac{\partial T}{\partial t} - \frac{\tau}{D}\frac{\partial^2 T}{\partial t^2} = 0. \tag{10}$$

The last term in equation (10) serves to overcome the paradox of instantaneous heat propagation associated with the parabolic heat diffusion equation. Note that as $\tau \to 0$ equation (10) reduces to equation (2). For the purpose of this paper it is interesting to point out that equation (10) is a wave equation, the so-called telegraph equation, similar to that found for electromagnetic waves propagating through conducting media. Consequently it is expected that, under the experimental conditions for which the thermal relaxation time is not negligible, heat propagates as a wave.

Let us now consider the same experimental configuration as we dealt with in section 2 and whose geometry is depicted in figure 1. As before the time-dependent component of the temperature of the material can be written as follows: $T'_{\mathrm{ac}}(x, t) = \mathrm{Re}[\theta'(x)\,\mathrm{e}^{\mathrm{i}\omega t}]$. Substituting this expression into equation (10) the spatial distribution of this time-dependent component satisfies Helmholtz's equation

$$\frac{\mathrm{d}^2\theta'(x)}{\mathrm{d}x^2} - q'^2\theta'(x) = 0, \tag{11}$$

where $q' = \sqrt{\frac{\mathrm{i}\omega}{D} - \frac{\tau\omega^2}{D}} = \frac{\mathrm{i}\omega}{D}(1 + \mathrm{i}\omega\tau)$. By solving equation (11) and using as boundary condition the heat flux continuity on the sample surface

$$-K\left.\frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\right|_{x=0} = \frac{I_{\mathrm{o}}}{2}\,(1 + \mathrm{i}\omega\tau)\,, \tag{12}$$

the time-dependent component of the temperature is obtained as

$$\begin{aligned} T'_{\mathrm{ac}}(x, t) &= \mathrm{Re}\left[\frac{I_{\mathrm{o}}}{2Kq'}\,(1 + \mathrm{i}\omega\tau)\,\mathrm{e}^{-q'x}\,\mathrm{e}^{\mathrm{i}\omega t}\right] \\ &= \frac{I_{\mathrm{o}}\sqrt{\tau}}{2\varepsilon}\sqrt[4]{\frac{1 + (\omega\tau)^2}{(\omega\tau)^2}}\exp\left(-x\sqrt{\frac{\omega}{2D}}\sqrt{\sqrt{1 + (\omega\tau)^2} - \omega\tau}\right) \\ &\quad\times\cos\left(x\sqrt{\frac{\omega}{2D}}\sqrt{\sqrt{1 + (\omega\tau)^2} - \omega\tau} - \omega t - \frac{\mathrm{arctg}(\omega\tau)}{2} + \frac{\pi}{4}\right). \end{aligned} \tag{13}$$

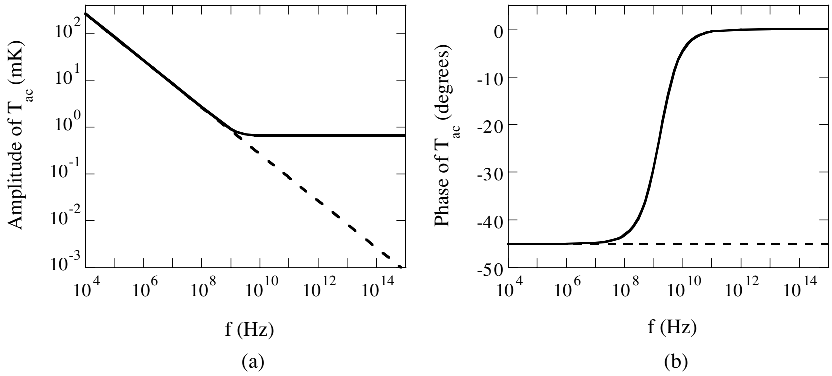


**Figure 3.** (a) Amplitude and (b) phase of the temperature oscillation on the surface of a semi-infinite stainless steel sample as a function of the modulation frequency. The solid line corresponds to the hyperbolic solution and the dotted line to the parabolic one. Calculations have been performed with $\tau = 10^{-10}$ s and $I_o = 10^6$ W m$^{-2}$.

This result is equivalent to that found by Yuen and Lee (see equation (12) in [11]). We show in figure 3 the amplitude and phase of the temperature oscillation on the surface of a semi-infinite stainless steel sample as a function of the modulation frequency. The solid line corresponds to the hyperbolic solution using equation (13) and the dotted line to the parabolic solution as given by equation (6). Calculations have been performed with $\tau = 10^{-10}$ s and $I_o = 10^6$ W m$^{-2}$. From this figure two different regimes can be distinguished:

(1) At low frequencies ($\omega\tau \ll 1$) equation (13) reduces to equation (6) and both parabolic and hyperbolic solutions coincide. This regime is characterized by the decrease of the amplitude of the surface temperature as the frequency increases and by a constant phase lag of $-45°$ between surface temperature and light excitation. On the other hand, as it was discussed in section 2, the penetration depth of the temperature oscillation (the improperly called thermal wave) depends on the modulation frequency of the light beam according to the expression of the thermal diffusion length $\mu = \sqrt{2D/\omega}$. The higher the frequency the lower the penetration depth is.

(2) At high frequencies ($\omega\tau \gg 1$) equation (13) reduces to

$$T'_{ac}(x, t) = \frac{I_o\sqrt{\tau}}{2\varepsilon}\,\mathrm{e}^{-\frac{x}{2\sqrt{D\tau}}}\cos\left(\sqrt{\frac{\tau}{D}}\omega x - \omega t\right). \tag{14}$$

As can be seen in figure 3 both parabolic and hyperbolic solutions clearly differ. According to the hyperbolic solution the amplitude of the surface temperature does not depend on the modulation frequency, while the phase lag goes to zero, indicating that the modulated light and the surface temperature are in phase. Moreover, the penetration depth $\mu' = 2\sqrt{D\tau}$ does not depend on the modulation frequency. In figure 2(b) we show the amplitude of $T'_{ac}$ as a function of the normalized depth $(x/\mu')$ for the same stainless steel sample as shown in figure 2(a) illuminated by the same light beam but now modulated at $5 \times 10^{10}$ Hz. Calculations have been performed for $\tau = 10^{-10}$ s and $t = 1.9 \times 10^{-11}$ s. Note that now, unlike the result found at low frequencies, the temperature oscillates many times before it vanishes. On the other hand, the penetration depth and the temperature amplitude, which are proportional to $\sqrt{\tau}$, are extremely small. These results are similar to those found by Galović and Kotoski [12] but differ from those presented by Marín and coworkers (see equation (18) in [13]) since they used equation (5) as boundary condition instead of equation (12).

Now we are interested in accounting for the energy carried by this temperature oscillation. To obtain the heat flux associated with $T'_{ac}$, the first-order differential equation (9) has to be solved. Using $j(0,t) = I_o\,\mathrm{e}^{\mathrm{i}\omega t}/2$ as boundary condition we obtain

$$\vec{j}_{ac} = \frac{I_o}{2}\exp\left(-\left(\frac{1}{2\sqrt{D\tau}} + \mathrm{i}\frac{\omega\sqrt{\tau}}{\sqrt{D}}\right)x\right)\cos(\omega t)\,\hat{\imath}, \tag{15}$$

whose time average taken over a period is $\langle\vec{j}_{ac}\rangle = \vec{0}$. Therefore, as in the case of the parabolic heat conduction equation, periodic illumination of an opaque material does not launch any travelling thermal wave even though its behaviour is governed by a wave equation.

## 4. Conclusions

In this work we have dealt with heat conduction in solid samples that are excited by periodic heat sources. Following a macroscopic approach we have shown that the resulting temperature oscillations inside the sample have the same mathematical expression as a thermal wave, but as they do not transport energy they cannot be considered as real travelling waves. This result is not surprising if we are working within the frame of the parabolic heat equation, for which heat conduction is a diffusive process. More interesting is the fact that in the frame of the hyperbolic heat conduction, when a time delay between temperature gradients and heat flux is taken into account, the temperature oscillations do not transport energy either, although they are the solution of a wave equation.

## Acknowledgments

This work has been supported by UPV/EHU (E-15928/2004) and by MEC (MAT2005-02999).

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