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A NOTE ON THE VIBRATIONS OF A CLAMPED-FREE BEAM WITH A MASS AT THE FREE END

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This note is concerned with the determination of natural frequencies and modal shapes of a clamped-free beam which carries a finite mass, M, at the free end. Accurate results are presented for different (concentrated mass)/(beam mass) ratios, M/M_v. It is also the purpose of this note to analyze the variation of the maximum dynamic stress as a function of the parameter M/M_v for the first mode of vibration.

1. INTRODUCTION

Characteristic functions and frequencies have been extensively tabulated for nearly all common types of beams: clamped–clamped, clamped–free, clamped–supported, free–supported and supported–supported [1, 2].

This note deals with the determination of the first ten natural frequencies of a clamped–free beam which carries a finite mass, M, at the free end (Figure 1). Shear and rotatory inertia effects are disregarded.

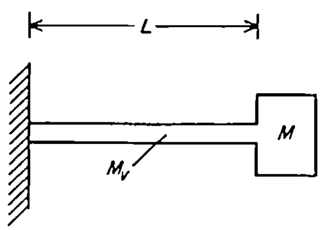

Figure 1. Beam-mass system studied in the present investigation.

It is important to point out that the frequency equation was obtained by Prescott [3] but no extensive numerical information has been published in the open literature.

Modal shapes and the effect of the mass M on the maximum dynamic stress when the structure vibrates in the first mode are also investigated.

2. MATHEMATICAL ANALYSIS

The problem under consideration is described by the following differential system:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0, \tag{2a, b}$$

$$\frac{\partial^2 w}{\partial x^2}(L,t) = 0, \tag{3}$$

$$-\left[-EI\frac{\partial^3 w}{\partial x^3}(L,t)\right] = M\frac{\partial^2 w}{\partial t^2}(L,t), \tag{4}$$

where E is the Young's modulus, I the constant moment of inertia, A the constant cross-sectional area, ρ the mass density and M the concentrated mass at the free end.

Using the standard method of separation of variables, one assumes

$$w(x,t) = X(x)T(t). \tag{5}$$

Substituting this form in equation (1) results in the equality

$$\frac{X^{(IV)}}{X} = -\frac{\rho A}{EI}\frac{T''}{T} = k^4. \tag{6}$$

The solution of the ordinary differential equation

$$X^{(IV)} - k^4 X = 0$$

is simply

$$X = C_1 \cos kx + C_2 \sin kx + C_3 \cosh kx + C_4 \sinh kx. \tag{7}$$

Substituting expression (7) in equation (2a, b) results in the equations

$$C_1 + C_3 = 0, \tag{8a}$$

$$C_2 + C_4 = 0. \tag{8b}$$

From equations (7) and (3) one obtains

$$-C_1 \cos kL - C_2 \sin kL + C_3 \cosh kL + C_4 \sinh kL = 0. \tag{8c}$$

Substituting equation (7) in equation (4) results in the algebraic expression

$$C_1 \sin kL - C_2 \cos kL + C_3 \sinh kL + C_4 \cosh kL =$$

$$= \frac{-1}{\rho A} kM(C_1 \cos kL + C_2 \sin kL + C_3 \cosh kL + C_4 \sinh kL). \tag{8d}$$

The solution of equations (8) leads to the following determinantal equation in the eigenvalues, k:

$$\begin{vmatrix} g_{11}(k) & g_{12}(k) \\ g_{21}(k) & g_{22}(k) \end{vmatrix} = 0, \tag{9}$$

where

$$g_{11}(k) = \cos kL + \cosh kL, \tag{10a}$$

$$g_{12}(k) = \sin kL + \sinh kL, \tag{10b}$$

$$g_{21}(k) = \sin kL - \sinh kL + \frac{(kL)\,M}{\rho AL}(-\cosh kL + \cos kL), \tag{10c}$$

$$g_{22}(k) = -\cos kL - \cosh kL + \frac{(kL)\,M}{\rho AL}(-\sinh kL + \sin kL). \tag{10d}$$

Equations (9) and (10) lead to the transcendental equation

$$\frac{1}{y}\frac{1+\cos y\cosh y}{\sin y\cosh y - \cos y\sinh y} = \frac{M}{M_v}, \tag{11}$$

where M_v is the beam mass ($M_v = \rho AL$) and $y = kL$.

The frequencies, f_i, are then given by

$$f_i = \frac{1}{2\pi}\left(\frac{y_i}{L}\right)^2\sqrt{\frac{EI}{\rho A}}. \tag{12}$$

3. DETERMINATION OF THE EIGENVALUES

In order to calculate the roots, y_i, of the transcendental equation (11) it is convenient to express it in the form

$$z(y) = \frac{M}{M_v}\,y(\cos y\sinh y - \sin y\cosh y) + \cos y\cosh y + 1 = 0. \tag{13}$$

The function $z(y)$ shows rapid oscillations, attaining very large values between successive roots. The slope of the function at each root is, therefore, very close to vertical.

This characteristic of the function suggests the use of the method of false position to ensure convergence in the determination of the roots [4]. In order to initiate the iterative process, the roots were first bracketed by means of a straight search process. The algorithmic procedure was implemented in a Hewlett–Packard 9810 A.

TABLE 1

Values of the first ten roots of equation (11) *for* $M/M_v = 0, 0{\cdot}20, 0{\cdot}40, \ldots, 10$

	M/M_v					
y_i	0	0·20	0·40	0·60	0·80	1·00
y_1	1·87510407	1·61639966	1·47240849	1·37566854	1·30408675	1·24791741
y_2	4·69409113	4·26706157	4·14443036	4·08665324	4·05307815	4·03113944
y_3	7·85475744	7·31837267	7·21548589	7·17252465	7·14898484	7·13413224
y_4	10·99554073	10·40156263	10·31780693	10·28498044	10·26748665	10·25662107
y_5	14·13716839	13·50670225	13·43667566	13·41020846	13·39631447	13·38775633
y_6	17·27875953	16·62335441	16·56341840	16·54128737	16·52977831	16·52272548
y_7	20·42035225	19·74686001	19·69456473	19·67556452	19·66574657	19·65975089
y_8	23·56194490	22·87475293	22·82841390	22·81177553	22·80321764	22·79800451
y_9	26·70353756	26·00561758	25·96403931	25·94924411	25·94166078	25·93704999
y_{10}	29·84513021	29·13858564	29·10089373	29·08757605	29·08076863	29·07663567

TABLE 1—*continued*

M/M_v

y_i	1·20	1·40	1·60	1·80	2·00
y_1	1·20206578	1·16355780	1·13051695	1·10168865	1·07619566
y_2	4·01568357	4·00420785	3·99535061	3·98830760	3·98257329
y_3	7·12390909	7·11644347	7·11075258	7·10627093	7·10265019
y_4	10·24921773	10·24384957	10·23977894	10·23658619	10·23401501
y_5	13·38195599	13·37776560	13·37459663	13·37211624	13·37012199
y_6	16·51796088	16·51452647	16·51193347	16·50990644	16·50827832
y_7	19·65570932	19·65280046	19·65060668	19·64889318	19·64751781
y_8	22·79449594	22·79197344	22·79007255	22·78858870	22·78739823
y_9	25·93395048	25·93172388	25·93004697	25·92873856	25·92768922
y_{10}	29·07385994	29·07186719	29·07036709	29·06919706	29·06825895

M/M_v

y_i	2·20	2·40	2·60	2·80	3
y_1	1·05340227	1·03283409	1·01412844	0·99700223	0·98123061
y_2	3·97781398	3·97380051	3·97037033	3·96740490	3·96481577
y_3	7·09966406	7·09715917	7·09502787	7·09319239	7·09159514
y_4	10·23190000	10·23012968	10·22862612	10·22733327	10·22620973
y_5	13·36848374	13·36711397	13·36595169	13·36495308	13·36408583
y_6	16·50694191	16·50582527	16·50487829	16·50406505	16·50335908
y_7	19·64638946	19·64544708	19·64464820	19·64396235	19·64336713
y_8	22·78642196	22·78560685	22·78491604	22·78432310	22·78380862
y_9	25·92682893	25·92611082	25·92550235	25·92498017	25·92452716
y_{10}	29·06749002	29·06684831	29·06630464	29·06583814	29·06543348

M/M_v

y_i	3·2	3·4	3·6	3·8	4
y_1	0·96663220	0·95305873	0·94038758	0·92851621	0·91735814
y_2	3·96253560	3·96051219	3·95870450	3·95707978	3·95561158
y_3	7·09019258	7·08895114	7·08784459	7·08685207	7·08595681
y_4	10·22522429	10·22435297	10·22357702	10·22288160	10·22225479
y_5	13·36332563	13·36265381	13·36205581	13·36152009	13·36103740
y_6	16·50274047	16·50219396	16·50170762	16·50127205	16·50087969
y_7	19·64284570	19·64238512	19·64197535	19·64160840	19·64127790
y_8	22·78335800	22·78296003	22·78260599	22·78228900	22·78200353
y_9	25·92413041	25·92378007	25·92346844	25·92318944	25·92293820
y_{10}	29·06507912	29·06476623	29·06448793	29·06423879	29·06401445

M/M_v

y_i	4·2	4·4	4·6	4·8	5
y_1	0·90683976	0·89689794	0·88747820	0·87853317	0·87002146
y_2	3·95427833	3·95306224	3·95194852	3·95092476	3·94998049
y_3	7·08514519	7·08440601	7·08373000	7·08310937	7·08253760
y_4	10·22168692	10·22117005	10·22069760	10·22026408	10·21986487
y_5	13·36060024	13·36020247	13·35983898	13·35950553	13·35919854
y_6	16·50052441	16·50020119	16·49990588	16·49963501	16·49938568
y_7	19·64097868	19·64070650	19·64045784	19·64022979	19·64001989
y_8	22·78174510	22·78151004	22·78129531	22·78109840	22·78091716
y_9	25·92271077	25·92250393	25·92231499	25·92214173	25·92198228
y_{10}	29·06381139	29·06362671	29·06345804	29·06330336	29·06316102

TABLE 1—*continued*

	M/M_v				
y_i	5·2	5·4	5·6	5·8	6
y_1	0·86190669	0·85415675	0·84674314	0·83964046	0·83282600
y_2	3·94910680	3·94829605	3·94754169	3·94683803	3·94618012
y_3	7·08200914	7·08151923	7·08106382	7·08063939	7·08024287
y_4	10·21949605	10·21915429	10·21883670	10·21854081	10·21826447
y_5	13·35891498	13·35865227	13·35840819	13·35818083	13·35796852
y_6	16·49915540	16·49894209	16·49874392	16·49855934	16·49838700
y_7	19·63982605	19·63964650	19·63947972	19·63932438	19·63917935
y_8	22·78074981	22·78059480	22·78045082	22·78031673	22·78019154
y_9	25·92183504	25·92169867	25·92157201	25·92145405	25·92134393
y_{10}	29·06302958	29·06290785	29·06279479	29·06268950	29·06259121

	M/M_v				
y_i	6·2	6·4	6·6	6·8	7
y_1	0·82627938	0·81998222	0·81391791	0·80807141	0·80242903
y_2	3·94556364	3·94498480	3·94444026	3·94392705	3·94344254
y_3	7·07987160	7·07952324	7·07919574	7·07888727	7·07859623
y_4	10·21800581	10·21776317	10·21753513	10·21732039	10·21711783
y_5	13·35776982	13·35758347	13·35740833	13·35724344	13·35708792
y_6	16·49822573	16·49807448	16·49793235	16·49779855	16·49767235
y_7	19·63904364	19·63891637	19·63879679	19·63868421	19·63857804
y_8	22·78007440	22·77996455	22·77986134	22·77976418	22·77967255
y_9	25·92124089	25·92114428	25·92105349	25·92096804	25·92088745
y_{10}	29·06249925	29·06241301	29·06233199	29·06225572	29·06218380

	M/M_v				
y_i	7·2	7·4	7·6	7·8	8
y_1	0·79697828	0·79170777	0·78660703	0·78166649	0·77687729
y_2	3·94298439	3·94255051	3·94213902	3·94174823	3·94137662
y_3	7·07832118	7·07806084	7·07781405	7·07757979	7·07735712
y_4	10·21692643	10·21674531	10·21657366	10·21641074	10·21625592
y_5	13·35694099	13·35680196	13·35667021	13·35654518	13·35642637
y_6	16·49755314	16·49744034	16·49733345	16·49723203	16·49713565
y_7	19·63847775	19·63838286	19·63829295	19·63820763	19·63812656
y_8	22·77958600	22·77950411	22·77942652	22·77935290	22·77928295
y_9	25·92081133	25·92073931	25·92067107	25·92060632	25·92054480
y_{10}	29·06211586	29·06205159	29·06199069	29·06193291	29·06187801

	M/M_v				
y_i	8·2	8·4	8·6	8·8	9
y_1	0·77223130	0·76772098	0·76333937	0·75908001	0·75493689
y_2	3·94102281	3·94068554	3·94036369	3·94005622	3·93976219
y_3	7·07714521	7·07694329	7·07675068	7·07656674	7·07639090
y_4	10·21610860	10·21596825	10·21583439	10·21570658	10·21558441
y_5	13·35631332	13·35620564	13·35610293	13·35600488	13·35591116
y_6	16·49704396	16·49695661	16·49687332	16·49679379	16·49671779
y_7	19·63804944	19·63797597	19·63790591	19·63783903	19·63777511
y_8	22·77921639	22·77915300	22·77909256	22·77903485	22·77897970
y_9	25·92048628	25·92043053	25·92037737	25·92032663	25·92027813
y_{10}	29·06182578	29·06177604	29·06172860	29·06168332	29·06164004

TABLE 1—*continued*

y_i	9·2	9·4	9·6	9·8	10
	M/M_v				
y_1	0·75090442	0·74697739	0·74315095	0·73942054	0·73578192
y_2	3·93948074	3·93921108	3·93895247	3·93870426	3·93846583
y_3	7·07622264	7·07606148	7·07590698	7·07575873	7·07561637
y_4	10·21546752	10·21535558	10·21524828	10·21514533	10·21504648
y_5	13·35582151	13·35573565	13·35565335	13·35557440	13·35549859
y_6	16·49664508	16·49657545	16·49650872	16·49644470	16·49638323
y_7	19·63771396	19·63765541	19·63759929	19·63754545	19·63749376
y_8	22·77892694	22·77887642	22·77882800	22·77878155	22·77873696
y_9	25·92023174	25·92018731	25·92014474	25·92010389	25·92006468
y_{10}	29·06159864	29·06155900	29·06152101	29·06148457	29·06144958

Table 1 shows the values of the first ten roots of equation (11) for $M/M_v = 0$, 0·20, 0·40, . . ., 10.

4. DETERMINATION OF MODAL SHAPES

From equations (8a), (8b) and (8c) one obtains

$$X(x/L) = C_1\left[\cos y_i\frac{x}{L} - \frac{\cos y_i + \cosh y_i}{\sin y_i + \sinh y_i}\sin y_i\frac{x}{L} - \cosh y_i\frac{x}{L} + \right.$$

$$\left. + \frac{\cos y_i + \cosh y_i}{\sin y_i + \sinh y_i}\sinh y_i\frac{x}{L}\right]. \tag{14}$$

The first five modes of vibration are shown in Figures 2 (a)–(f) for $M/M_v = 0$, 0·20, 0·40, 0·60, 0·80 and 1·00.

In the case where $M/M_v = 0$ the calculated co-ordinates of the nodes are in good agreement with those available in the technical literature. The only noticeable difference is in the case of the second node corresponding to the second mode since reference [5] indicates $(x/L)_{2-2} = 0{\cdot}774$ while the result obtained in the present investigation is 0·783.

5. VARIATION OF THE MAXIMUM DYNAMIC STRESS WITH THE PARAMETER M/M_v

For the first mode one has

$$W_1 = C_1\, X(x/L)\, T(t). \tag{15}$$

Taking $C_1 = 1/X(x/L)|_{x=L}$ and replacing this in equation (15) results in the expression

$$W_1 = \frac{1}{X(x/L)|_{x=L}}X(x/L)\,T(t). \tag{16}$$

In this form, the tip deflection is equal to unity.

Using well-known results from the Bernoulli–Navier beam theory one can easily show

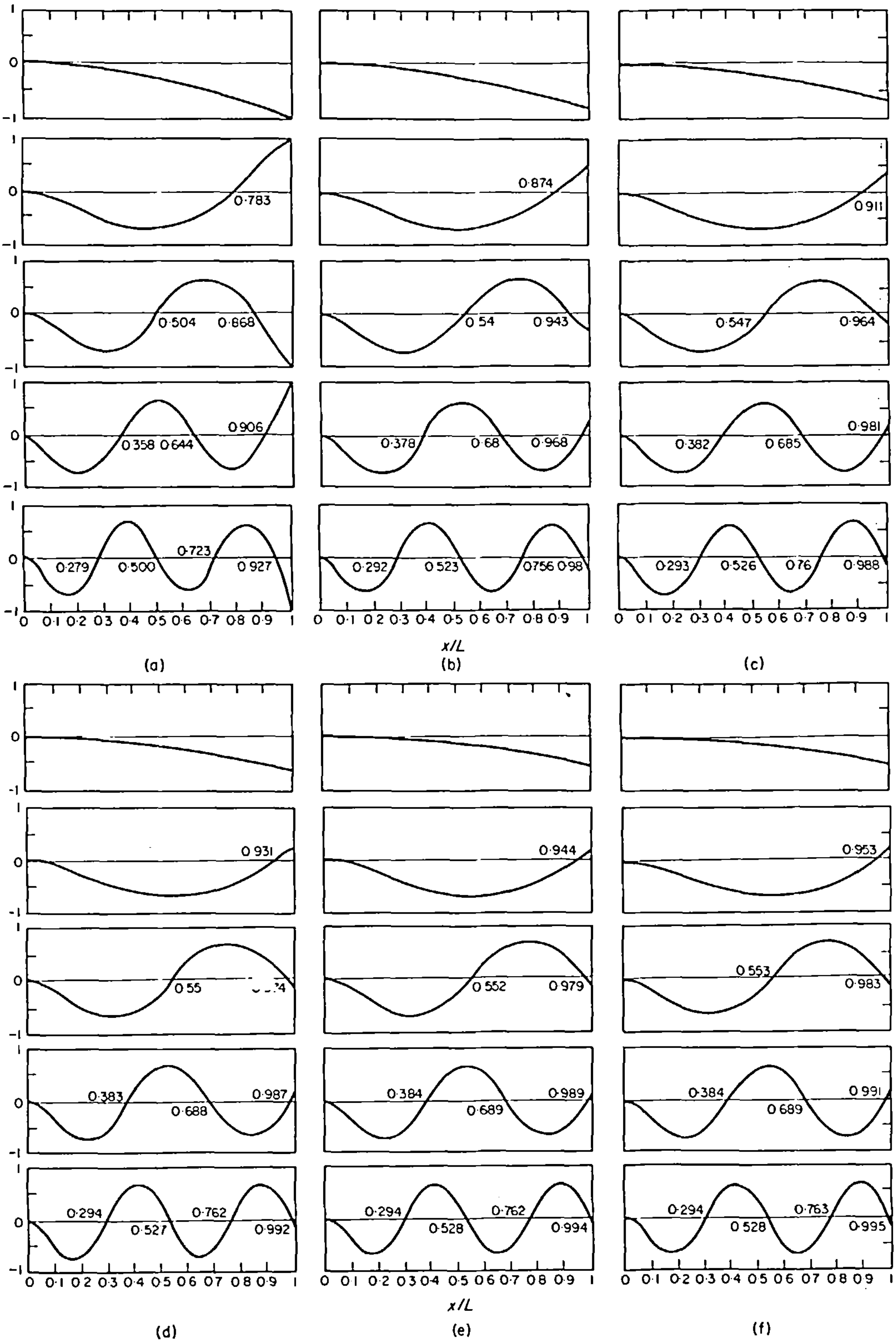

Figure 2. Modes of vibration. (a) $M/M_v = 0$; (b) $M/M_v = 0{\cdot}20$; (c) $M/M_v = 0{\cdot}40$; (d) $M/M_v = 0{\cdot}60$; (e) $M/M_v = 0{\cdot}80$; (f) $M/M_v = 1{\cdot}00$.

that the amplitude of the maximum dynamic stress is given by

$$|\sigma_x|_{max} = Eh/2 \frac{X''(0)}{X(x/L)|_{x=L}}. \quad (17)$$

Evaluating $X''(0)$ by means of equation (14), substituting this in equation (17) and conveniently defining a dimensionless parameter

$$\gamma = \frac{|\sigma_x|_{max} L^2}{Eh/2},$$

one obtains

$$\gamma = -2 \frac{(kL)^2}{X(x/L)|_{x=L}}. \quad (18)$$

The numerical results are tabulated in Table 2 for several values of the ratio M/M_v. It is

TABLE 2
Variation of the amplitude of the dynamic stress as a function of the parameter M/M_v

M/M_v	(k_1L)	$\gamma = \dfrac{\lvert\sigma_x\rvert_{max} L^2}{Eh/2}$
0	1·87510407	3·52
0·2	1·61639966	3·28
0·4	1·47240849	3·19
0·6	1·37566854	3·14
0·8	1·30408675	3·11
1	1·24791741	3·10
2	1·07619566	3·05
3	0·98123061	3·04
4	0·91735814	3·03
5	0·87002146	3·02
6	0·83282600	3·02
7	0·80242903	3·02
8	0·77687729	3·01
9	0·75493689	3·01
10	0·73578192	3·01

observed that γ decreases as M/M_v increases, as is to be expected, in view of the fact that the frequency coefficient decreases.

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