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Measurement of electrical resistivity by a mutual inductance method

by R. G. CHAMBERS, M.A., Ph.D., and J. G. PARK,* M.A., D.Phil., H. H. Wills Physics Laboratory, Bristol

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Abstract

The resistivity of a sample can be deduced from the change in mutual inductance between two coils when the sample is inserted. It is shown that with simple equipment for measuring mutual inductance over a range of frequencies, the method can be used to measure resistivities from 2×10^{-9} *ohm cm upwards, and the necessary functions are tabulated.*

Introduction

THE self-inductance L of a coil, or the mutual inductance M between a pair of coils, is reduced when a non-ferromagnetic conductor is placed in the magnetic field generated by alternating current, and the losses are at the same time increased. We may represent the decrease in L or M on bringing up the conductor by $\delta L = \delta L' + i\delta L''$ or $\delta M = \delta M' + i\delta M''$. By measuring δL or δM at a number of frequencies it is possible, knowing the geometry of the system, to deduce the resistivity of the conductor. Using simple equipment, the accuracy of such measurements is limited to about 1%, but the method has the advantage that no direct electrical connections need to be made to the sample. The method has been described previously by a number of authors (Kouwenhoven and Daiger 1934, Grube and Speidel 1940, Laurmann and Shoenberg 1949, Fraser and Shoenberg 1949, Rorschach and Herlin 1951, Van den Berg and Van der Marel 1954); the object of the present paper is to discuss the interpretation of the results in rather more detail.

Theory

If the sample is isotropic in resistivity, the alternating field $H = H_0 e^{i\omega t}$ inside it satisfies the equation

$$\nabla^2 H = 2iH/\delta^2 \qquad (1)$$

subject to the appropriate boundary conditions. Here δ is the skin depth for a plane surface: if ρ is measured in microhm cm, f in kc/s and δ in cm,

$$\delta = \frac{1}{2\pi}\left(\frac{\rho}{f}\right)^{1/2}.$$

Solutions of the related equation $\nabla^2 H = H/\lambda^2$ are given, for instance, by Shoenberg (1952) for an infinite circular cylinder parallel to the applied field, for a thin plate with the field parallel to its surface, and for a sphere. The solutions of (1) follow on replacing λ by $\delta/(2i)^{1/2}$.

A well-defined geometry is most simply achieved by using a mutual inductance with a long primary winding, and a short secondary coil wound over the centre of the primary, and measuring the change δM in the mutual inductance on inserting a long cylindrical specimen. (The primary and secondary may, of course, be interchanged without affecting δM). The flux through the secondary is then not appreciably perturbed by end effects, and the solution of (1) for an infinite circular cylinder may be used. This solution may be expressed in terms of an effective volume susceptibility χ or permeability μ:

$$\chi = \frac{1}{4\pi}\frac{J_2(xi^{3/2})}{J_0(xi^{3/2})}; \quad \mu = \frac{2J_1(xi^{3/2})}{xi^{3/2}J_0(xi^{3/2})}$$

Here the J_ν are Bessel functions of the first kind, and $x = \sqrt{2}a/\delta$, where a is the specimen radius. Thus

$$\rho/a^2 = 8\pi^2 f/x^2 \qquad (2)$$

where again ρ is expressed in $\mu\Omega$ cm, f in kc/s and a in cm. If A_c and A_s are the cross-sectional areas of the primary coil and the specimen and the mutual inductance changes from M_c to $M_c - \delta M$ when the specimen is inserted, we have $\delta M = \delta M' + i\delta M'' = -4\pi\chi M_c A_s/A_c$. At high frequencies, where $\delta \ll a$, little flux penetrates into the specimen, χ tends to $-1/4\pi$ and δM tends to the limiting value $\delta M_0 = M_c A_s/A_c$. If the specimen becomes superconducting, $\chi = -1/4\pi$ at all frequencies (assuming that the penetration depth λ is much less than a) and δM_0 can be measured directly. Writing $\delta M'/\delta M_0 = m'$, $\delta M''/\delta M_0 = m''$, we have

$$m' + im'' = -J_2(xi^{3/2})/J_0(xi^{3/2}). \qquad (3)$$

The variation of m' and m'' with x is shown in the Table and in Fig. 1; the Table also shows the values of

$$\alpha = x^2 m''/6m'.$$

For small x,

$$m' = \frac{x^4}{48}\left(1 - \frac{57x^4}{1920}\dots\right), \quad m'' = \frac{x^2}{8}\left(1 - \frac{55x^4}{1920}\dots\right)$$

and for large x,

$$m' = 1 - \frac{\sqrt{2}}{x} - \frac{\sqrt{2}}{8x^3}\dots, \quad m'' = \frac{\sqrt{2}}{x} - \frac{1}{x^2} - \frac{\sqrt{2}}{8x^3}\dots$$

(Fraser and Shoenberg 1949); these series are useful for $x \leqslant 1$ and $x \geqslant 5$ respectively. At $x = 2{\cdot}49_6$, $m' = m''$, and m'' passes through a maximum at $x_m = 2{\cdot}51_8$. The corresponding frequency is given from (2) by

$$f_m = 0{\cdot}0803\rho/a^2. \qquad (4)$$

* Now at the Department of Physics, Imperial College, London, S.W.7.

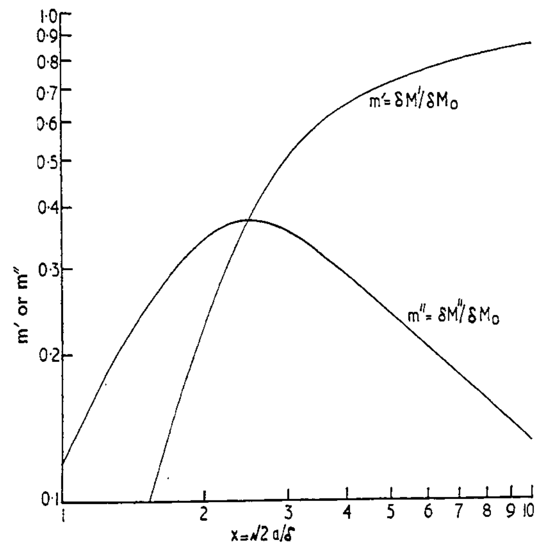

Fig. 1. Variation of m' and m'' with x (x is proportional to the square root of the frequency).

x	m'	m''	α
0·0	0·0000	0·0000	1·0000
0·2	0·0000	0·0050	1·0000
0·4	0·0005	0·0200	1·0000
0·6	0·0027	0·0448	1·0001
0·8	0·0084	0·0791	1·0004
1·0	0·0202	0·1214	1·0010
1·2	0·0407	0·1679	1·0022
1·4	0·0718	0·2205	1·0041
1·6	0·1142	0·2695	1·0069
1·8	0·1667	0·3121	1·0109
2·0	0·2262	0·3449	1·0163
2·2	0·2884	0·3660	1·0235
2·4	0·3495	0·3760	1·0330
2·6	0·4063	0·3768	1·0451
2·8	0·4569	0·3707	1·0601
3·0	0·5009	0·3600	1·0780
3·2	0·5387	0·3469	1·0988
3·4	0·5710	0·3327	1·1226
3·6	0·5987	0·3186	1·1494
3·8	0·6224	0·3050	1·1792
4·0	0·6429	0·2921	1·2118
4·2	0·6609	0·2802	1·2467
4·4	0·6769	0·2693	1·2838
4·6	0·6912	0·2593	1·3229
4·8	0·7041	0·2500	1·3637
5·0	0·7159	0·2415	1·4059
5·2	0·7268	0·2337	1·4493
5·4	0·7370	0·2265	1·4934
5·6	0·7464	0·2196	1·5380
5·8	0·7553	0·2132	1·5826
6·0	0·7635	0·2070	1·6267

The ratio ρ/a^2 can be determined experimentally in three ways:

(*a*) If δM_0 is known, by measuring $\delta M'$ or $\delta M''$ at a single frequency. From the ratio m' or m'', the value of x can be found from the Table, and ρ/a^2 from (2). If $\delta M''$ alone is measured, it is also necessary to know whether the measuring frequency is above or below f_m.

(*b*) By measuring $\delta M'$ or $\delta M''$ over a range of frequencies, and scaling the experimental graph of $\delta M'$ (or $\delta M''$) against $f^{1/2}$ to fit the theoretical graph of m' (or m'') against x. This is most simply done by plotting both graphs logarithmically, as in Fig. 1. Then the vertical scaling factor gives δM_0 (which therefore need not be known beforehand), and the horizontal scaling factor gives $f^{1/2}/x$, and hence ρ/a^2 from (2) or (4).

(*c*) The simplest and most convenient method, which again involves no knowledge of δM_0, is to measure both $\delta M'$ and $\delta M''$ at a single frequency. Then the ratio $\delta M''/\delta M' = m''/m' = 6\alpha/x^2$ is a unique function of x, and an accurate value of x is readily found by successive approximations, using the Table. The value of ρ/a^2 follows as before from (2), and δM_0 can be found from

$$\delta M_0 = \delta M'/m'(x) = \delta M''/m''(x).$$

The analysis is particularly simple at low frequencies ($f \ll f_m$) where $\alpha \simeq 1$. This method has the advantage that residual end-effect errors, which chiefly affect δM_0 (in a frequency-dependent fashion) are virtually eliminated.

If end-effect errors can be neglected, and the coil constant M_c/A_c is known, the value of a^2 can be deduced at once from the measured δM_0 using $\delta M_0 = \pi a^2 M_c/A_c$ so that it is unnecessary to measure a directly. In particular, if method (*b*) is used, we have $m' \simeq m'' = 0{\cdot}3774$ at $f = f_m$ so that from Eqn (4) the impedance changes at f_m give ρ directly:

$$\omega\delta M' \simeq \omega\delta M'' = 0{\cdot}598\rho M_c/A_c \qquad (5)$$

where $\omega\delta M$ is expressed in $m\Omega$, ρ in $\mu\Omega$ cm, and M_c/A_c in μH cm^{-2}.

In practice it is desirable not to work at frequencies much higher than f_m, since at such frequencies the field penetration is small and the result is determined only by the resistivity of the surface layers of the specimen, rather than by the bulk resistivity, and may be affected by surface roughness. Moreover, very pure specimens at low temperatures will exhibit anomalous skin effect behaviour when the electronic mean free path l becomes comparable with δ and under these conditions Eqn (1) breaks down and the present analysis becomes inapplicable. The condition for (1) to be valid is $l/\delta \lesssim 10^{-1}$ as shown by Reuter and Sondheimer (1948). For most metals $\rho l \sim 10^{-5}\,\mu\Omega\,\text{cm}^2$ (Chambers 1952) so that this condition can be written

$$f \lesssim 2 \times 10^6 \rho^3 \qquad (6)$$

with f in kc/s, ρ in $\mu\Omega$ cm. Similarly, size-effect errors will arise if l is comparable with the specimen radius, and to avoid these we require $l/a \lesssim 10^{-2}$. Thus, using the above estimate of ρl we find that we need $a \gtrsim 10^{-3}/\rho$. With the aid of Eqn (4) this can be put in a form similar to (6):

$$f_m \lesssim 10^5 \rho^3. \qquad (7)$$

If errors of 5–10% in the determination of ρ can be tolerated, these frequency limits can probably be exceeded by factors of 10 and 100 respectively (corresponding to $l/\delta \lesssim 0{\cdot}3$, $l/a \lesssim 0{\cdot}1$).

Experimental details

As a practical example, we describe briefly the apparatus we have used to measure the residual resistivity of dilute Hg–Cd alloys (Chambers and Park 1960). The samples were 2 cm long and 0·5–1 mm in diameter, and their resistivities ranged between 0·0025 and 0·5 $\mu\Omega$ cm. The 200 μH mutual inductance M into which the samples were inserted had an effective cross-sectional area A_c of about $0\cdot5$ cm^2, so that δM_0 was about 1–3 μH. To measure δM, a Hartshorn bridge was used (Fig. 2). From the changes in R and M_b

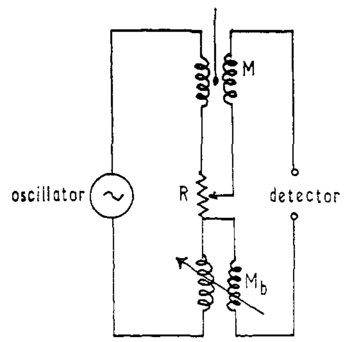

Fig. 2. The Hartshorn bridge.

needed to restore balance on inserting the sample, we have $\delta M' = \delta M_b$, $\delta M'' = \delta R/\omega$. Provision was made for inserting and withdrawing the sample from outside the cryostat, so that δR and δM_b could be measured directly. This is necessary because the empty-coil balance point itself varies with temperature and frequency. The compensating inductance M_b was in three parts: a main fixed compensator, similar to M, inside the cryostat, a small uncalibrated variable for zero-setting, and a carefully calibrated variable with ranges $+0\cdot45n$ to $-0\cdot45n$ μH, where $n = 1, 3, 10, \ldots 1000$. The variable resistance R was a conventional $0\cdot1$ ohm low-inductance slide-wire potentiometer, supplemented by $\times10^{-1}$ and $\times10^{-2}$ attenuators when necessary. The bridge was fed through a power amplifier by an audio oscillator (20 c/s–20 kc/s), and the output from the bridge was fed through a conventional audio amplifier to an oscilloscope.

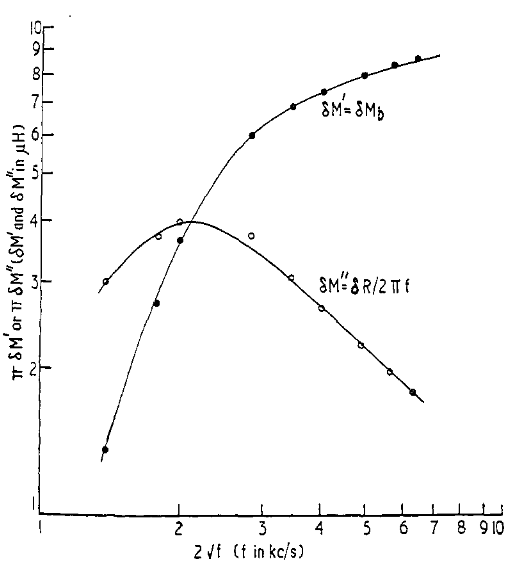

Fig. 3. An example of the use of method (b). The theoretical curves that fit the points best are shown superimposed. (The values of $\delta M'$, $\delta M''$ and $\sqrt{f}$ have been multiplied by factors which bring all the points within one cycle of logarithmic graph paper, for convenience in fitting the curves.)

If δM_b is measured in μH, δR in $m\Omega$ and f in kc/s, then $m''/m' = \delta R/2\pi f\delta M_b$, and from Eqn (2) we have simply $\rho/a^2 = (2\cdot094/\alpha)\delta R/\delta M_b$ for method (c) of measurement. Thus in this method it is not necessary to measure the frequency accurately, if one works at low frequencies were $\alpha \simeq 1$. Finally, as an example of the use of method (b), Fig. 3 shows typical plots of $\delta M'$ and $\delta M''$ against $f^{1/2}$, with the theoretical curves fitted to them. For this specimen $f_m = 1\cdot12$ kc/s and $a = 0\cdot0519$ cm, so that from (4), $\rho = 0\cdot0378$ $\mu\Omega$ cm.

Discussion

If the resistivity of the specimen is not isotropic, the value of ρ measured by the present method is some average over directions normal to the specimen axis. If the specimen is a single uniaxial crystal, with principal resistivities $\rho_{\parallel}$ and $\rho_{\perp}$ parallel and perpendicular to the crystal axis, and the anisotropy is not too great, the apparent resistivity is approximately (Fraser and Shoenberg 1949)

$$\bar{\rho} = \tfrac{1}{2}\{\rho_{\perp} + (\rho_{\perp}\cos^2\phi + \rho_{\parallel}\sin^2\phi)\},$$

where ϕ is the angle between the specimen axis and the crystal axis. Alternatively, if a specimen were used in the form of a long thin strip, with one of the principal axes of the material (the k axis say) parallel to the width of the strip, the apparent resistivity would be approximately ρ_k. The details of the above analysis, of course, would have to be replaced by that appropriate to the altered geometry.

The range of resistivities that can be measured by the present method depends on the frequency range available and the bridge sensitivity. With simple apparatus, measurements are not readily extended below about 20 c/s, so that if anomalous skin-effect difficulties are to be avoided the lowest resistivity that can be measured is about $0\cdot002$ $\mu\Omega$ cm, from (6). To avoid size-effect errors, f_m should then be less than 1 c/s, from (7); i.e. a should be greater than $0\cdot4$ cm. Under these conditions, however, the measured resistivity will be that of the surface layers rather than the bulk material, since $f \gg f_m$. Size effect errors will probably not be serious if a specimen of radius $0\cdot1$ cm is used; then $l/a \sim 0\cdot05$ and $f_m \sim 20$ c/s so that $f \sim f_m$ and the measured resistivity will be that of the bulk material. Taking a coil constant M_c/A_c of 400 μH cm^{-2}, a specimen of resistivity $0\cdot002$ $\mu\Omega$ cm will then give an impedance change of about $0\cdot5$ mΩ from (5); large enough to measure with fair accuracy.

For materials of very high resistivity, such that $f \ll f_m$ even at the highest measuring frequency, we can use the approximations $m' = x^4/48$, $m'' = x^2/8$, and we then find for the impedance change on inserting the specimen:

$$\omega\delta M' = (M_c/A_c)8\pi^6a^6f^3/3\rho^2,\quad \omega\delta M'' = (M_c/A_c)2\pi^4a^4f^2/\rho.$$

Taking $f = 20$ kc/s, $M_c/A_c = 400$ μH cm^{-2}, and $a = 0\cdot1$ cm again, we find that $\omega\delta M' = 1$ mΩ for $\rho \sim 10^2$ $\mu\Omega$ cm, and $\omega\delta M'' = 1$ mΩ for $\rho \sim 3 \times 10^3$ $\mu\Omega$ cm. Thus if an impedance change of 1 mΩ is measurable at 20 kc/s, method (c) can be used for resistivities up to 10^2 $\mu\Omega$ cm, and method (a) up to 3×10^3 $\mu\Omega$ cm. These limits can be greatly extended by increasing the specimen radius above $0\cdot1$ cm, though very high resistivities can be more simply measured by dielectric loss measurements.

A closely related method of measuring resistivity has recently been described by Bean, de Blois and Nesbitt (1959), in which the rate at which magnetic flux escapes from a conducting cylinder is measured, after an external steady field has been switched off. The rate of flux expulsion determines the e.m.f. induced in a pick-up coil wound round the specimen. After a time of the order of τ, the induced e.m.f. decays exponentially with a time constant $\tau = 1 \cdot 10/2\pi f_m$. Using suitable equipment, values of τ between 50 s and about $0 \cdot 5$ μs can be measured, corresponding in f_m to $0 \cdot 003$ c/s and 300 kc/s, and for a specimen 1 cm in diameter this corresponds to a range of resistivities from 10^{-5} $\mu\Omega$ cm to 10^3 $\mu\Omega$ cm. This method is therefore particularly powerful for materials of very low resistivity, though not quite so powerful as this example suggests: a sample of resistivity 10^{-5} $\mu\Omega$ cm would have $l \sim 1$ cm, so that size-effect errors would be considerable in a rod 1 cm in diameter. Moreover, the radius of the electron orbits in typical metals (in cm) is related to the field H (in oersted) by $r \sim 5/H$ so that further complications due to orbit curvature and magnetoresistance would arise in fields of 1 oersted or more. A safer lower limit for this method would be 10^{-4} $\mu\Omega$ cm. (Since the decaying e.m.f. contains appreciable Fourier components up to at least $10 f_m$, condition (6) for avoiding anomalous skin-effect errors becomes identical with condition (7) for avoiding size-effect errors: we need at least $l/a < 10^{-1}$ and preferably $l/a < 10^{-2}$, to avoid such errors.) It remains true that for materials of the lowest resistivity the flux decay method is superior, but for resistivities above 10^{-2} $\mu\Omega$ cm, the mutual inductance method as discussed in the present paper may be simpler and more convenient.

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Notes and Comments

The Second International Conference on Stress Analysis

The Second International Conference on Stress Analysis organized by the Groupment pour l'avancement des Methods d'Analyse des Contraintes (G.A.M.A.C.) will be held in Paris from 10th–14th April 1962. It is not intended to include review papers in the Conference. Contributions will deal mainly with the results of experimental work, and there will be no publication of the complete texts by G.A.M.A.C. The Joint British Committee for Stress Analysis is arranging for the scrutiny of the British contributions on behalf of G.A.M.A.C. Abstracts and summaries must be submitted by 30th September 1961. Further information may be obtained from the Secretary, The Joint British Committee for Stress Analysis, The Institution of Mechanical Engineers, 1 Birdcage Walk, Westminster, S.W.1.

Symposium on Carbohydrate Chemistry

An International Symposium on Carbohydrate Chemistry, sponsored by the Chemical Society, will be held in Birmingham from 10th–20th July 1962. Further particulars will be published by the Society in due course. Copies of the announcement may be obtained, when available, from the General Secretary, The Chemical Society, Burlington House, London, W.1. Papers read at this meeting will not be published in collected form.

Third International Symposium on Rarefied Gas Dynamics

The Third International Symposium on Rarefied Gas Dynamics will be held at the University of Paris, on 26th–29th June 1962. The programme will range from topics of immediate significance for upper atmosphere and space flight to basic scientific studies and will include the following areas: Studies of the limits of the continuum theory or the quasi-equilibrium kinetic theory of gases; Problems in kinetic theory of gases, particularly attempts to solve the Boltzmann equation; Free-molecule and near-free-molecule flow in neutral and ionized gases; The physics of surface interactions between gases and solids; Boundary conditions for rarefied gas equations—slip flow; Experimental techniques and instrumentation developments bearing on the above, whether applied to laboratory or field experiments.

Inquiries from the U.S. should be addressed to L. Talbot, Department of Aeronautical Sciences, University of California, Berkeley, and from Europe to Laboratoire d'Aerothermique, 4c, route des Gardes, Meudon (S. and O.), France.