AMERICAN JOURNAL of PHYSICS

Frequencydomain description of a lockin amplifier

John H. Scofield

Citation: American Journal of Physics **62**, 129 (1994); doi: 10.1119/1.17629
View online: http://dx.doi.org/10.1119/1.17629
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/62/2?ver=pdfcov
Published by the American Association of Physics Teachers

We close showing that statistical mechanics enables a direct confirmation of Eq. (29). For a 1D relativistic ideal gas with $q \equiv pv$,

$$\langle q^2 \rangle = Z^{-1} \int_0^\infty q^n \exp(-\beta E) dp,$$

where Z is the partition function, defined by

$$Z = \int_0^\infty \exp(-\beta E) dp,$$

and $E = \sqrt{(pc)^2 + (mc^2)^2}$. Notice that $dE/dp = pc^2/E = v$ (speed), and $q = p dE/dp = (pc)^2/E$. Writing $u = -\beta^{-1} p^n \times (dE/dp)^{n-1}$ and $w = \exp(-\beta E]$, $\langle q^n \rangle = Z^{-1} \int u dw$, which can be integrated by parts, giving

$$\beta \langle q^n \rangle = (2n-1) \langle q^{n-1} \rangle - (n-1) \langle q^n/E \rangle.$$

Use of the identity $q^n/E = q^{n-1}(1 - (mc^2)^2/E^2)$ then yields Eq. (29) directly.

ACKNOWLEDGMENTS

I am grateful to Frank Crawford for the inspiration generated by his creative ideas and for his helpful correspondence, and to John Mallinckrodt and Thomas Marcella for their valuable, constructive comments on drafts of this manuscript.

[1] F. S. Crawford, "Relativistic equipartition via a massive damped sliding partition," Am. J. Phys. **61**, 317–326 (1993).

[2] H. S. Leff and A. F. Rex, *Maxwell's Demon: Entropy, Information, Computing* (Princeton University, Princeton, 1990, and Adam Hilger, Bristol, 1990).

[3] This configuration is motivated by an earlier pedagogical study on how heat engines do external work. See H. S. Leff, "Heat engines and the performance of external work," Am. J. Phys. **46**, 218–224 (1978).

[4] An excellent discussion of these limitations may be found in T. V. Marcella, "Entropy production and the second law of thermodynamics: An introduction to second law analysis," Am. J. Phys. **60**, 888–895 (1992).

[5] In all the examples considered by Crawford, demonlike actions either increase the system entropy or leave it unaltered; i.e., $\Delta S > 0$. This differs from the traditional Maxwell's demon, whose task is to lower a system's entropy.

[6] These forms hold for nonrelativistic gases, where $E \approx mc^2 + p^2/2m$ with m = rest mass and c = speed of light, and for photons, where $E = cp$ with $m = 0$.

[7] A. S. Kompaneets, "The establishment of thermal equilibrium between quanta and electrons," Sov. Phys. JETP **4**, 730–737 (1957). Note that addition of a speck of carbon, rather than an electron, would lead to the absorption and emission of photons. This generally *changes* the number of photons and leads to the Planck, rather than the Wien, distribution.

[8] For a general discussion of equilibrium between matter and radiation, which explains and extends Einstein's path-breaking work, see F. Munley, "Approach of gas and radiation to equilibrium," Am. J. Phys. **58**, 357–362 (1990).

[9] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics* (Wiley, New York, 1985), pp. 53–54.

[10] A. E. Curzon and H. S. Leff, "Resolution of an entropy maximization controversy," Am. J. Phys. **47**, 385–387 (1979).

[11] V. J. Menon and S. C. Agrawal, "Concept of relativistic temperature via the Crawford technique," Am. J. Phys. **59**, 258–260 (1991).

[12] P. T. Landsberg, "Equipartition for a relativistic gas," Am. J. Phys. **60**, 561 (1992).

[13] F. S. Crawford, "Elementary examples of adiabatic invariance," Am. J. Phys. **58**, 337–344 (1990); D. E. Neuenschwander and S. R. Starkey, "Adiabatic invariance derived from the Rund–Trautman identity and Noether's theorem," Am. J. Phys. **61**, 1008–1013 (1993).

[14] Use of the mixing metaphor to interpret entropy was proposed in H. S. Leff, "A mixing route to thermodynamics," Am. J. Phys. **61**, 667 (1993); "A modern approach to thermodynamics," AAPT Announcer **22**, 100 (1992); and "Entropy, mixing, and information," *ibid.* **21**, 93 (1991).

[15] H. S. Leff, "Available work from a finite source and sink: How effective is a Maxwell's demon?" Am. J. Phys. **55**, 701–705 (1987).

Frequency-domain description of a lock-in amplifier

John H. Scofield
Department of Physics, Oberlin College, Oberlin, Ohio 44074

(Received 18 February 1993; accepted 3 June 1993)

The basic principles behind the operation of a lock-in amplifier are described. Particular emphasis is placed on looking at the frequency components of the signal present at the various stages of the lock-in during a typical measurement. The description presented here has been used successfully to explain lock-in operation to upper-level laboratory students at Oberlin College.

I. INTRODUCTION

While not as important as the oscilloscope, the lock-in amplifier is a laboratory instrument whose operation should be familiar to undergraduate physics majors, especially those planning to pursue graduate studies in experimental physics. Several lock-in experiments have been proposed for inclusion in intermediate-level laboratory courses.[1,2] At Oberlin College, students are introduced to the operation of a lock-in amplifier in the mandatory junior course in laboratory techniques. The following year, in the mandatory senior lab, students are expected to use a lock-in to perform measurements associated with several solid-state experiments; these include experiments to measure the temperature dependence of the resistance of a pure metal and an yttrium-barium-copper-oxide (YBCO) superconductor, and the Hall effect in a metallic thin film.

The theory behind the operation of a lock-in amplifier has become a standard topic included in many introduc-

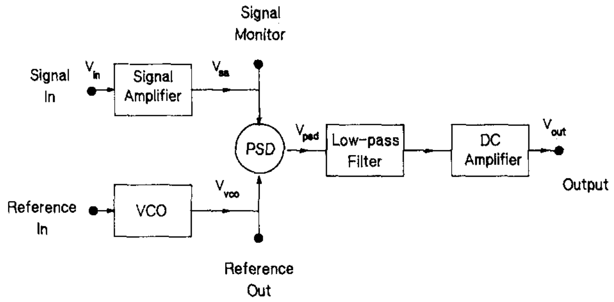

Fig. 1. Block diagram of a lock-in amplifier.

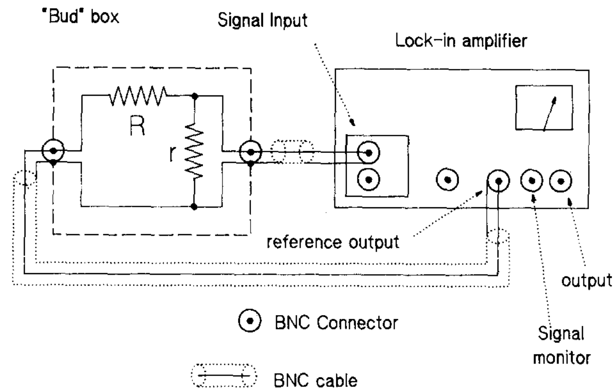

Fig. 2. Measurement circuit.

tory electronics textbooks for scientists.[3] Nevertheless, many students continue to have difficulty grasping how a lock-in amplifier works. Here, I describe the operation of a lock-in amplifier from a nonstandard perspective, emphasizing the frequency, rather than time domain. This treatment has been successful in introducing Oberlin students to lock-ins.

II. PURPOSE OF A LOCK-IN AMPLIFIER

In a nutshell, what a lock-in amplifier does is measure the amplitude V_0 of a sinusoidal voltage,

$$V_{\text{in}}(t) = V_0 \cos(\omega_0 t), \tag{1}$$

where $\omega_0 = 2\pi f_0$ and f_0 is the ordinary frequency of the signal. You supply this voltage to the signal input of the lock-in and its meter displays the amplitude V_0, typically calibrated in V-rms. To accomplish its task the lock-in must also be supplied with a decent size (say 1 $V_{p\text{-}p}$) *reference input* voltage synchronous with the signal whose amplitude is to be measured.[4] The lock-in uses this signal (like an external trigger for an oscilloscope) to "find" the signal to be measured while ignoring anything that is not synchronized with the reference.

In simple cases, particularly those in which the signal to be measured is dominant, an oscilloscope or ac voltmeter might be used to determine V_0 more easily than a lock-in amplifier. In many cases however, the desired signal is only a minor component of the voltage present, i.e.,

$$V_{\text{in}}(t) = V_0 \cos(\omega_0 t) + V_n(t) \tag{2}$$

with $\langle V_n^2 \rangle$ much greater than V_0^2. In such cases $V_n(t)$ is considered to be interference or "background noise." It is no easy task to determine V_0 in the presence of a large background noise. The lock-in can measure voltage amplitudes as small as a few nanovolts while ignoring noise signals thousands of times larger. In contrast, an ac voltmeter or oscilloscope responds to the entire sum above.

III. BLOCK DIAGRAM

While a lock-in amplifier is an extremely important and powerful measuring tool, it is also quite simple. The block diagram of a lock-in amplifier is shown in Fig. 1. The lock-in consists of five stages: (1) an ac amplifier, called the signal amplifier; (2) a voltage controlled oscillator (VCO); (3) a multiplier, called the phase sensitive detector (PSD); (4) a low-pass filter; and (5) a dc amplifier. The signal to be measured is fed into the input of the ac amplifier. The output of the dc amplifier is a dc voltage proportional to V_0. This voltage is displayed on the lock-in's meter and is also available at the output connector. I now elaborate on the functions of the five stages.

(1) The *ac amplifier* is simply a voltage amplifier combined with variable filters. Some lock-in amplifiers let you change the filters as you wish, others do not. Some lock-in amplifiers have the output of the ac amplifier stage available at the *signal monitor* output. Many do not.

(2) The *voltage controlled oscillator* is just an oscillator, except that it can synchronize with an external reference signal (i.e., trigger) both in phase and frequency. Some lock-in amplifiers contain a complete oscillator and need no external reference. In this case they operate at the frequency and amplitude that you set and you must use their oscillator output in your experiment to derive the signal that you ultimately wish to measure. Virtually all lock-in amplifiers are able to synchronize with an external reference signal. The VCO also contains a phase-shifting circuit that allows the user to shift its signal from 0° to 360° with respect to the reference.

(3) The *phase sensitive detector* is a circuit which takes in two voltages as inputs V_1 and V_2 and produces an output which is the product $V_1^* V_2$. That is, the PSD is just a multiplier circuit. (This is not quite true, since the PSD is a very specialized multiplier, but more on that later.)

(4) The *low pass filter* is an RC filter whose time constant may be varied by the user. In many cases you may choose to have one RC filter stage (single pole filter) or two RC filter stages in series (two-pole filter). In newer lock-in amplifiers, this might be a digital filter with the attenuation of a multipole filter.

(5) The *dc amplifier* is just a low-frequency amplifier similar to those frequently assembled with op amps. It differs from the ac amplifier in that it works all the way down to zero frequency (dc) and is not intended to work well at higher frequencies, say above 10 kHz.

IV. HOW A LOCK-IN WORKS (TIME-DOMAIN DESCRIPTION)

Briefly, a lock in works like this.[5] A signal $V_0 \cos(2\pi f_0 t)$ is fed into the signal channel input and amplified by the ac amplifier stage. Since this signal is at a frequency f_0, it is not necessary to amplify other frequencies. In fact, significant extraneous signals may be eliminated by placing filters in the ac amplifier to pass only a narrow band of frequencies around f_0. The ac amplifier

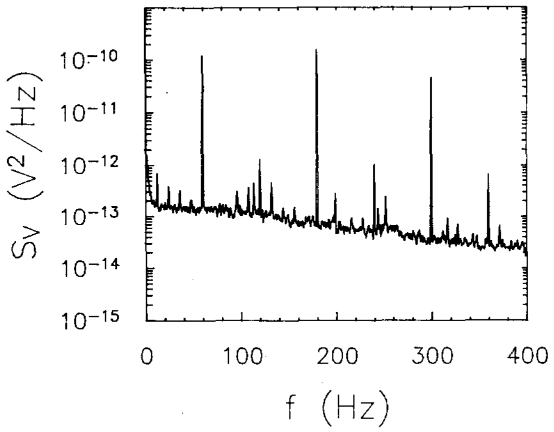

Fig. 3. Graph of the power spectral density of the input signal. The desired signal is the spike at 199 Hz. Other spikes are due to unwanted pickup from the 110 VAC line at 60 Hz and its harmonics. The broad spike at zero frequency is $1/f$ noise. The flat background noise is the unavoidable thermal noise.

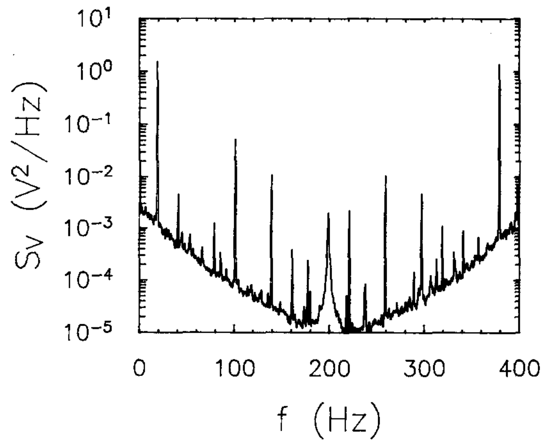

Fig. 5. Power spectral density of the output of the multiplier stage. The effect of beating has shifted each spike at frequency f_j in Fig. 4 to two new positions, 199 Hz$+f_j$ and 199 Hz$-f_j$. The desired signal is now located at zero frequency and at 398 Hz. The broad spike now located at 199 Hz is mostly associated with the "$1/f$ noise" of the signal amplifier.

has a voltage gain G_{ac} that is determined by the sensitivity setting of the lock-in.[6] The output of the ac amplifier becomes one of the two inputs to the multiplier stage, namely

$$V_{ac}(t)=G_{ac}V_0\cos(\omega_0 t). \quad (3)$$

The multiplier stage is the heart of the lock-in amplifier. A multiplier produces an output voltage that is the product of the voltages at its two inputs, $V_1(t)$ and $V_2(t)$. As mentioned above, the output of the ac amplifier is one of the multiplicands. The other is a voltage

$$V_{VCO}(t)=E_0\cos(\omega_0 t+\phi) \quad (4)$$

furnished by the voltage controlled oscillator. The multiplier output is then

$$V_{PSD}(t)=G_{ac}V_0E_0\cos(\omega_0 t)\cos(\omega_0 t+\phi). \quad (5)$$

For simplicity, assume $\phi=0$, i.e., that the VCO output is in phase with the signal that is being measured. Recall that when two sinusoidal signals at frequencies f_1 and f_2 are multiplied together, their product may be represented by the sum of two new sinusoids, one having a frequency equal to the sum f_1+f_2 and the other at the difference frequency, f_2-f_1.[7] For this application the two frequencies f_1 and f_2 are identical, so the output of the multiplier has components at the second harmonic (i.e., $2f_0$) and at dc (i.e., at $0=f_0-f_0$). Using the appropriate trigonometric identity, the above equation may be rewritten

$$V_{PSD}(t)=\tfrac{1}{2}G_{ac}V_0E_0[1+\cos(2\omega_0 t)]. \quad (6)$$

The amplitudes of the second harmonic and the dc voltage are both proportional to V_0, the amplitude we are trying to measure. We can throw away the redundant information at $2f_0$ and concentrate only on the dc component. This is accomplished by feeding the multiplier output into a low-pass filter whose time constant is chosen so that the signal at $2f_0$ is strongly attenuated. The output of the low-pass filter may not yet be large enough, so it is ampli-

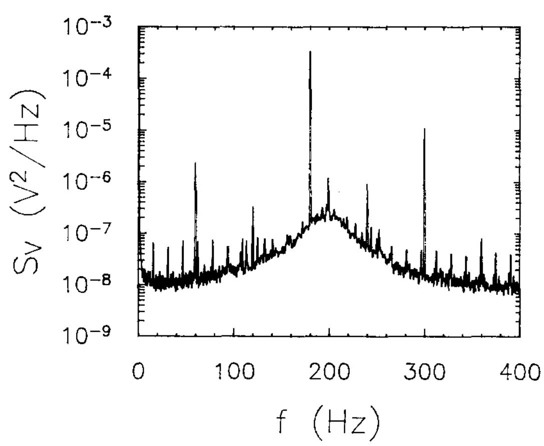

Fig. 4. Power spectral density of the output of the ac amplifier stage. The bandpass character allows the desired frequency of 199 Hz to be amplified while frequencies well away from 199 Hz are attenuated. In particular we have reduced the effects of line frequency pickup and the $1/f$ noise near dc.

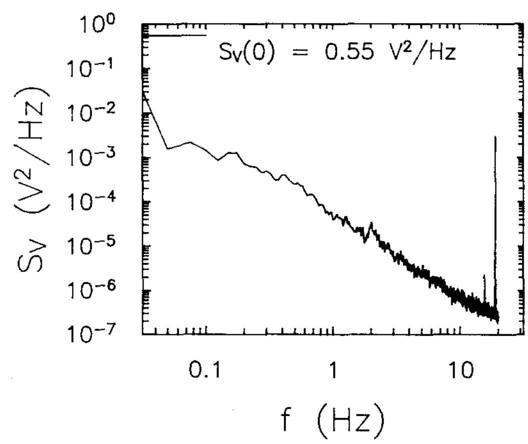

Fig. 6. Power spectral density at the output of the dc amplifier. Note the change in frequency scale from the earlier plots. The desired signal at dc, $S_V(0)=0.55$ V^2/Hz (not shown on the logarithmic scale) is now the only significant signal since the low-pass filter has removed everything above 1 Hz.

fied further with a dc amplifier. The final output voltage is a dc voltage directly proportional to the amplitude V_0 we were trying to measure, namely

$$V_{out}=\frac{1}{2}G_{dc}G_{ac}V_0E_0, \quad (7)$$

where G_{dc} is the voltage gain of the dc amplifier stage.[8] This voltage is both displayed on the meter, calibrated in rms voltage, and also made available at the output connector.

The low-pass filter serves several functions. First, as already mentioned, it eliminates the second-harmonic signal produced by the multiplier. Second, it is an exponential integrator.[9] Much of the random noise that ends up in the dc output of the multiplier will integrate to zero, and thus contribute negligibly to the measurement.

V. FREQUENCY-DOMAIN DESCRIPTION

To understand better what is going on it is useful to look at the frequency components present in the signal at various stages of the lock-in amplifier. Figure 2 shows the circuit that was used to make these measurements. The internal oscillator of a lock-in amplifier furnished a very small current to series resistors $R\approx 100\ k\Omega$ and $r\approx 1\ k\Omega$ at a frequency $f_0=199$ Hz. The voltage drop across the smaller resistor, r, was connected to the signal input of the lock-in amplifier. A spectrum analyzer was used to measure the power spectral density of the voltages present at the various stages of the lock-in amplifier.[10]

Figure 3 contains a semilog graph of the power spectral density (S_V) vs frequency (f) of the signal present at the input of the lock-in.[11] (Note that the power in a certain bandwidth is not just the area under the curve because of the logarithmic vertical scale.) In addition to the small desired signal at frequency f_0, there were also present large interfering signals at the power line frequency, 60 Hz, and its (mainly odd) harmonics. In addition to these, the input signal contains broadband noise which tends to be frequency independent at high frequencies but increases at low frequencies as $1/f$.[12]

Figure 4 shows the output of the ac amplifier. The ac amplifier was set to operate as a bandpass filter centered at $f_0\approx 200$ Hz with a "quality factor," $Q=5$. Note how the desired signal is amplified while unwanted frequencies, notably those associated with 60 Hz and its harmonics, are significantly attenuated. Despite this attenuation the third harmonic of the line frequency remains the largest signal present.

The output of the multiplier stage is shown in Fig. 5. All of the "spikes" in the spectrum have now been shifted from their original frequencies f_j to f_0+f_j and f_0-f_j. This phenomenon is called "beating." The desired signal is now contained in both the spikes at zero frequency (dc) and $2f_0=398$ Hz. The interference from the third harmonic of the line frequency now appears at 19 and 379 Hz.

Figure 6 shows the output of the dc amplifier with the low-pass filter time constant set to 1 s. Note the different frequency scale from those of Figs. 3–5. All of the frequency components present in the output in the range 0–1 Hz are present in the input signal at frequencies 198–200 Hz![13] Thus the low-pass filter has, in some sense, allowed the lock-in to act like a narrow-band amplifier with a high $Q\approx(200\ \text{Hz})/(2\ \text{Hz})=100$. If we are willing to settle for a slower response time, a time constant of 100 s gives an effective Q of 10^5!

VI. IMPORTANT COMPLICATIONS

Real lock-in amplifiers differ from that described above in two important ways. First, the reference may be shifted in phase from the desired signal so that an accurate phase-shifting circuit is included in the voltage controlled oscillator stage. This is important since, in practice, we may not know the phase of the input signal. More precisely, the input signal of Eq. (1) is more accurately written as $V_{in}(t)=V_0\cos(\omega_0t+\delta)$, where δ is unknown. With this change, the final output represented by Eq. (7) should be written as

$$V_{out}=\frac{1}{2}G_{dc}G_{ac}V_0E_0\cos(\delta+\phi). \quad (8)$$

In this case it is not desirable to set the VCO phase angle ϕ to zero, but instead, one adjusts ϕ so as to maximize V_{out}, i.e., so that $\delta+\phi$ is zero.

Second, for many commercial lock-in amplifiers, the multiplier does not multiply by a sinusoid at all, but rather by a square wave

$$V_{VCO}(t)=\sum_{n=0}^{\infty}\frac{4E_0}{(2n+1)}\sin[(2n+1)2\pi f_0t+\phi_n] \quad (9)$$

at frequency f_0.[14] Since a square wave can be represented by a Fourier series whose dominant component is the fundamental, this does not alter the result much *provided* that the ac amplifier is operated in a narrow-band mode, strongly attenuating any input signal at the higher harmonics. However, a square-wave multiplier certainly complicates the analysis.[15]

The reason for multiplying by a square wave is a practical one. Until recently, it has not been possible to build sufficiently stable multiplier circuits. It has been possible, however, to make very stable transistor switches that turn "on" and "off" like a square wave. In practice, most lock-in amplifiers use these in place of the multiplier stage. The input of the PSD is just the output of the ac amplifier stage as described previously. When the PSD is switched on its output is equal to its input. When it is switched off its output is zero. The net result is that the output of the PSD is equal to its input multiplied by a square wave. Thus most analog lock-in amplifiers multiply by square waves rather than sine waves, and the PSD is intimately connected with the VCO.

VII. SUMMARY

In summary, I have described the operation of a lock-in amplifier. The lock-in amplifier consists of five stages: the ac amplifier, the VCO, the PSD, the low-pass filter, and the dc amplifier. The functions of the various stages have been illustrated by looking at power spectral densities of signals present at each them during a typical measurement.

[1] P. A. Temple, "An introduction to phase-sensitive amplifiers: An inexpensive student instrument," Am. J. Phys. **43**, 801–807 (1975).

[2] R. Wolfson, "The lock-in amplifier: a student experiment," Am. J. Phys. **59**, 569–572 (1991).

[3] For instance, see R. E. Simpson, *Introductory Electronics for Scientists and Engineers*, 2nd ed. (Allyn and Bacon, Boston, 1987), pp. 344–355;

D. W. Preston and E. R. Dietz, *The Art of Experimental Physics* (Wiley, New York, 1991), pp. 367–375; R. A. Dunlop, *Experimental Physics* (Oxford University, New York, 1988), pp. 102–107.

[4]Actually, the reference signal does not have to be sinusoidal, the only requirement is that it be periodic, synchronous with the desired signal.

[5]A variety of useful technical notes are available from EG&G Princeton Applied Research. Three of these are Technical Note 115, "Explore the lock-in amplifier," Technical Note 116, "Specifying lock-in amplifiers," and "Lock-in Applications Anthology," 1985.

[6]More precisely, the ac amplifier is characterized by a complex transfer function, $G_{ac}(\omega)$, where $\omega=2\pi f$.

[7]This, of course, is easily shown using the trigonometric identity $\cos(\alpha)\cos(\beta)=\frac{1}{2}\cos(\alpha-\beta)+\frac{1}{2}\cos(\alpha+\beta)$.

[8]Actually, it makes more sense to consider the low-pass filter and dc amplifier as a single, low-frequency amplifier having a complex transfer function $G_{dc}(\omega)$.

[9]The exponential and linear integrators have different properties. For more see D. C. Champeney, *Fourier Transforms and Their Physical Applications* (Academic, New York, 1973), pp. 122–125.

[10]The power spectral density or power spectrum, $S_V(f)$, of a voltage $V(t)$ is defined to be the mean square voltage per unit frequency. Just as the power contained between frequencies f_1 and f_2 for black-body radiation is obtained by integrating the black-body spectral distribution over this bandwidth, the mean-square voltage in the bandwidth f_1 to f_2 is given by $\langle v^2\rangle = \int_{f_1}^{f_2} s_V(f)df$.

[11]This was actually measured by connecting the spectrum analyzer to the signal monitor connector and setting the frequency response of the signal amplifier to be flat. The measured spectrum was then divided by the square of the known voltage gain of the signal amplifier to obtain the spectrum for the signal at the input.

[12]This, so-called, "$1/f$ noise," is actually added by the ac amplifier stage and is present in virtually all electrical components.

[13]This "folding" of a bandwidth $2\Delta f$ of the input signal to a bandwidth Δf of the output occurs through the demodulation process of the PSD. The beating of the 199 Hz reference with any signal present at 200 Hz produces outputs at both 399 Hz and 1 Hz (sum and difference frequencies). Similarly, the beating of the reference with any signal present at 198 Hz produces outputs at 397 and 1 Hz. Both "sum" frequencies (397 and 399 Hz) are removed by the low-pass filter of the dc amplifier.

[14]One vendor, *Stanford Research Instruments*, sells a "digital" lock-in that multiplies by a pure sinusoid. Because of their simplicity, greater dynamic reserve, the cost-effectiveness, digital lock-ins will likely render analog lock-ins obsolete.

[15]For a more detailed analysis see John H. Scofield, "ac method for measuring low-frequency resistance fluctuation spectra," Rev. Sci. Instrum. **58**, 985–993 (1987).

Charge-coupled device detection of two-beam interference with partially coherent light

B. Tom King and W. Tobin
Department of Physics & Astronomy, University of Canterbury, Private Bag 4800, Christchurch, New Zealand

(Received 15 February 1993; accepted 2 June 1993)

In a demonstration of the advantages of charge-coupled device detectors in the undergraduate laboratory the Thompson and Wolf experiment on the effect of partial coherence on fringe visibility was repeated. Confirmation of the van Cittert–Zernike theorem to a few percent accuracy was attained.

I. CHARGE-COUPLED DEVICE DETECTORS

A charge-coupled device (CCD) is a two-dimensional array of identical semiconductor units each of which is photosensitive. Incident photons create electron–hole pairs in the doped semiconductor substrate. The majority charge carriers are conducted away. The minority carriers are trapped in depletion regions created by the potentials of an overlying electrode structure and form a signal equal to the number of detected photons. After an exposure has been made a serial process of cycling the electrode potentials allows the charges in each photosensitive unit (or "pixel") to be transferred across the chip to an output amplifier. Fuller descriptions of CCD operation are available elsewhere at an elementary[1] or more advanced level.[2]

In the last decade research-grade CCDs have revolutionized optical astronomy because of their great sensitivity resulting from their high quantum efficiency and low readout noise. CCDs have replaced photographic plates for almost all observations where a CCD's relatively small format is not a disadvantage.

Recently a number of companies have started producing inexpensive CCD camera systems (US $400–$3000) aimed primarily at the amateur astronomer and astronomy education markets.[3] Systems typically include the CCD sensor

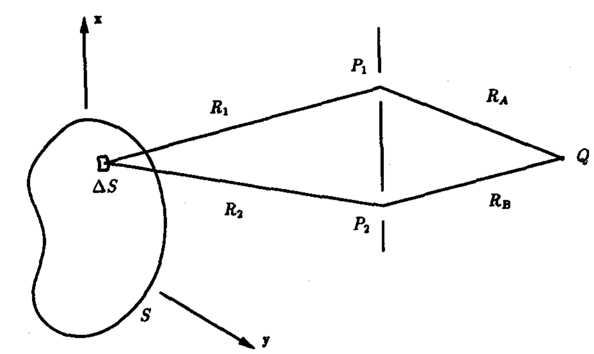

Fig. 1. Coherence schematic. The spatially incoherent, quasimonochromatic source radiates over the surface S; the apertures P_1 and P_2 are secondary sources.

